# ULRICH PARTITIONS FOR TWO-STEP FLAG VARIETIES

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Abstract. Ulrich bundles play a central role in singularity theory, liaison theory and Boij-Söderberg theory. Coskun, Costa, Huizenga, Miró-Roig and Woolf proved that Schur bundles on flag varieties of three or more steps are not Ulrich and conjectured a classification of Ulrich Schur bundles on two-step flag varieties. By the Borel-Weil-Bott Theorem, the conjecture reduces to classifying integer sequences satisfying certain combinatorial properties. In this paper, we resolve the first instance of this conjecture and show that Schur bundles on $F(k, k+3; n)$ are not Ulrich if $n > 6$ or $k > 2$.

Contents



## 1. Introduction

Let $j, k, l > 0$ be positive integers. Let

$$P = (a_1, \dots, a_k | b_1, \dots, b_j | c_1, \dots, c_l)$$

be a strictly increasing sequence of integers divided into 3 nonempty subsequences $a_\bullet, b_\bullet, c_\bullet$. Let $P(t)$ denote the sequence

$$P(t) = (a_1 + t, \dots, a_k + t | b_1, \dots, b_j | c_1 - t, \dots, c_l - t)$$

obtained by adding $t$ to each of the entries in the sequence $a_\bullet$ and subtracting $t$ from each of the entries in the subsequence $c_\bullet$. Set $N = kj + kl + jl$.

**Definition 1.1.** The partition $P$ is called an *Ulrich partition* if the sequences $P(t)$ have exactly two equal entries for $1 \leq t \leq N$.

Note that $P(t)$ can have repeated entries for at most $N$ values of $t$. We will refer to $P(t)$ as the time evolution of $P$ at time $t$. Hence, Ulrich partitions are those for which there are a maximum number of collisions among the entries during their time evolution and these collisions all occur at consecutive times.

Two partitions $P_1$ and $P_2$ are *equivalent* if they differ by adding a constant to all the entries. If $P_1$ and $P_2$ are equivalent, then $P_1$ is Ulrich if and only if $P_2$ is. We always consider partitions up to equivalence. Our main theorem is the following.

**Theorem 1.2.** *If $P = (a_1, \dots, a_k | b_1, b_2, b_3 | c_1, \dots, c_l)$ is an Ulrich partition, then $k + l \leq 3$.*

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Given a partition $P = (a_1, \dots, a_k | b_1, \dots, b_j | c_1, \dots, c_l)$, we obtain a new partition $P^s$ called the *symmetric partition* by multiplying all the entries by $-1$ and listing the entries in the reverse order

$$P^s = (-c_l, \dots, -c_1 | -b_j, \dots, -b_1 | -a_k, \dots, -a_1).$$

The partition $P$ is Ulrich if and only if $P^s$ is Ulrich. Similarly, there is a dual partition $P^*$ obtained by

$$P^* = (c_1 - (N+1)t, \dots, c_l - (N+1)t | b_1, \dots, b_j | a_1 + t(N+1), \dots, a_k + t(N+1)).$$

This is the partition $P(N+1)$ reordered so that the entries are increasing. By running the time evolution backwards, it is clear that $P$ is Ulrich if and only if $P^*$ is Ulrich (see [CCHMW, §3] for more details). We can also form $(P^s)^*$, which is Ulrich if and only if $P$ is.

As a consequence of the proof, we obtain a complete classification of Ulrich partitions where the $b_\bullet$ subsequence has length 3. Up to equivalence and these symmetries, they are

$$(0|1,2,3|8), \quad (-8,0|1,2,3|8), \quad (0|1,2,5|8), \quad (-1|1,2,6|7), \quad (0|1,3,6|8).$$

