Moduli spaces of sheaves on surfaces

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Schedule

1. *Moduli spaces of sheaves—an overview of the geography*
2. *Constructions and local properties: the Cayley-Bacharach condition*
3. *Ascending induction and O'Grady's method*
4. *Structure of the moduli spaces and their boundaries*

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Remark on references

References to authors given in these slides are not by any means complete or exhaustive. That would go way beyond the possible scope. Some attempt to do that was made in our survey paper “Moduli of Sheaves”, but that was already some time ago and the current state of research has moved forward a lot in the mean time.

Moduli spaces of sheaves—an overview of the geography

In this first talk we will look more globally at the properties of moduli spaces of vector bundles and sheaves on various kinds of varieties, then specializing to the case of surfaces and in particular the case of rank 2 bundles on quintic hypersurfaces in $\mathbb{P}^3$.

Consider a smooth projective surface X, with hyperplane class denoted $[H]$. We recall that the *degree* and *slope* of a torsion-free sheaf are

$$\deg(E) := c_1(E).[H], \quad \mu(E) := \frac{\deg(E)}{\mathrm{rk}(E)}.$$

We say that E is *stable* (resp. semistable) if, for any subsheaf of smaller nonzero rank, $\mu(F) < \mu(E)$ (resp. $\leq$).

These definitions of "slope (semi)stability" may be replaced by *Gieseker (semi)stability* by using the Hilbert polynomial instead of the degree.

Theorem (Mumford, Gieseker, Maruyama, Langer)

The set of semistable sheaves is bounded, and there exists a GIT construction of the coarse moduli space $\overline{M}_X^{\mathrm{tf}}(r, c_1, c_2)$ *of Gieseker semistable torsion-free sheaves on* X *with given Chern invariants.*

This may be extended to a construction of moduli of pure sheaves supported in any dimension. Mukai first introduced the moduli space of simple sheaves in such a context.

Deformation and obstruction theory, coupled with Serre duality, imply that if X is a K3 or abelian surface then the obstruction space vanishes for stable sheaves; in that case, the moduli space is smooth and the Serre duality pairing on the tangent space leads to a natural symplectic form.

These cases have been studied carefully by Yoshioka following upon the original studies by Mukai that had been continued by Markman and others.

An important general theme has been *Brill-Noether theory*, on which much is known for K3 surfaces, with recent work for many other kinds of surfaces too.[1]

[1] cf today's preprint of Battacharya and references therein.

The moduli spaces of sheaves on rational and ruled surfaces have been studied from several viewpoints. In those situations, purely algebraic methods such as "monads", "helices" are available to give descriptions of the moduli spaces (Drézet, Le Potier, ...).

In the research project that is the subject of our talk series, joint with Nicole Mestrano, we wanted to look in the direction of surfaces of general type. For this, we focus on a first case, that of hypersurfaces of degree $d = 5$ in $\mathbb{P}^3$. We note that hypersurfaces of degree 4 are K3 and in degree ≤ 3 they are rational. Degree 5 is the first general-type case, with $K_X = \mathcal{O}_X(1)$.

Sheaves on curves

We go back and comment on a much more basic case, the moduli spaces of vector bundles on curves. This started with the work of Narasimhan and Seshadri, who showed the equivalence with moduli spaces of unitary representations of the fundamental group, and Tyurin, Narasimhan-Ramanan who gave explicit descriptions for curves of genus 2, relating the theory to the famous "quadric line complex".

The space of sections $H^0(M, \mathcal{L}^{\otimes k})$ of the powers of the determinant line bundle has dimension given by the *Verlinde formula* and mathematical physics interprets these sections as spaces of "conformal blocks". WZW and Hitchin defined a projectively flat connection on these spaces as the curve moves over moduli.

These properties for curves could be relevant for suggesting questions to look at in higher dimensions.

For example, the Verlinde formula exhibits an example of "strange duality" or "rank-level duality" (between the rank of the bundle and the level k) that was studied by Le Potier, Sorger and others also in higher-dimensional cases.

Bogomolov-Gieseker inequality

The first main new feature when we pass from curves to higher dimensions is the *Bogomolov-Gieseker inequality*. This is best phrased in terms of the Δ-invariant designed to be stable under tensoring with a line bundle:

$$\Delta(E) := c_2(E) - \frac{r-1}{2r} c_1(E)^2.$$

$$\Delta(E \otimes L) = \Delta(E)$$

Bogomolov-Gieseker inequality

Theorem (Bogomolov-Gieseker, Donaldson, Mehta-Ramanathan)

If E is a slope-semistable torsion-free sheaf then $\Delta(E) \geq 0$, with equality if and only if it is a bundle carrying a projectively flat unitary connection.

Donaldson theory

Donaldson's theory of special metrics was one of the major developments. Given an hermitian metric on E, there is a unique unitary connection compatible with the holomorphic structure. We say that the metric is *Hermitian-Einstein* if $\Lambda F = \lambda \cdot 1$ where Λ is the adjoint of the Kähler operator (here X should be given a Kähler metric in the class of $[H]$).

Donaldson theory

Theorem (Donaldson, Uhlenbeck-Yau)

If E is slope-stable then it admits a (unique up to scalar) Hermitian-Einstein metric. An integration formula implies the previous $\Delta(E) \geq 0$ and if $\Delta(E) = 0$ then the curvature is a scalar two-form.

One of Donaldson's motivations was to use the complex vector bundle moduli spaces in order to study his more general invariants for differential-geometric manifolds. His invariants were defined using a generic gauge equation analogous to the equations defining the complex moduli spaces. One important question from the beginning of this theory was whether the complex moduli spaces would also yield calculations of the gauge-theoretic invariants.

This is, in general, a difficult question because the complex moduli spaces always have some singular points, whereas the gauge-theoretical spaces are assumed to be smooth. It continues as a subject of study today.

We can nonetheless say that this motivation spurred a flurry of work on the structure of the complex moduli spaces of vector bundles. The basic idea was to say that for large values of c_2, these spaces should approach as much as possible their gauge-theoretic cousins.

In particular, one main type of theorem was to show that the moduli spaces of vector bundles would have the expected dimension, be irreducible, and be generically smooth (with, optimally, some kind of bound on the codimension of the singular locus).

Nice properties for $c_2 \gg 0$

This kind of result was indeed achieved in a series of works by Donaldson, Friedman, Li, Zuo, O'Grady and others. The optimum theoretical bound was obtained by O'Grady, in a series of papers introducing various useful techniques that we'll meet later.

