MATH 553: MIDTERM PROBLEM SET

This problem set is due Wednesday March 3, 2021. You may work on the problem set in groups; however, the final write-up must be yours and reflect your own understanding. Let k be an algebraically closed field.

Problem 1. Let $X \subset \mathbb{P}^n_k$ be a nonsingular hypersurface of degree d. Let $\mathcal{O}_X(e)$ denote the pullback of $\mathcal{O}_{\mathbb{P}^n}(e)$ under the inclusion i of X in $\mathbb{P}^n_k$.

(1) For all $e \in \mathbb{Z}$ and all $i \geq 0$ compute $H^i(X, \mathcal{O}_X(e))$.
(2) Show that $\omega_X \cong \mathcal{O}_X(d-n-1)$ and compute $H^i(X, \omega_X)$.

Problem 2. Do Exercises III.5.1 and III.5.2 in Hartshorne. Compute the Hilbert polynomials of the following projective varieties using the definition in III.5.2:

(1) A rational normal curve of degree d.
(2) The Veronese image of $\mathbb{P}^n$ under the d-upple Veronese embedding.
(3) A hypersurface of degree d in $\mathbb{P}^n$.

Problem 3. Prove the following theorem by following the given steps.
Theorem *Any vector bundle V of rank r on $\mathbb{P}^1_k$ is isomorphic to a direct sum of line bundles $\oplus_{i=1}^r \mathcal{O}_{\mathbb{P}^1}(a_i)$ for a uniquely determined sequence of integers $a_1 \leq a_2 \leq \cdots \leq a_r$.*

(1) Show the uniqueness.
(2) Show that there exists a nonzero homomorphism $\mathcal{O}_{\mathbb{P}^1}(a) \to V$ for some $a \in \mathbb{Z}$.
(3) Let
$$a_r := \max\{a | \operatorname{Hom}(\mathcal{O}_{\mathbb{P}^1}(a), V) \neq 0\}.$$
Show that in the exact sequence induced by a nonzero homomorphism
$$0 \to \mathcal{O}_{\mathbb{P}^1}(a_r) \to V \to W \to 0,$$
W is locally free. Hence, by induction on the rank we may assume that the theorem holds for W and $W = \oplus_{i=1}^{r-1} \mathcal{O}_{\mathbb{P}^1}(b_i)$.
(4) Do exercise III.6.1 in Hartshorne.
(5) Prove that $\operatorname{Ext}^1(W, \mathcal{O}_{\mathbb{P}^1}(a_r)) = 0$ and conclude the theorem by induction. (Hint: Tensor the sequence defining W by $\mathcal{O}_{\mathbb{P}^1}(-a_r - 1)$ and compute cohomology.)

Problem 4. Let $n \geq 2$ and let $T\mathbb{P}^n$ denote the tangent bundle of $\mathbb{P}^n_k$.

(1) Prove that if $\operatorname{Hom}(\mathcal{O}_{\mathbb{P}^n}(a), T\mathbb{P}^n) \neq 0$, then $a \leq 1$.
(2) Show that $T\mathbb{P}^n$ for $n \geq 2$ is not a direct sum of line bundles. Hence, the theorem in the previous problem does not generalize to $\mathbb{P}^n$ for $n > 1$.
(3) Show that $T\mathbb{P}^n(-i)$ is acyclic (has no cohomology) for $i = 2, 3, \ldots, n$ and $i = n+2$.

Problem 5. Prove the following:

(1) Let I_Z be the ideal sheaf of 4 points on $\mathbb{P}^2_k$. Show that $H^1(\mathbb{P}^2, I_Z(2)) = 0$ if and only if the 4 points are not collinear.
(2) Let I_Z be the ideal sheaf of 9 points which are the complete intersection of 2 cubic curves. Let W be any 8 of the points in Z. Show that $H^1(\mathbb{P}^2, I_W(3)) = 0$ and $H^1(\mathbb{P}^2, I_Z(3)) = k$.