We now explain the significance of Ulrich partitions. Let $X \subset \mathbb{P}^m$ be an arithmetically Cohen–Macaulay projective variety of dimension $d$. A vector bundle $\mathcal{E}$ on $X$ is called an *Ulrich bundle* if $H^i(X, \mathcal{E}(-i)) = 0$ for $i > 0$ and $H^j(X, \mathcal{E}(-j-1)) = 0$ for $j < d$ (see [BaHU], [BHU] and [ESW]). These are the bundles whose Hilbert polynomials have $d$ zeros at the first $d$ negative integers. They play a central role in singularity theory, liaison theory and Boij–Söderberg theory. For example, if $X$ admits an Ulrich bundle, then the cone of cohomology tables of $X$ coincides with that of $\mathbb{P}^m$ [ES]. Consequently, classifying Ulrich bundles on projective varieties is an important problem in commutative algebra and algebraic geometry (see [CKM], [CCHMW], [F] for more details and references). In particular, it is interesting to decide when representation theoretic bundles on flag varieties are Ulrich.

Let $0 < k_1 < k_2 < n$ be three positive integers. Set $k_0 = 0$ and $k_3 = n$. Let $V$ be an $n$-dimensional vector space. The two-step partial flag variety $F(k_1, k_2; n)$ parameterizes partial flags $W_1 \subset W_2 \subset V$, where $W_i$ has dimension $k_i$. The variety $F(k_1, k_2; n)$ has a minimal embedding in projective space corresponding to the ample line bundle with class the sum of the two Schubert divisors. We will always consider $F(k_1, k_2; n)$ in this embedding and $\mathcal{O}(1)$ will refer to the hyperplane bundle in this embedding.

The variety $F(k_1, k_2; n)$ has a collection of tautological bundles

$$0 = T_0 \subset T_1 \subset T_2 \subset T_3 = \underline{V} = V \otimes \mathcal{O}_{F(k_1,k_2;n)},$$

where $\underline{V}$ is the trivial bundle of rank $n$ and $T_i$, for $i = 1$ or 2, is the subbundle of $\underline{V}$ of rank $k_i$ which associates to a point $[W_1 \subset W_2]$ the subspace $W_i$. Let $U_i = T_i / T_{i-1}$. Given $\lambda = (\lambda_1 | \lambda_2 | \lambda_3)$ a concatenation of partitions $\lambda_i$ of length $k_i - k_{i-1}$, the Schur bundle $E_\lambda$ is defined by

$$E_\lambda = \mathbb{S}^{\lambda_1} U_1^* \otimes \mathbb{S}^{\lambda_2} U_2^* \otimes \mathbb{S}^{\lambda_3} U_3^*,$$

where $\mathbb{S}^\lambda$ is the Schur functor of type $\lambda$.

Costa and Miró-Roig in [CMR] initiated the study of determining when Schur bundles are Ulrich. They showed every Grassmannian admits Ulrich Schur bundles and classified these bundles. In [CCHMW], the authors showed that Schur bundles on flag varieties with three or more steps are never Ulrich for their minimal embedding. They also constructed several infinite families of Ulrich Schur bundles on specific two-step flag varieties and showed that many two-step flag varieties do not admit Ulrich Schur bundles. They conjectured a complete classification of Ulrich Schur bundles on two-step flag varieties. Their main conjecture is the following.

**Conjecture 1.3.** [CCHMW, Conjecture 5.9] *A two-step flag variety $F(k_1, k_2; n)$ does not admit an Ulrich Schur bundle with respect to $\mathcal{O}(1)$ if $k_2 \geq 3$ and $n - k_2 \geq 3$.*

The Borel–Weil–Bott Theorem computes the cohomology of Schur bundles and allows one to determine whether a Schur bundle is Ulrich. There is a bijective correspondence between equivalence classes of Ulrich partitions of type $(n-k_2, k_2-k_1, k_1)$ and Schur bundles $E_\lambda$ on $F(k_1, k_2; n)$ which are Ulrich [CCHMW, Proposition 3.5]. Hence, classifying Ulrich Schur bundles is equivalent to classifying Ulrich partitions. Consequently, as a corollary of Theorem 1.2, we resolve the first case of Conjecture 1.3.