Theorem (Donaldson, Friedman, Gieseker, Li, Zuo, O'Grady)

There is a computable bound C depending on X, H, r and c_1 such that for $c_2 \geq C$, the moduli space $M_X(r, c_1, c_2)$ is good *(generically smooth of the expected dimension) and irreducible.*

Higher dimensional varieties

A significant direction of research has been the study of moduli of sheaves on threefolds and in bigger dimensions. For example, the topic of *arithmetically Cohen-Macaulay sheaves*, namely those for which every twist has vanishing interior cohomology groups, has generated numerous results for varieties of dimensions 3 and 4.

In turn, the study of bundles can lead to results on subvarieties of codimension ≥ 2. In the Serre-type constructions in higher dimensions, the role played here by a zero-dimensional subscheme P will concern curves and other subvarieties.

This leads to one indication of the potential difficulty in higher dimensions, namely the classification of bundles runs up against problems such as *Hartshorne's conjecture*.

Stability conditions

Bridgeland's general notion of a *stability condition on a derived category* leads to a range of generalized kinds of slope-stability. It hasn't been easy to construct examples of these stability conditions over varieties of dimension ≥ 3.

The search for them has led to a new direction: exotic Bogomolov-Gieseker inequalities. Bayer, Macri and Toda conjectured a generalized Bogomolov-Gieseker inequality for tilt-stable objects, involving c_3. This has been proven by Macri for projective space, by Schmidt for the quadric threefold, by Bayer-Macri-Stellari for abelian and CY threefolds, by C. Li for the quintic threefold, by Bayer-Lahoz-Macri-Stellari on some Kuznetzov components in derived categories, and more.

We should also mention the phenomenon of wall-crossing: there can be parameters determining a stability condition (such as the Kähler class, but it could also be another parameter such as shows up in the theory of coherent systems). Typically the space of parameters is divided into chambers, and the moduli space undergoes a transformation when the parameter crosses some wall.

The study of wall-crossing has led to many distinct directions that we couldn't describe in detail here.

Intermediate values of c_2

Turn now to the motivation for our research project. We are looking at bundles on surfaces. From the above discussion, we can see that for surfaces with K_X trivial, the structure is known for all values of c_2.

In the direction of rational surfaces, a lot is known but since there are a wide range of possibilities, this leads to interesting questions. Coskun-Huizenga have recently obtained the criterion for existence of stable bundles on Hirzebruch surfaces, for example. Similarly for del Pezzo surfaces by Levine.

The case of elliptic surfaces, subject of Friedman's original papers, undoubtedly remains a good source of questions on the fine structure (Yamada, Yoshioka, . . .).

In the case of surfaces of general type, the structure is well understood for large values of c_2, and of course the moduli spaces are empty for small values. It is therefore natural to investigate what happens for intermediate values of c_2.

Vakil proposes a general "Murphy's Law" principle saying that various types of behaviors not ruled out by general theorems, should eventually occur as the invariants of the underlying variety get big enough.

Examples of this type of phenomena were known: one can get generically non-reduced moduli spaces, there were examples of reducible moduli spaces (Friedman, Mestrano, Coskun-Huizenga), and more recently examples of disconnected moduli spaces (leading up to Coskun-Huizenga-Kopper for example).

We were motivated to try to understand the full picture of the moduli spaces for all values of c_2, in a first case of surfaces of general type, namely hypersurfaces of degree 5 in $\mathbb{P}^3$. For this case, it turns out that the irreducibility results from the K_X-trivial case persist. As we'll see at the end, for degree 6 already we get a reducible moduli space.

Setup and notations

Let us now start with the general setup and notations that will be in effect throughout most of the series. We consider a very general hypersurface $X \subset \mathbb{P}^3$ of degree 5. The very general hypothesis includes, at least, the condition that $\mathrm{Pic}(X) \cong \mathbb{Z}$ with generator $\mathcal{O}_X(1)$. The canonical sheaf is the generator: $K_X = \mathcal{O}_X(1)$.

We recall that $H^1(\mathcal{O}_X(n)) = 0$, and $H^0(\mathcal{O}_X(n)) = H^0(\mathcal{O}_{\mathbb{P}^3}(n))$ for $n < 5$.

- Let $M(c_2)$ denote the moduli space of stable rank 2 bundles E with $c_1(E) = [H] =: c_1(\mathcal{O}_X(1))$, whose second Chern class is c_2.
- Let $\overline{M}^{\mathrm{tf}}(c_2)$ denote the moduli space of stable torsion-free sheave of rank 2 with the same Chern invariants. It contains $M(c_2)$ as an open subset.
- Denote by $\overline{M(c_2)}$ the closure of $M(c_2)$ in $\overline{M}^{\mathrm{tf}}(c_2)$. This will often be a proper closed subset.
- We note that stability and semistability coincide for bundles of degree 1.

The Serre construction

Our main tool is the Serre construction,[2] expressing a rank 2 bundle as an extension of an ideal sheaf by a line bundle. Suppose for example that $H^0(E) \neq 0$, then we have an exact sequence

$$0 \to \mathcal{O}_X \to E \to J_{P/X}(1) \to 0.$$

Here $J_{P/X}$ is the ideal sheaf of a 0-dimensional subscheme $P \subset X$. In this case $c_2(E) = |P|$ (if the subbundle has a different degree then that would be modified).

[2] The Serre construction was used notably by I. Reider in his thesis.

The condition for a subscheme P to occur in such an exact sequence is that P should be locally a complete intersection, and it should satisfy the *Cayley-Bacharach condition* $CB(2)$ for quadrics. We'll discuss that more in the next talk.

One of our main techniques will be to use geometric arguments to try to understand what positional properties P should have.

Main theorem

Theorem

The moduli space $M(c_2)$ is empty for $c_2 \leq 3$, and irreducible for $c_2 \geq 4$. For $c_2 \geq 10$ it is good, *i.e. generically smooth of the expected dimension $4c_2 - 20$.*
For $c_2 \geq 10$ the moduli space of torsion-free sheaves $\overline{M}^{\mathrm{tf}}(c_2)$ is also good, and for $c_2 \geq 11$ it is irreducible. For $c_2 = 10$ it has two irreducible components.

For $4 \leq c_2 \leq 9$ we'll give explicit descriptions of the Cayley-Bacharach subschemes P that show up for general points of $M(c_2)$.

Future directions

We could comment on some general indications about future study suggested by these results. We notice that there is a very nice explicit description of the moduli spaces for very low values of c_2 (in our case, $c_2 = 4, 5$). It would be interesting to see if that could persist for hypersurfaces of higher degree and other surfaces.

This leads also to the question of characterizing Cayley-Bacharach subschemes $P \subset \mathbb{P}^3$ in general, without supposing that P lies in a particular surface. Along the way of our constructions, we meet a lot of geometric constructions giving reasons why some P should satisfy a $CB(n)$ property. Could those be made more systematic?

