1. Practice Problems for Midterm 1

Problem 1.1. Find all solutions of the following system of equations.

$$\begin{aligned} 8x + 3y + 5z - w + u &= 2 \\ 3x + 2y + z + 2w - u &= 4 \\ x - 3y + z - 3w + u &= 6 \end{aligned}$$

Problem 1.2. Calculate the rank of the following matrix.

$$\begin{pmatrix} 3 & 2 & -2 & 1 \\ 2 & 3 & -1 & -1 \\ 5 & 2 & 2 & 4 \end{pmatrix}$$

Find a basis of its row space. Find a basis of its column space.

Problem 1.3. Determine whether the following sets S are subspaces of $\mathbb{R}^4$. If S is a subspace, find a basis and determine the dimension of that subspace.

(1)

$$S = \{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \mid x - 3y + z = 0, 2x + 5w = 0\}.$$

(2)

$$S = \{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \mid x + y + z = 0, x + y + w = 0, -x - y - 3z + 2w = 0\}.$$

(3)

$$S = \{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \mid xy = 0\}.$$

Problem 1.4. Solve the system of equations

$$\begin{aligned} ax + y &= a^2 \\ x + ay &= 1 \end{aligned}$$

where $a \in \mathbb{R}$ is a parameter. For which values of a does the system have no solutions? For which valued of a does the system have infinitely many solutions?

Problem 1.5. Let S be the following subset of the space of 2×2 matrices

$$S = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid 3a - b = 2c - 3d = 0\}.$$

Prove that S is a subspace. Find a basis for S. What is the dimension of S. Complete the basis for S to a basis of the vector space of 2×2 matrices.

Problem 1.6. Let V be the subspace of the space of polynomials spanned by

$$x + x^3,\ x - 2x^2 + 3x^3 + 4x^4,\ x^2 - x^3 + 2x^4.$$

Determine a basis of V. What is the dimension of V. Complete the basis you found to a basis of the space of polynomials of degree at most 6.

Problem 1.7. Determine whether the following statements are true or false. If the statement is true give a proof. If the statement is false provide a counterexample.

(1) A homogeneous system of linear equations with fewer equations than unknowns always has infinitely many solutions.
(2) A homogeneous system of linear equations with more equations than unknown has only the zero vector as a solution.
(3) An arbitrary system of linear equations with fewer equations than unknowns always has infinitely many solutions.
(4) An arbitrary system of linear equations with fewer equations than unknowns has infinitely many solutions provided that it has at least one solution.
(5) If a subset S of a finitely dimensional vector space V spans V, then the cardinality of S is greater than or equal to the dimension of V.
(6) If a subset S of a finite dimensional vector space V is linearly independent, then S contains a basis of V.
(7) If a set of vectors $v_1, \dots, v_r$ is linearly independent and c is a non-zero scalar, then the set of vectors $cv_1, \dots, cv_r$ is also linearly independent.
(8) $\mathbb{Q} \subset \mathbb{R}$ is a subfield.
(9) $\mathbb{Q} \subset \mathbb{R}$ is a subspace.
(10) Let V and W be two vector spaces over a field F. Then the space of isomorphisms from V to W is a vector space.

Problem 1.8. Let $h : P_2 \to \mathbb{R}$ be a homomorphism from the space of polynomials of degree at most two to $\mathbb{R}$. Suppose that $h(x-1) = 1$, $h(x^2-1) = 3$ and $h(5x) = 10$. Calculate $h(x^2 + x - 7)$.

Problem 1.9. Determine the dimension of the space of homomorphisms between P_5, the space of polynomials with real coefficients of degree at most 5, and $M_{2\times 3}$, 2×3 matrices with real entries. Are the two vector spaces isomorphic? If so, find an explicit isomorphism.

Problem 1.10. Let the map $\mathbb{R}^4 \to \mathbb{R}^3$ be given by multiplication by the matrix

$$\begin{pmatrix} 3 & 2 & 5 & -1 \\ 7 & -4 & 3 & 15 \\ 9 & 2 & 11 & 5 \end{pmatrix}$$

Determine a basis for the range space. Determine a basis for the kernel/null-space. Check that your answer satisfies the rank-nullity theorem.