**Theorem 1.4.** *The flag variety $F(k, k+3; n)$ does not admit an Ulrich Schur bundle with respect to $\mathcal{O}(1)$ if $n > 6$ or $k > 2$.*

In particular, the only two step flag varieties of the form $F(k, k+3; n)$ that admit Ulrich Schur bundles are $F(1, 4; 5)$, $F(1, 4; 6)$ and $F(2, 5; 6)$. All the Ulrich Schur bundles on these varieties have been classified in [CCHMW]. There has been work on classifying Ulrich Schur bundles on other homogeneous varieties using the same strategy (see [Fo]).

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## 2. The proof of the main theorem

In this section, we prove our main theorem.

**Theorem 2.1.** *There does not exist Ulrich partitions $(a_1, \dots, a_k | b_1, b_2, b_3 | c_1, \dots, c_l)$ with $k+l > 3$.*

We begin with the following simple observation, which is a special case of [CCHMW, Lemma 4.3].

**Lemma 2.2.** *If $P = (a_1, \dots, a_l | b_1, \dots, b_j | c_1, \dots, c_k)$ is an Ulrich partition, then all the entries in the sequences $a_\bullet$ and $c_\bullet$ are equal modulo 2.*

*Proof.* If $P$ is Ulrich, the $a_p$ and $c_q$ entries of $P(t_{pq})$ must be equal at some time $t_{pq}$. From now on, we will express this by saying $a_p$ and $c_q$ collide at time $t = t_{pq}$. Hence $a_p + t_{pq} = c_q - t_{pq}$ or, equivalently, $c_q - a_p = 2t_{pq}$. Consequently, $a_p$ and $c_q$ are equal modulo 2. Since this holds for each $1 \leq p \leq l$ and $1 \leq q \leq k$, we conclude that all the entries in the sequences $a_\bullet$ and $c_\bullet$ have the same parity. Furthermore, their parities remain equal in $P(t)$ for all $t$. □

Let $P = (a_1, \dots, a_k | b_1, b_2, b_3 | c_1, \dots, c_l)$ be an Ulrich partition. Recall that we always assume $k, l > 0$. Up to symmetry and duality, there are three possibilities:

(1) The sequence $b_1, b_2, b_3$ may be consecutive.
(2) Only the entries $b_1, b_2$ may be consecutive.
(3) Finally, no two of the entries in $b_\bullet$ are consecutive.

We will analyze each of these cases separately.

**The $b_\bullet$ sequence is consecutive.** In this case, we will see that $k + l \leq 3$ and up to symmetry and duality the two possible partitions are $(0|1, 2, 3|8)$ or $(-8, 0|1, 2, 3|8)$. In fact, we can analyze sequences where the $b_\bullet$ sequence is consecutive more generally.

**Proposition 2.3.** *Let $P$ be an Ulrich partition of the form $(a_1, \dots, a_k | 1, 2, \dots, r | c_1, \dots, c_l)$, where the $b_\bullet$ sequence consists of $r$ consecutive integers. Assume that $r \geq 3$. Then $k + l \leq 3$.*

*Proof.* Without loss of generality, we may assume that at $t = 1$, the collision is $a_k b_1$. Then for $1 \leq t \leq r$, the collision is $a_k b_t$. We claim that at $t = r+1$, the collision must be $a_k c_1$. The collision must be either $a_{k-1} b_1$ or $a_k c_1$. If $r$ is odd, then it cannot be $a_{k-1} b_1$ since otherwise $a_{k-1}$ and $a_k$ would have different parities. If $r$ is even and the collision is $a_{k-1} b_1$, we obtain a contradiction as follows. Let $t_0$ be the time of the collision $a_k c_1$. Until that time all the collisions must be between an entry from $a_\bullet$ and an entry from $b_\bullet$. We conclude that $t_0 = ir+1$ for some $i$. At time $t = t_0 + 1$, the collision cannot be $a_k c_2$. Otherwise, we would have $c_2 - c_1 = 2$ and the collisions $c_1 b_1$ and $c_2 b_3$ would occur at the same time. If $i > 1$, the collision at $t = t_0 + 1$ cannot be $b_r c_1$. Hence, at $t = t_0 + 1$, the collision must be $a_{k-i} b_1$. This violates parity since $a_k$ is even while $a_{k-i}$ is odd. We conclude that at $t = r+1$, the collision is $a_k c_1$.