There might also be a metric geometry counterpart to the previous questions. We know that when the Bogomolov-Gieseker inequality is attained, it comes from a flat connection. When the values of c_2 are small but not equal to the Bogomolov-Gieseker bound, could that mean there would exist a metric with small curvature?

That could go on to give a corresponding isoperimetric inequality where the size of a region contracting a given curve would have to be big compared to the length of the curve.

One might conjecture that a Higgs-bundle analogue of the previous comments, applied to the Higgs bundle $\mathcal{O}_X \oplus \Omega^1_X$, would tell us that simply connected surfaces whose Chern invariants are close to, but not on the line $c_1^2 = 3c_2$, could look “approximately non-simply connected”.

This in the context of the recent result of Roulleau-Urzúa showing that there are simply connected surfaces whose Chern slope is arbitrarily close to 3.

Constructions and local properties: the Cayley-Bacharach condition

In the second talk, we look at the Serre construction of rank 2 vector bundles using the Cayley-Bacharach condition on zero-dimensional subschemes. Topics include the local deformation theory, how it interacts with the Serre construction, and the interpretation of co-obstructions as Higgs fields.

Kuranishi theory

Notations are conserved from above.

Let's start with some general considerations. We are looking at bundles of odd degree, so semistability and stability coincide; and our stable bundles have no non-scalar automorphisms.

The tangent space to the moduli space at a point E is $H^1(\mathrm{End}^0(E))$ where $\mathrm{End}^0(E)$ is the rank 3 bundle of trace-free endomorphisms.

The space of *obstructions* is $H^2(\mathrm{End}^0(E))$. We recall that *Kuranishi theory* says that locally in a formal or complex-analytic sense, the moduli looks like the germ at 0 of the zero-set of the "Kuranishi map"

$$\Phi : H^1(\mathrm{End}^0(E)) \to H^2(\mathrm{End}^0(E)).$$

This is an analytic map with trivial linear term, and the Kuranishi identification coincides to first order with the identification of $H^1(\mathrm{End}^0(E))$ as the Zariski tangent space of M at E.

The *expected dimension* of M at E is by definition

$$\mathrm{e.d.}(M) := h^1(\mathrm{End}^0(E)) - h^2(\mathrm{End}^0(E)).$$

It only depends on the Chern invariants of E and for our case ($c_1(E) = [H]$ on $X \subset \mathbb{P}^3$ a hypersurface of degree 5) we have

$$\mathrm{e.d.}(M(c_2)) = 4c_2 - 20.$$

From the above, we immediately deduce that if $h^2(\mathrm{End}^0(E)) = 0$ then M is *good* at E, that is to say it is generically smooth of the expected dimension. On the other hand, the dimension of any irreducible component of $M(c_2)$ is $\geq \mathrm{e.d.}(M(c_2))$. Therefore, if we can bound the dimension of the locus of bundles E such that $h^2(\mathrm{End}^0(E)) > 0$, by a bound that is strictly smaller than the expected dimension, this shows that $M(c_2)$ is good.

One good way of getting a hold of $H^2(\mathrm{End}^0(E))$ is by Serre duality: it is dual to the space of *co-obstructions* defined to be $H^0(\mathrm{End}^0(E) \otimes K_X)$. One notices, then, that a co-obstruction for a bundle E on a surface may be interpreted as being like a Higgs field *à la* Hitchin, that is an endomorphism-valued section of the canonical line bundle.

This is different from the Higgs fields that take part in "nonabelian Hodge theory", those having coefficients in the rank 2 bundle Ω^1_X. One can apply the classical theory of "spectral varieties" here to say that a bundle E with a nontrivial co-obstruction ϕ is going to have a spectral variety that is a subvariety of the total space of K_X, finite over X.

This type of reasoning allows us to obtain bounds for the number of co-obstructions and the dimension of the locus of bundles E having a co-obstruction. Here is the statement we can get, although the bound here depends on many of the other arguments going into the classification of components below.

Theorem

The moduli space $M(c_2)$ is good for $c_2 \geq 10$.

We would next like to view a bundle $E \in M(c_2)$ as coming from the Serre construction. Before getting there, let's consider a simple argument with the Euler characteristic that helps in our situation. We note that $\chi := h^0 - h^1 + h^2$ has formula

$$\chi(E(n)) = 5n^2 + 10 - c_2.$$

Furthermore, $E^* \cong E(-1)$ so $E(n)^* \otimes K_X \cong E(-n)$ and Serre duality gives

$$h^i(E(n)) = h^{2-i}(E(-n)).$$

For $n = 0$ we get $2h^0(E) = h^1(E) + \chi(E)$ so as soon as $\chi(E) > 0$ it implies $h^0(E) > 0$. From the above formula, this gives $h^0(E) \geq 1$ whenever $c_2 \leq 9$, and similarly $h^0(E) \geq 2$ whenever $c_2 \leq 7$.

In conclusion, for the cases $c_2 \leq 9$ we may assume that E has a section, so it may be viewed as an extension given by the Serre construction with sub-line bundle $\mathcal{O}_X \hookrightarrow E$.

To discuss Cayley-Bacharach in general, suppose $s \in H^0(E(m))$ is a nonzero section. We get an exact sequence

$$0 \to \mathcal{O}(-m) \xrightarrow{s} E \to J_{P/X}(m+1) \to 0.$$

Let's assume that s doesn't come from a section of $H^0(E(m-1))$. Under our assumption that $\mathrm{Pic}(X)$ is generated by $\mathcal{O}_X(1)$, this implies that s doesn't vanish along a divisor. Hence, the subscheme P has dimension 0.

As a first approximation one may think of P as consisting of a distinct set of points, beware however that this hides some of the main subtleties in the proof where we need to rule out cases where the generic P could be a non-reduced subscheme.

The first observation is that the ideal of P has locally 2 generators, since E has rank 2. Therefore, each local piece of P is a local complete intersection subscheme.

Once P has the lci property, the other constraint it has to satisfy in order to come from an exact sequence such as the above is the *Cayley Bacharach condition* with respect to the line bundle $K_X(2m+1)$.

Cayley Bacharach:

- Suppose $P' \subset P$ is a subscheme defined by an ideal of length 1 (so, roughly, any collection of all points except one[3]);
- Suppose $f \in H^0(K_X(2m+1))$ vanishes on P';
- Then f should vanish on P.

[3] More precisely, the local pieces of P are Gorenstein so they have a "socle", that is to say a unique ideal of length 1. This is the ideal that should be used, at a single location, to define the test $P' \subset P$.