Problem 6. Let X be a smooth, integral, projective surface over k. Let Z be a zero-dimensional scheme of length n on X and let I_Z denote its ideal sheaf. Show that $\dim_k(\operatorname{Hom}_{\mathcal{O}_X}(I_Z, \mathcal{O}_Z)) = 2n$ by following the given steps:

(1) Observe that $\text{Ext}^i(\mathcal{O}_X, \mathcal{O}_Z) = 0$ for $i > 0$.
(2) Compute $\text{Hom}(\mathcal{O}_Z, \mathcal{O}_Z)$ and $\text{Ext}^2(\mathcal{O}_Z, \mathcal{O}_Z)$. Then use a locally free resolution of $\mathcal{O}_Z$ to compute

$$\chi(\mathcal{O}_Z, \mathcal{O}_Z) = \sum_{i=0}^{2} (-1)^i \dim(\text{Ext}^i(\mathcal{O}_Z, \mathcal{O}_Z))$$

directly and conclude that $\dim_k(\text{Ext}^1(\mathcal{O}_Z, \mathcal{O}_Z)) = 2n$.
(3) Apply $\text{Hom}(-, \mathcal{O}_Z)$ to the standard exact sequence

$$0 \to I_Z \to \mathcal{O}_X \to \mathcal{O}_Z \to 0$$

to finish the computation.

Remark: Combined with the connectedness of the Hilbert scheme of points on a smooth, projective, integral surface X, this exercise shows that the Hilbert scheme of points on X is smooth.

Problem 7. A sheaf $\mathcal{F}$ on a projective scheme X over k is called *exceptional* if $\text{Hom}(\mathcal{F}, \mathcal{F}) = k$ and $\text{Ext}^i(\mathcal{F}, \mathcal{F}) = 0$ for $i > 0$. A *strong exceptional collection* of sheaves is an ordered set of exceptional sheaves $\mathcal{F}_1, \dots, \mathcal{F}_r$ such that $\text{Ext}^i(\mathcal{F}_m, \mathcal{F}_l) = 0$ unless $m \leq l$ and $i = 0$.

(1) Show that $\mathcal{O}_{\mathbb{P}^n}, \mathcal{O}_{\mathbb{P}^n}(1), \mathcal{O}_{\mathbb{P}^n}(2), \dots, \mathcal{O}_{\mathbb{P}^n}(n)$ is a strong exceptional collection on $\mathbb{P}^n$.
(2) Show that $\mathcal{O}_{\mathbb{P}^2}, T\mathbb{P}^2(-1), \mathcal{O}_{\mathbb{P}^2}(1)$ is a strong exceptional collection on $\mathbb{P}^2$.
(3) Let X be the blowup of $\mathbb{P}^2$ at r distinct points $p_1, \dots, p_r$ and let E_i denote the exceptional divisor over p_i. Let $\mathcal{O}_X(H)$ denote the pullback of $\mathcal{O}_{\mathbb{P}^2}(1)$ to X. Show that

$$\mathcal{O}_X(-2H), \mathcal{O}_X(-H), \mathcal{O}_X(-E_1), \mathcal{O}_X(-E_2), \dots, \mathcal{O}_X(-E_r), \mathcal{O}_X$$

is a strong exceptional collection on X. (Hint: Recall that $\mathcal{O}_{E_i}(E_i) \cong \mathcal{O}_{\mathbb{P}^1}(-1)$.)

Problem 8. Compute the genus of the unique nonsingular projective model of the following complex plane curves

(1) $y^2 = x^7 - 1$
(2) $y^3 = x^5 - 1$

Problem 9. Compute the genus of a smooth, complete intersection curve of degrees $d_1, \dots, d_{n-1}$ in $\mathbb{P}^n$. Deduce that a smooth curve of genus 8 cannot be realized as a complete intersection in projective space.

Problem 10. Prove that a genus 2 curve cannot be realized as a smooth, complete intersection curve in projective space.

Problem 11. Prove that there does not exist a nonsingular curve of degree 9 and genus 11 in $\mathbb{P}^3$.

Problem 12. For this problem work over the complex numbers. Show that a line, a conic, a twisted cubic curve and an elliptic quartic curve in $\mathbb{P}^3$ have no multi-secant lines. Prove that every other smooth, irreducible, projective curve in $\mathbb{P}^3$ has infinitely many multisecant lines.

Problem 13. Show that any two smooth, nondegenerate rational curves of degree 4 in $\mathbb{P}^4$ are projectively equivalent. Show that a rational curve of degree 4 in $\mathbb{P}^3$ is always contained in a unique smooth quadric surface. The class of the curve on the quadric is $(1,3)$. Show that two such curves need not be projectively equivalent (Hint: Think of the cross-ratio of the branch points of the degree 3 projection to $\mathbb{P}^1$).

Problem 14. Show that a rational curve of degree 5 in $\mathbb{P}^3$ is always contained in a cubic surface. Exhibit examples of such curves which are contained in a quadric surface. Prove that the general degree 5 rational curve is not contained in a quadric surface.

Problem 15. Show that the smallest degree embedding of a genus 2 curve has degree 5. Show that a genus 2 degree 5 curve has to lie on a quadric surface. Does the quadric surface have to be smooth?