Hence, for $t = r+1+i$ with $1 \leq i \leq r$, the collisions are $b_{r+1-i} c_1$. If the progression stops at time $t = 2r+1$, we obtain the Ulrich partition $(0|1, 2, \ldots, r|2r+2)$. Else, at time $t = 2r+2$, the collision must be $a_{k-1} c_1$. Otherwise, the collision would have to be $a_k c_2$. At time $t = 2r+3$, since the collision could not be $a_k c_3$, the collision would have to be $a_{k-1} c_1$. Then at time $t = 3r+3$, $a_{k-1}, b_r$ and $c_2$ would collide simultaneously. This contradiction shows that the collision at $t = 2r+2$ must be $a_{k-1} c_1$. Hence, for times $t = 2r+2+i$ with $1 \leq i \leq r$, the collisions must be $a_{k-1} b_i$. If the progression stops at $t = 3r+2$, we obtain the Ulrich partition $(-2r-2, 0|1, 2, \ldots, r|2r+2)$.

Otherwise, at time $t = 3r+3$, the collision must either be $a_k c_2$ or $a_{k-2} c_1$. Then at time $t = 3r+4$, the only possible collisions are $a_{k-2} c_1$ or $a_k c_2$, respectively, since the distance between consecutive entries in $a_\bullet$ or $c_\bullet$ has to be at least $r > 2$. If the order is $a_k c_2$ and $a_{k-2} c_1$, then at time $t = 3r+4$ the entry $c_2$ is $3r+2$ and $a_{k-2}$ is $-r-2$. The entries $a_{k-2}$, $b_r$ and $c_2$ collide simultaneously at time $t = 5r+5$. Hence, the order of collisions must be $a_{k-2} c_1$ at time $t = 3r+3$ and $a_k c_2$ at time $3r+4$. If $r \geq 5$, then at time $t = 3r+5$, there cannot be any collisions. If $3 \leq r \leq 4$, the only possible collision at time $t = 3r+5$ is $a_{k-3} c_1$. But then $a_{k-3}, b_r$ and $c_2$ collide simultaneously at time $t = 5r+8$. This is a contradiction. Hence, the time evolution must stop at time $t = 3r+2$ and we conclude the proposition. □

In particular, we conclude that up to equivalence and symmetries, the only Ulrich partitions where the $b_\bullet$ sequence consists of three or more consecutive integers are $(0|1, 2, \ldots, r|2r+2)$ and $(-2r-2, 0|1, 2, \ldots, r|2r+2)$.

2.1. **Exactly two of the $b_\bullet$ entries are consecutive.** Up to symmetry and duality, we may assume that $b_1$ and $b_2$ are consecutive.

**Lemma 2.4.** *Assume that $b_1$ and $b_2$ are the only two consecutive entries in the $b_\bullet$ sequence and $P = (a_1, \ldots, a_k|b_1, b_2, b_3|c_1, \ldots, c_l)$ is Ulrich. Then the $b_\bullet$ sequence up to equivalence and symmetry must be $1, 2, 5$ or $1, 2, 6$. In the first case, at time $t = 1$ the collision is $a_k b_1$. In the second case, at time $t = 1$ the collision is $b_3 c_1$.*

*Proof.* At time $t = 1$, the collision is either $a_k b_1$ or $b_3 c_1$. First, assume that at time $t = 1$ the collision is $b_3 c_1$. Since $b_2$ and $b_3$ are not consecutive, the collision at time $t = 2$ cannot be $c_1 b_2$. By parity, the collision cannot be $b_3 c_2$. Consequently, at time $t = 2$ the collision must be $a_k b_1$. Hence, at time $t = 3$, the collision is $a_k b_2$. If at time $t = 4$ the collision is $a_k c_1$, then the $b_\bullet$ sequence is $1, 2, 6$. Otherwise, the only possible collision is $a_{k-1} b_1$ since $a_k b_3$ or $b_2 c_1$ cannot occur before $a_k c_1$ and $b_3 c_2$ is excluded by parity. Moreover, the distance $|b_3 - b_2| \geq 8$ and $a_k - a_{k-1} = 2$.