Recall that we are looking at a hypersurface $X \subset \mathbb{P}^3$ of degree 5, so

$$K_X = \mathcal{O}_X(1), \quad K_X(2m+1) = \mathcal{O}_X(2m+2).$$

Furthermore for the treatment of the cases $c_2 \leq 9$ we are interested in the Serre construction with $m = 0$. Hence, we'll be looking for subschemes P that satisfy $CB(2)$.

We now proceed to illustrate the Cayley-Bacharach condition by giving some examples. Since these will also be important elements of the future arguments, we'll give all the examples of $CB(2)$ subschemes that serve to define generic points of the irreducible components of our moduli spaces $M(c_2)$ for $c_2 \leq 9$.

Start by noting that P cannot have length 3 (or less). A length 2 subscheme P' would be either two distinct points or a double curvilinear point, and in any case this defines a line. If P is contained in the line then we can find a degree 2 polynomial vanishing on P' but not on P, whereas if P isn't contained in the line, a general plane through that line will not contain P. Either way, such a P cannot satisfy $CB(2)$.

Consider the case $|P| = 4$. Suppose P is contained in a line $\ell \subset \mathbb{P}^3$, that is $P \subset \ell \cap X$. Now, a section of $\mathcal{O}_X(2)$ (those are the same as sections of $\mathcal{O}_{\mathbb{P}^3}(2)$) restricts to a degree 2 polynomial on the line. For any subscheme $P' \subset P$ of length 3, if the degree 2 polynomial vanishes on P' then it must vanish on the whole line, so in particular it vanishes on P. This proves the $CB(2)$ property for subschemes of length 4 on a line.

Before getting to the next examples, let's look at the proof that all the $CB(2)$ subschemes of length 4 arise in this way. This proof is an easy representative for the proofs of similar statements that are needed along the way.

So, suppose $P \subset X$ is an lci subscheme of length 4 satisfying $CB(2)$. Any length 3 subscheme $P' \subset P$ defines a plane, and that plane would have to contain P (by $CB(1)$ in fact).

Suppose we can choose a plane H meeting P in a subscheme of length ≥ 2, but not containing it. Let h be the equation of the plane in local coordinates. We may introduce the *residual subscheme* of P with respect to the plane. This is the subscheme R defined by the ideal of functions f such that $f \cdot \mathcal{O}_P \subset h \cdot \mathcal{O}_P$, and $|R| + |P \cap H| = |P|$.

If $|P \cap H| = 3$ then the $CB(1)$ condition implies that $P \subset H$, so we must have $|P \cap H| = 2$. But then, there is another plane H' such that $|R \cap H'| = 1$, and taking as quadric $Q = H \cup H'$ we get a quadric such that $|P \cap Q| = 3$, contradicting $CB(2)$ for P. Thus, we conclude that P has to be contained in any plane H meeting it in a subscheme of length 2, which implies that P is contained in a line.

Residual subscheme

The notion of residual subscheme is often useful.

Given a Cartier divisor $D \subset X$ and a subscheme $P \subset X$, define

$$\mathcal{O}_R := \mathrm{im}\left(\mathcal{O}_P \to \mathcal{O}_P \otimes_{\mathcal{O}_X} \mathcal{O}_X(D)\right).$$

This is seen to be the structure sheaf of a subscheme whose ideal is the kernel of the surjection $\mathcal{O}_X \to \mathcal{O}_R$.

We also have $J_{D \cap P/P} \cong \mathcal{O}_R \otimes_{\mathcal{O}_X} \mathcal{O}_X(-D)$, so

$$|P| = |D \cap P| + |R|.$$

The residual subscheme enjoys nice transitivity properties with respect to the Cayley-Bacharach condition.

Getting back to our examples of Cayley-Bacharach subschemes, consider the case of length 5.

Similarly to the previous case, we can have a subscheme of length $|P| = 5$ contained in a line. In fact, this is uniquely determined by the line with $P = \ell \cap X$ since $X \subset \mathbb{P}^3$ has degree 5. As before, P satisfies $CB(2)$—in fact it satisfies $CB(3)$.

Furthermore, a proof similar to the one above serves to show that all $CB(2)$ subschemes of length 5 arise in this manner.

Turn next to the case $|P| = 6$. Here, P can no longer be contained in a line. One can see that it must be contained in a plane, by the previously mentionned symmetry consideration on the Euler characteristic of the bundle E.

If $h^0(E) \geq 2$ it means $h^0(J_{P/X}(1)) \geq 1$.

In other words, we have

$$P \subset C := H \cap X$$

for some plane H.

A subscheme $P' \subset P$ of length 5 will be contained in a plane conic inside H. If P is going to satisfy $CB(2)$ it therefore has to be contained in that conic. In other words, we have a quadric Q defining the plane conic as $Q \cap H$ such that $P \subset Q \cap H \cap X$.

If Q is chosen so that $Q \cap H$ is a smooth conic, then it is isomorphic to $\mathbb{P}^1$ and a calculation similar to the one for the line shows that any subscheme of length 6 satisfies $CB(2)$. It amounts to saying that a subscheme of $\mathbb{P}^1$ of length 6 satisfies Cayley Bacharach for the line bundle $\mathcal{O}_{\mathbb{P}^1}(4)$.

One shows that the case where P lies on a smooth plane conic is general among the possibilities. This is somewhat similar to the proof for 4 points above, but more involved. The use of dimension counts, to rule out certain types of configurations as not being able to correspond to irreducible components of the moduli space, starts here.

Consider next the case $|P| = 7$. Here again, by the same argument P is contained in a plane H. The general case is when the curve $C = H \cap X$ is smooth, and P consists of 7 points in general position on C. This satisfies $CB(2)$.

Indeed suppose $P' \subset P$ is a collection of 6 points; they are also in general position on C. It follows that they aren't all contained in a plane conic, indeed 5 of them would define a unique plane conic and then the 6th point is not contained in that conic intersected with C. Thus, a plane conic containing P' must vanish as a section of $\mathcal{O}_H(2)$, so it must contain P.

The case $|P| = 8$ is somewhat different. The subscheme is no longer contained in a plane. However, a subscheme $P' \subset P$ of length 7 is going to define a subspace of $H^0(\mathcal{O}_X(2)) \cong H^0(\mathcal{O}_{\mathbb{P}^3}(2)) \cong \mathbb{C}^{10}$ of dimension at least 3.

The generic case is that this subspace does have dimension 3, and is spanned 3 quadrics whose common intersection consists of 8 reduced points. As usual, the statement that such is indeed the general case in any complete irreducible family, is something that needs to be proven.

Suppose that we start with three quadrics Q_1, Q_2, Q_3 in $\mathbb{P}^3$, such that their intersection $P = Q_1 \cap Q_2 \cap Q_3$ consists of 8 distinct points. Then P satisfies $CB(2)$. This example was called the *Cayley octad* by Dolgachev.

The intersection of two $Q_1 \cap Q_2$ is a smooth elliptic curve, and consideration of linear systems there tells us that if a new quadric vanishes on 7 of the points of P then its restriction to the elliptic curve must be proportional to Q_3, so vanishing on the 8th point. This is how we prove the $CB(2)$ condition for the generic case.

We do need to worry, however, about getting $P \subset X$. For a general choice of Q_1, Q_2, Q_3 the intersection will of course not meet X. Thus, there is a subvariety of the parameter space consisting of choices such that $Q_1 \cap Q_2 \cap Q_3 \subset X$.

Using a monodromy argument suggested by A. Hirschowitz, for a given general X, the space of choices of Q_1, Q_2, Q_3 is irreducible. We'll do a version of this argument at another place later.

This then defines our irreducible family of $CB(2)$ subschemes of X of length 8.

The case $|P| = 9$ gets us back more closely to the type of argument used for $|P| = 7$. Indeed, we expect to have a 2-dimensional space of quadrics passing through X (that is, those passing through a length 8 subscheme $P' \subset P$). So, to get the general point we choose quadrics Q_1 and Q_2 that meet in a smooth elliptic curve $C = Q_1 \cap Q_2$. Then let P be a choice of 9 out of the 20 points of $C \cap X$. This is seen to be in sufficiently general position so that any section of $\mathcal{O}_C(2)$ that vanishes on P' of length 8, has to vanish on C and in particular on P.

This defines the family of $CB(2)$ subschemes of length 9.

We have now seen several examples of subschemes satisfying $CB(2)$ for various reasons. A sort of "meta-conjecture" is that subschemes satisfying the Cayley-Bacharach condition, at least ones moving in sufficiently big families, should do so because of some geometric reason.

In the above examples, that reason often involved the position of P inside an auxiliary curve $C \subset \mathbb{P}^3$ not contained in X. Other possible reasons, more in the direction of the Cayley octads for example, should most certainly be expected.

Putting together the above constructions and counting dimensions, we obtain irreducible components of the moduli spaces $M(c_2)$ with the following dimensions. We record also the expected dimensions, dimension of the generic obstruction space, and whether the component is generically reduced or non-reduced.

Table: Components of $M(c_2)$ for $4 \leq c_2 \leq 9$

c_2	4	5	6	7	8	9
dim	2	3	7	9	13	16
e.d.	-4	0	4	8	12	16
obs	6	3	3	3	1	1
r/nr	r	r	r	nr	r	nr

The non-reduced cases are those where the obstruction dimension plus the expected dimension is greater than the dimension. The discrepancy is the number of extra tangent dimensions due to the non-reduced directions.

We didn't give the details of the computations for the dimensions and obstruction dimensions of the families that we have constructed above. This is fairly straightforward, noting however that the same bundle might occur as an extension in several different ways, that needs to be taken into account in counting the dimension of moduli.

The main work that needs to be done for these cases $c_2 < 10$ is to show that the irreducible components constructed explicitly, are the only ones. Namely, it needs to be shown that more special cases for the position of P, and/or the positions of the planes, quadrics and curves that occur in the above discussion, don't add distinct irreducible components.

That basically requires checking a lot of cases. Let's look at a few aspects in a specific example $c_2 = 9$.

Recall that the subschemes in our family impose 8 conditions on quadrics. Let us show that a general point of an irreducible component doesn't consist of subschemes P that impose ≤ 7 conditions on quadrics. If it did, we would have a subspace of quadrics containing at least 3 linearly independent ones Q_1, Q_2, Q_3. However, P has to be contained in $I := Q_1 \cap Q_2 \cap Q_3$ so this intersection cannot be 0-dimensional (if it was it would have length only 8).

Therefore, I has a component I^+ of positive dimension. One rules out the possibility that the Q_i all contain a common plane, so we may assume $C := Q_1 \cap Q_2$ is a degree 4 curve. It is reducible, since it contains I^+ as an irreducible component. By considering the possible decompositions of C, we see that either C is contained in a union of two planes, or it contains a rational normal cubic curve.

Suppose $C = L \cup T$ where L is a line and T a rational normal cubic curve. if $I^+ = L$ then, either all the Q_i are singular, in which case we can reduce to one of the other cases considered below, or say Q_1 is smooth. If it is smooth, taking the residual of $L \cap P$ with respect to L in Q_1 we would get a subscheme of the intersection of two $(1, 2)$ divisors on Q_1, having length at most 4. Thus $|L \cap P| = 5$, but then taking a plane through L and a plane through three of the four remaining points by $CB(2)$ we would get P contained in two planes.

If $I^+ = T$ then L moves as we vary C among double intersections of our quadrics, from which we get that $P \subset T$.
Finally, we are reduced to two cases: either P is contained in the union of two planes, or it is contained in a rational normal curve. Dimension counts show that the possible dimensions of the moduli points of bundles E in these cases, are < 16 whereas the expected dimension is 16, therefore these cases can't contribute irreducible components.

The proof continues using pretty much these same kinds of arguments, to show that there are also no new irreducible components consisting of subschemes P that impose 8 conditions on quadrics, completing the case $c_2 = 9$.

The proofs for the other cases $c_2 = 8, 7, \ldots$ are similar.

In the next talk we'll look at methods for the cases $c_2 \geq 10$.

Ascending induction and O'Grady's method

In the third talk, we introduce O'Grady's method of deforming to the boundary by creating deformations from bundles to torsion-free sheaves. Combining that with explicit constructions for low values of c_2, we look at the implications for the how the collection of moduli spaces fits together as c_2 increases.

Recall that the full moduli space of semistable torsion-free sheaves with $c_1 = H$ and given c_2 is denoted by $\overline{M}^{\mathrm{tf}}(c_2)$. This contains the open subset of stable bundles $M(c_2)$, and we let $\overline{M(c_2)}$ denote the closure of $M(c_2)$. This will, in general, be a smaller closed subset of $\overline{M}^{\mathrm{tf}}(c_2)$, as does happen in some cases.

The *full boundary* of the moduli space is defined to be the closed subset $\overline{M}^{\mathrm{tf}}(c_2) - M(c_2)$. The *boundary divisor* is defined to be $\partial M(c_2) := \overline{M(c_2)} - M(c_2)$. While it would be interesting to provide these closed subsets with a natural subscheme structure measuring the extent to which the "non-locally-freeness" persists in deformations, we don't do that here. The boundary divisor is instead just provided with its reduced structure.