The last collision at time $t = N$ is either $a_1 b_3$ or $b_1 c_l$. If it is $b_1 c_l$, then the collisions at time $t = N-1$ and $t = N-2$ must be $b_2 c_l$ and $a_l b_3$, respectively. Note that at time $t = N-2$, the collision cannot be $b_1 c_{l-1}$. Otherwise, $c_l - c_{l-1} = 2$ and $c_l$ would collide with $a_k$ at the same time as $c_{l-1}$ collides with $a_{k-1}$. Then at time $t = N-3$, the collision cannot be $a_{k-1} b_3$ or $c_{l-1} b_1$ by parity. Since $b_3 - b_2 \geq 8$, the collision cannot be $a_1 c_l$. We conclude that at $t = N-3$ there are no possible collisions. This is a contradiction.

If the last collision is $a_1 b_3$, then the two previous collisions must be $b_1 c_l$ and $b_2 c_l$ by parity. At time $t = N-3$, the collision cannot be $b_1 c_{l-1}$ since $c_l - c_{l-1}$ cannot be 2. The collision cannot be $a_2 b_3$ by parity. It cannot be $a_1 c_l$ since $b_3 - b_2 \geq 8$. We obtain a contradiction. We conclude that if at $t = 1$ the collision is $b_3 c_1$, then at $t = 4$ the collision must be $a_k c_1$ and the $b_\bullet$ sequence is up to equivalence $1, 2, 6$.

Next assume that the collision at $t = 1$ is $a_k b_1$. Let $t = 2j + 1$ be the first odd time when the collision is not of the form $a_i b_1$. If $j = 1$, since the entries in $b_\bullet$ are not consecutive, at time $t = 3$ the collision must be $b_3 c_1$. Then at time $t = 4$, by parity, the only possible collision is $a_k c_1$. Therefore, the $b_\bullet$ sequence is $1, 2, 5$. If $j > 1$, then $a_k - a_{k-1} = 2$. The collision at time $t = 2j + 1$ must be $b_3 c_1$. Otherwise, the collision would have to be $a_k b_3$. Then at time $t = 2j + 2$, by parity the collision would have to be $a_k c_1$. Then the collisions $a_{k-1} b_3$ and $b_3 c_1$ would happen at the same time at $t = 2j + 3$. We conclude that at time $t = 2j + 1$ the collision is $b_3 c_1$. At time $t = 2j + 2$, by parity we cannot have a collision of the form $a_i b_1$ or $b_3 c_{l-1}$. We conclude that the collision must be $a_k c_1$. If $j > 1$, then at time $r = 2j + 2$ the collisions $a_{k-1} c_1$ and $a_k b_3$ occur at the same time leading to a contradiction. We conclude that $j = 1$ and the $b_\bullet$ sequence is $1, 2, 5$. This concludes the proof of the lemma. □

We thus obtain two standard Ulrich partitions of type $(1, 3, 1)$ given by $(0|1, 2, 5|8)$ and $(-1|1, 2, 6|7)$. To conclude the analysis in this case, we argue that these Ulrich partitions cannot be extended to longer Ulrich partitions.

**Lemma 2.5.** *The only Ulrich partition of the form*

$$(a_1, \dots, a_{k-1}, a_k = 0|b_1 = 1, b_2 = 2, b_3 = 5|c_1 = 8, c_2, \dots, c_l)$$

*is* $(0|1, 2, 5|8)$. *The only Ulrich partition of the form*

$$(a_1, \dots, a_{k-1}, a_k = -1|b_1 = 1, b_2 = 2, b_3 = 6|c_1 = 7, c_2, \dots, c_l)$$

*is* $(-1|1, 2, 6|7)$.