The terminology "boundary divisor" is justified by the following lemma of O'Grady.

Lemma (O'Grady)

Suppose $Z \subset M(c_2)$ is an irreducible component. Define $\partial Z := \overline{Z} - Z$. Then the boundary ∂Z has codimension 1 in $\overline{Z}$.

An important first question, for a given irreducible component Z, is whether ∂Z is nonempty.

Recall that if F is a semistable torsion-free sheaf then its double-dual $E := F^{**}$ is a semistable bundle fitting into an exact sequence

$$0 \to F \to E \to S \to 0$$

where S is a sheaf of finite length. It follows that $c_1(E) = c_1(F)$ and $c_2(E) \leq c_2(F)$ with the last inequality being strict unless F was already a bundle.

In other words, the values of c_2 of the double dual go down along the boundary. It follows that the component(s) of smallest values of c_2 can't have any boundary, that is to say they are compact. So the equation $\partial Z \neq \emptyset$ can't always be true.

O'Grady gives a very nice construction showing that $\partial Z \neq \emptyset$ under some very general hypotheses.

O'Grady's deformation to the boundary

Let us briefly recall O'Grady's construction. It is based on the idea of restriction to a curve. Suppose $C \subset X$ is a curve. If we are given an irreducible component of moduli of bundles Z such that $\partial Z = \emptyset$, then for every bundle $E \in Z$ we obtain a bundle $E|_C$ on C.

If we assume, furthermore, that these restricted bundles are all semistable, it gives a morphism $\rho : Z \to M_C$ from Z to the moduli space of semistable bundles on C of the appropriate degree.
Let $d(C)$ denote the dimension of M_C. There is an ample *theta-divisor* Θ_C on M_C, and from the dimension it follows that

$$\Theta_C^{d(C)+1} = 0, \text{ therefore } \rho^*(\Theta_C)^{d(C)+1} = 0.$$

On the other hand, one can calculate that the pullbacks of the theta-divisor to Z depend only on the class of C. Taking it to be a hyperplane section, the dependence on the degree is also controlled so that the theta-divisors in the various cases are proportional. For curves of very high degree, the map is an embedding so the theta-divisor pulls back to an ample bundle on Z. It follows from the proportionality that $\rho^*(\Theta_C)$ is ample on Z, and from the vanishing above we get

$$\dim(Z) \leq \dim(M_C).$$

We conclude that if $C \subset X$ is a curve such that $\dim(M_C) < \dim(Z)$, then some bundle E in Z must have restriction $E|_C$ that isn't semistable.

The second part of O'Grady's method is to take the destabilizing subsheaf of $E|_C$ and use it to write down a deformation to a non-locally free semistable torsion-free sheaf. We have an exact sequence

$$0 \to L \to E|_C \to Q \to 0.$$

Let T be the elementary transform (kernel of $E \to Q$). We have the transformed exact sequence

$$0 \to Q(-1) \to T|_C \to L \to 0.$$

The idea will be to try to write down a deformation of the quotient map, in the *Quot* scheme of quotients of $T|_C$, to a map $T|_C \to L_1$ such that L_1 has torsion.

O'Grady shows by an ingenious argument that this always happens if the *Quot* scheme has dimension ≥ 2. Furthermore we can give a specialized treatment in our case if the dimension is 1.

Let's just calculate to see that the dimension of the *Quot* scheme at our quotient $T|_C \to L$ is positive. A deformation of the quotient is a map from the subbundle to the quotient bundle, that is

$$T(Quot)_{T|_C \to L} = \mathrm{Hom}(Q(-1), L) = H^0(Q^{-1} \otimes L(1)).$$

The genus of C is 6 and the degree of $\mathcal{O}_C(1)$ is 5, so the Euler characteristic is

$$\chi(Q^{-1} \otimes L(1))) = \deg(L) - \deg(Q).$$

This is strictly positive exactly when L is a destabilizing subsheaf.

Recall that $E(-1)$ is the kernel of $T \to L$. Once the map $T \to L$ has been deformed to $T \to L_1$, let $E_1(-1)$ be the kernel of the new map. The torsion-free sheaf E_1 is a deformation of E.

We get a deformation from E to a boundary point. The precise discussion requires ruling out the case of bundles that deform into a subset V that we'll see next.

O'Grady introduces the subset V consisting of bundles with $h^0(E) \neq 0$, hence fitting into an exact sequence of the form

$$0 \to \mathcal{O}_X \to E \to J_P(1) \to 0$$

with $|P| = c_2$.

This V is the main and largest piece of the obstructed locus of the moduli space. A bundle $E \in V$ has a nilpotent co-obstruction $\phi : E \to E \otimes K_X = E(1)$ that factors

$$E \to J_P(1) \hookrightarrow \mathcal{O}_X(1) \to E(1).$$

For $c_2 \leq 9$ the Euler characteristic argument says that V is the whole moduli space.

For $c_2 \geq 10$, any general subscheme $P \subset X$ satisfies the required $CB(2)$ condition. In this case, V is irreducible of dimension $3c_2 - 11$. Hence, it has dimension 19 when $c_2 = 10$.

Proof. The space of extensions has dual fitting into an exact sequence

$$0 \to H^0(J_{P/X}(2)) \to H^0(\mathcal{O}_X(2)) \to \mathbb{C}^{|P|} \to H^1(J_{P/X}(2)) \to 0$$

so the dimension of Ext^1 is $|P| + h^0(J_{P/X}(2)) - 10$. Adding $2c_2$ for the choice of $P \subset X$ and subtracting 1 for projectivization of the space of extensions gives $3c_2 - 11$. One must analyze the possibly degenerate cases where P imposes less conditions on quadrics to see that those only contribute in smaller dimension.

This leads up to *Nijsse's connectedness theorem*:

Theorem (Nijsse)

For $c_2 \geq 11$*, any irreducible component of* $M(c_2)$ *meets the boundary. For* $c_2 = 10$*, any irreducible component meets either the boundary or the subset* V*. For* $c_2 \geq 10$ *the moduli spaces* $\overline{M(c_2)}$ *and* $\overline{M}^{\mathrm{tf}}(c_2)$ *are connected.*

Corollary

For $c_2 = 10$, the general point E in any irreducible component of $M(c_2)$ has seminatural cohomology, which in this case is equivalent to saying $h^1(E(n)) = 0$ for any n.

Proof.

This is by observing that general points of V have seminatural cohomology, and by the inductive argument on boundary components that we'll see below. One calculates that general points of any possible boundary divisors have seminatural cohomology. □

Theorem

The space of bundles $M^{\mathrm{sn}}(10)$ *with* $c_2 = 10$ *having seminatural cohomology is irreducible.*

Combining with the previous corollary, we conclude that $M(10)$ is irreducible and good of dimension 20.

We can look at some aspects of the proof here. The *seminatural cohomology* condition says that for each n, at most one of the $H^i(E(n))$ is nonzero. In case $c_1 = 1$ and $c_2 = 10$, we have $\chi(E(n)) = 5n^2$, so at $n = 0$ the condition says that all $H^i(E) = 0$. As the Euler characteristic is positive, we have $H^1(E(n)) = 0$ (this is "arithmetically Cohen-Macaulay"), and at most one of H^0, H^2 are going to be nonzero in any case from duality.

To view E by the Serre construction we must look at an element of $H^0(E(1))$, a 5-dimensional space.

Each such element leads to an exact sequence

$$0 \to \mathcal{O}_X(-1) \to E \to J_{P/X}(2) \to 0$$

with P an lci subscheme of length $|P| = 20$ satisfying $CB(4)$.

As $h^0(\mathcal{O}_X(4)) = 35$ and P imposes no more than 19 conditions, the space of sections through P has dimension ≤ 16. One shows that in fact $h^0(J_{P/X}(4)) = 16$.

The choice of P depends on the choice of section in $H^0(E(1))$. With this lattitude, we can insure that most pieces of P are moveable and consist of disjoint reduced points. To see this, we should investigate the *base locus*.

Let B_2 be the subset of points where all sections of $H^0(E(1))$ vanish. We'll show that it has at most one point. This will imply that there is at most one point of P that doesn't move as we change the section.

Consideration of B_1, the subset where sections don't generate E, allows us to conclude furthermore that at the base locus, a general P has at worst multiplicity two.

Let's look at the proof that B_2 has at most a single point. The proofs of what is needed about B_1 are somewhat similar.

Suppose we had two points $p, q \in B_2$. Take a general plane H passing through them and set $Y := H \cap X$, a smooth curve. Let ℓ be the line through p and q, so $\ell \cap X = p + q + u + v + w$.

Assume for here that at least two of the points u, v, w are distinct. The case where all three are at the same location requires a longer proof.

Consider the evaluation map

$$\mathbb{C}^5 \cong H^0(E(1)) \overset{\cong}{\to} H^0(E_Y(1)) \to H^0(E(1)_{\ell\cap X}) \cong \mathbb{C}^{10}.$$

One shows that the image has dimension 5, however by hypothesis everything in the image vanishes at p, q. We can impose four conditions of vanishing at u, v, yielding a section s that doesn't vanish at w.

Let $M \subset E_Y$ be the saturated sub-line bundle generated by our section. Viewed as a section of M, s vanishes at $p + q + u + v$.

Viewed as an inclusion $\mathcal{O}_X(-1) \to E$, our section yields a new $CB(4)$ subscheme P' such that $p + q + u + v \subset P'$. But then, using that to calculate $H^0(E(1))$ by $H^0(J_{P'/X}(3))$, we notice that any cubic vanishing along P' has to also vanish at w, showing that all sections of $H^0(E(1))$ have to take values in M_w at the point w. This shows $w \in B_1$.

In our current hypotheses we obtain at least two new points in B_1 along ℓ, in addition to p and q, contradicting a dimension count for the image of the restriction map above. This contradiction shows (almost…the case $u = v = w$ needing to be treated separately with a more complicated argument) that we can't have two points $p, q \in B_2$.

With some more work for the structure of B_1, we obtain the following conclusion: for a general section of $H^0(E(1))$ the resulting $CB(4)$ subscheme decomposes $P = P' \cup P''$ where the "movable part" P'' consists of at least 18 distinct points that move in a doubly transitive way as we change the section, whereas P' if nonempty consists of a single point, of length 1 or 2, located at the single point of B_2.

The next step is to look at the cubic surfaces containing P. We have $h^0(J_{P/\mathbb{P}^3}(3)) = 4$. Choosing two general elements defines a curve Z of degree 9 that is a complete intersection of two cubics.

Using what we know about the structure of P, we are able to rule out the possibility that a general Z would decompose as a union of two irreducible components, one fixed and one varying. Doing that requires a little classification of the possibilities for Z. The need for such classifications seems to be rather general in this subject.

One concludes that the general Z is irreducible. Along the way, it turns out that P' if nonempty has to also be just a single nonreduced point.

Let $\mathbf{H}^{\mathrm{sn}}_{\mathbb{P}^3}$ denote the Hilbert scheme of subschemes $P \subset \mathbb{P}^3$ consisting of 20 distinct points that satisfy $CB(4)$ with $h^0(J_{P/\mathbb{P}^3}(3)) = 4$ and $h^0(J_{P/\mathbb{P}^3}(2)) = 0$, and contained in at least one smooth quintic hypersurface.

We would like to show that it is irreducible.

Up to now we can say that the general element lies on an irreducible curve Z complete intersection of two cubics. Let $\mathbf{H}^{\mathrm{sn}}_{\mathbb{P}^3}[2]$ denote the bundle over $\mathbf{H}^{\mathrm{sn}}_{\mathbb{P}^3}$ that includes the information of Z.

Some arguments are needed to deal with possible singularities of Z.

Let's ignore that problem and suppose that Z is smooth. Then $P \subset Z$ is a divisor. Put $L := \mathcal{O}_Z(4)(-P)$. Some calculations with Serre duality tell us that[4] $h^0(J_{P/\mathbb{P}^3}(4)) = 16$ if and only if $h^0(K_Z \otimes L^{-1}) = 1$. The line bundle $K_Z \otimes L^{-1}$ has degree 2. Therefore, we look to parametrize effective line bundles of degree 2 on Z; this is given by $Z^{(2)}$. We obtain an irreducible family of the required dimension, altogether 48, in the Hilbert scheme.

Under our hypothesis that a general Z is smooth, this covers a dense open set of $\mathbf{H}^{\mathrm{sn}}_{\mathbb{P}^3}[2]$ and we get the desired irreducibility. More work is needed to deal with singularities of Z.

[4] Note that deformations preserving this condition will also preserve the $CB(4)$ condition.

Look now at how to go from an irreducible Hilbert scheme parametrizing $P \subset \mathbb{P}^3$, to irreducibility for a given quintic hypersurface. This is an argument of the style that A. Hirschowitz explained to us, also used on several other occasions (such as for the Cayley octads).

Denote by $\{P, Z\} := \mathbf{H}^{\mathrm{sn}}_{\mathbb{P}^3}[2]$, irreducible of dimension 48, fibering over $\{P\} := \mathbf{H}^{\mathrm{sn}}_{\mathbb{P}^3}$ which is thus irreducible of dimension 44. Let $\{(P, X)\}$ denote the incidence variety of pairs (P, X) where X is a smooth quintic hypersurface containing P.