*Proof.* Suppose there exists an Ulrich partition of the form $(a_1, \dots, a_{k-1}, 0|1, 2, 5|8, c_2, \dots, c_l)$ with $k$ or $l$ bigger than 1. Then the last collision at time $t = N$ must be either $a_1 b_3$ or $b_1 c_l$. If the collision is $a_1 b_3$, then by parity the collision at time $t = N - 1$ must be $b_1 c_l$. Then $a_1$ and $c_l$ have different parities and can never collide. We obtain a contradiction. We conclude that at $t = N$ the collision must be $b_1 c_l$. Hence, at time $t = N - 1$ the collision is $b_2 c_l$. If the collision at $t = N - 2$ is $a_1 b_3$, then the distance between $a_1$ and $a_k$ (which is equal to $N - 7$) is equal to the distance between $c_1$ and $c_l$. Hence, these pairs collide simultaneously leading to a contradiction. We conclude that at time $t = N - 2$, the collision must be $b_1 c_{l-1}$. Hence the collisions at times $t = N - 3, N - 4$ must be $b_2 c_{l-1}$ and $b_3 c_l$, respectively. However, at time $t = N - 5$ there are no possible collisions. The collision cannot be $b_1 c_{l-2}$ by parity. There are no collisions between $c_{l-1}$, $c_l$ and any entries in the $b_\bullet$ sequence. On the other hand, if $a_1$ collides with $c_l$, then at time $t = N - 4$ the $a_1 b_3$ collision coincides with the $b_2 c_{l-1}$ collision. This contradiction shows that $k = l = 1$.

Suppose there exists an Ulrich partition of the form $(a_1, \dots, a_{k-1}, -1|1, 2, 6|7, c_2, \dots, c_l)$ with $k$ or $l$ bigger than 1. The argument is almost identical to the previous case. The last collision at time $t = N$ cannot be $a_1 b_3$. Otherwise, at time $t = N - 1$ the collision would have to be $b_1 c_l$ and the distance between $a_1$ and $a_k$ would equal to the distance between $c_1$ and $c_l$. We conclude that the collision at time $t = N$ is $b_1 c_l$. Hence, at time $t = N - 1$ the collision is $b_2 c_l$. At time $t = N - 2$, the collision cannot be $a_1 b_3$, otherwise at that time $c_l$ would be at position 3 and would have different parity from $a_1$. We conclude that at time $t = N - 2$ the collision must be $b_1 c_{l-1}$. This determines the collisions at $t = N - 3, N - 4$ which must be $b_2 c_{l-1}$ and $b_3 c_l$. Then, as in the previous case, at time $t = N - 5$, there cannot be any collisions leading to a contradiction. This shows that $k = l = 1$. □

2.2. **None of the $b_\bullet$ entries are consecutive.** In this case, we have the following lemma.

**Lemma 2.6.** *Let* $(a_1, \dots, a_k|b_1, b_2, b_3|c_1, \dots, c_l)$ *be an Ulrich partition with* $k, l > 0$ *and none of the entries in the* $b_\bullet$ *sequence are consecutive. Then up to equivalence and symmetry the* $b_\bullet$ *sequence is* $1, 3, 6$.

*Proof.* Without loss of generality, we may assume that at $t = 1$ the collision is $a_k b_1$. By parity and the fact that $b_2 - b_1 > 1$, we conclude that at $t = 2$ the collision must be $b_3 c_1$. Similarly, by parity and the fact that $b_3 - b_2 > 1$, at time $t = 3$ the collision is either $a_k b_2$ or $a_{k-1} b_1$. If the collision is $a_k b_2$, then the collision at $t = 4$ has to be $a_k c_1$. By parity, it cannot be $a_{k-1} b_1$. It cannot be $b_3 c_2$ otherwise the collisions $b_1 c_1$ and $b_2 c_2$ would occur at the same time. We conclude that at time $t = 0$ the $b_\bullet$ sequence must be $1, 3, 6$ and $a_k = 0$ and $c_1 = 8$.

If the collision at time $t = 3$ is $a_{k-1} b_1$, then by parity the collision at $t = 4$ may only be one of $a_k b_2$, $b_2 c_1$ or $b_3 c_2$. It cannot be $b_2 c_1$, otherwise $a_k b_3$ and $a_{k-1} b_2$ would occur at the same time since both $a_{k-1}, a_k$ and $b_2, b_3$ would be two apart. Similarly, it cannot be $b_3 c_2$, otherwise $a_k c_2$ and $a_{k-1} c_1$ would occur at the same time. We conclude that at $t = 4$, the collision is $a_k b_2$. At time $t = 5$, the collision cannot be $b_3 c_2$ by parity. Hence, it is either $a_{k-2} b_1$ or $a_k c_1$. It cannot be $a_k c_1$, otherwise at time $t = 6$ all three $a_{k-1}$, $b_2$ and $c_1$ collide. Hence, at $t = 5$ the collision is $a_{k-2} b_1$. In this case, we have that $b_3 - b_2 \geq 5$. Now consider the last two collisions at $t = N$ and $N - 1$. They are either $a_1 b_3$ at $t = N$ and $b_1 c_l$ at $t = N - 1$, or $b_1 c_l$ at $t = N$ and $a_1 b_3$ at $t = N - 1$. Notice that it cannot be the latter. Otherwise, the distance between $a_1$ and $a_k$ would be equal to the distance between $c_1$ and $c_l$ and the pair would collide simultaneously. We conclude that the collisions at $t = N$ and $N - 1$ must be $a_1 b_3$ and $b_1 c_l$, respectively. Then at time $t = N - 3$, the collision cannot be $a_2 b_3$ by parity. It cannot be $a_1 b_2$ or $b_2 c_l$ because of the distances between the entries in the $b_\bullet$ sequence. Finally, it cannot be $b_1 c_{l-1}$ since otherwise the distance between $c_l$ and $c_{l-1}$ would be 2 and they would collide with the pair $a_k$ and $a_{k-1}$ simultaneously. We conclude that this case is not possible. This concludes the proof of the lemma. □

We thus obtain the standard Ulrich partition of type $(1, 3, 1)$ given by $(0|1, 3, 6|8)$. To conclude the analysis in this case, we argue that this Ulrich partition cannot be extended to longer Ulrich partitions.

**Lemma 2.7.** *The only Ulrich partition of the form*

$$(a_1, \dots, a_{k-1}, a_k = 0 | b_1 = 1, b_2 = 3, b_3 = 6 | c_1 = 8, c_2, \dots, c_l)$$

*is* $(0|1, 3, 6|8)$.

*Proof.* Suppose there were a longer Ulrich partition. Then the last two collisions at time $t = N$ and $t = N-1$ must be $a_1 b_3$ and $b_1 c_l$, respectively. Otherwise, as in the previous cases, the distance between $a_1$ and $a_k$ would equal the distance between $c_1$ and $c_l$. But then at time $t = N - 2$ there cannot be any collisions. The entries $c_l$ and $a_k$ do not collide with any entries in the $b_\bullet$ sequence or with each other by the distribution of the $b_\bullet$ sequence. The collision cannot be $b_1 c_{l-1}$ and it cannot be $a_{k-1} b_3$. Otherwise, the distance between $a_k$ and $a_{k-1}$ would be 2 and the collisions $a_k b_1$ and $a_{k-1} b_2$ would be at the same time. This contradiction concludes the proof. □

*Proof of Theorem 1.2.* Let $P = (a_1, \dots, a_k | b_1, b_2, b_3 |, c_1, \dots, c_l)$ be an Ulrich partition. If the $b_\bullet$ sequence is consecutive, then by Proposition 2.3, up to symmetry, duality and equivalence $P = (-8, 0|1, 2, 3|8)$ or $(0|1, 2, 3|8)$. If only two entries in the $b_\bullet$ sequence are consecutive, then by Lemmas 2.4 and 2.5, $P = (0|1, 2, 5|8)$ or $P = (-1|1, 2, 6|7)$. Finally, if none of the entries in the $b_\bullet$ sequence are consecutive, then by Lemmas 2.6 and 2.7, $P = (0|1, 3, 6|8)$. In all cases we have that $k + l \leq 3$. This concludes the proof. □

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