Each given P imposes 20 conditions on the 56 dimensional space of quintic hypersurfaces (this doesn't jump, due to our conditions of seminatural cohomology). Thus, $\{(P, X)\} \to \{P\}$ is a bundle of 35-dimensional projective spaces.

In the other direction, the map $\{(P, X)\} \to \{X\} = \mathbb{P}^{55}$ has fibers of dimension 24. We would like to show that they are irreducible. What we know is that the total space is irreducible, therefore the action of the Galois group of the function field of the base, on the set of generic irreducible components of the fibers, is transitive.

The idea to finish the proof is to observe that there is a naturally defined choice of irreducible component of the fibers depending on X. This gives a Galois-invariant choice of single irreducible component in each fiber. Since the Galois action on the set of irreducible components is transitive, this in turn implies that each fiber has only one irreducible component, as desired.

The naturally defined component is the one that contains the subset V of bundles with $H^0(E) \neq 0$. We can show using deformation theory arguments that bundles in E generize to bundles with seminatural cohomology, in a uniquely defined irreducible component. This completes the proof.

Structure of the moduli spaces and their boundaries

The fourth talk combines the previous strands to obtain the picture of the moduli spaces of rank 2 bundles and sheaves of odd degree on a quintic hypersurface. Further questions are explored.

In view of Nijsse's theorem, it is important to look at the structure of the boundary. Recall that if F is a torsion-free sheaf and we let $E := F^{**}$ be its double-dual, then E is a vector bundle. This is because we are on a surface.

If F is semistable then so is E, with the same c_1. The exact sequence we saw before implies that $c_2(F) \geq c_2(E)$ with equality if and only if $E = F$ that is to say if F is a bundle.

The number $c_2(E)$ depends on F in a constructible and semi-continuous way. Therefore we get a stratification

$$\overline{M}^{\mathrm{tf}}(c_2) = M(c_2) \sqcup \coprod_{c_2' < c_2} \overline{M}(c_2, c_2')$$

where $\overline{M}(c_2, c_2')$ is the locally closed subset of the moduli space consisting of points F such that $c_2(F^{**}) = c_2'$.

The results of Li and Ellingsrud-Lehn allow us to understand the dimensions of the strata.

Theorem (Li, Ellingsrud-Lehn)

The map

$$F \mapsto F^{**} \ : \ \overline{M}(c_2, c_2') \to M(c_2')$$

is an etale-locally trivial fibration with connected smooth projective fibers, whose fibers have dimension $3(c_2 - c_2')$.

An open subset consists of F given by disjoint collections of $(c_2 - c_2')$ quotients of rank 1 of the base bundle E, the 3 parameters for such a quotient are 2 for the location in X and 1 for the direction in the projectivized fiber of E.

Coupled with the previous specific results on $M(c_2)$ for $c_2 \leq 9$ we can describe the stratification for $c_2 = 10$.

Table: Dimensions of strata for $\overline{M}(10)$

stratum	dimension	$\dim M(c_2')$	$3(c_2 - c_2')$
$\overline{M}(10,4)$	20	2	18
$\overline{M}(10,5)$	18	3	15
$\overline{M}(10,6)$	19	7	12
$\overline{M}(10,7)$	18	9	9
$\overline{M}(10,8)$	19	13	6
$\overline{M}(10,9)$	19	16	3
$M(10)$	20		

Consideration of the previous table is one of the ingredients in the proof of the corollary about seminatural cohomology: the possible boundary divisors to an irreducible component of $M(10)$ are $\overline{M}(10,9)$, $\overline{M}(10,8)$, $\overline{M}(10,6)$, and some divisor in $\overline{M}(10,4)$.

The $\overline{M}(10,4)$ case needs to be examined more carefully. We'll be able to identify the divisor there where it intersects $\overline{M(10)}$.

The above table also gives information about what $\overline{M}(10)$ looks like, namely it has two irreducible components $\overline{M(10)}$ and $\overline{M}(10,4)$, both of dimension 20, meeting along a 19 dimensional component that is a divisor in $\overline{M}(10,4)$.

In turn, this yields the boundary for $\overline{M}(11)$: namely the 23-dimensional pieces of the boundary stratification for $\overline{M}(11)$ are $\overline{M}(11,10)$ and $\overline{M}(11,4)$.

Since every component of $M(11)$ meets the boundary, and the boundary has pure codimension 1, it follows that no component of $M(11)$ can have dimension > 24, thus the dimension of $M(11)$ is the expected one.

Furthermore, the boundary pieces are already good in the sense that their generic points have vanishing obstruction space, so it shows that $M(11)$ is good.

To finish the proof of irreducibility of $M(11)$, we just have to look at the torsion-free sheaves where $\overline{M}(11, 10)$ and $\overline{M}(11, 4)$ meet. Indeed, it suffices to show by direct calculation that they do indeed meet, equivalently that $\overline{M(10)}$ meets $\overline{M}(10, 4)$.

Since the dimension equals the expected dimension, Kuranishi theory tells us that the moduli space is a local complete intersection. (This works equally well for neighborhoods of torsion-free sheaves.)

Now, if there were one component of $M(11)$ whose boundary was $\overline{M}(11, 10)$ and a different component whose boundary was $\overline{M}(11, 4)$, those would form two 24 dimensional varieties meeting in a subvariety of dimension ≤ 22. That isn't possible in a local complete intersection.

We have shown that whatever components of $M(11)$ are adjacent to $\overline{M}(11, 10)$, are also adjacent to $\overline{M}(11, 4)$. The proof finishes by noting that the space of co-obstructions vanishes at a general point of $\overline{M}(11, 10)$, using the same fact over $M(10)$. Therefore, the global moduli space $\overline{M}(11)$ is smooth along a general point of the divisor $\overline{M}(11, 10)$. Thus, there can be only one component of $M(11)$.

In fact one can show that the two boundary components meet at an unobstructed and hence smooth point, giving an alternative to the argument on the previous slide.

The proof of irreducibility and goodness for $\overline{M}(c_2)$ for $c_2 \geq 12$ proceeds along the same lines but more easily since there now can be only one boundary divisor and we know inductively the vanishing of co-obstructions at general points.

Turn next to a more explicit description of the boundary strata such as $\overline{M}(10,4)$. For this, let's start by looking at $M(4)$. Recall that the bundles in $M(4)$ were obtained via the Serre construction with a subscheme $P \subset \ell \cap X$ for a line $\ell \subset \mathbb{P}^3$. However, the line isn't uniquely determined by E.

It turns out that the invariant quantity in this situation is the location of the 5th point $y \in \ell \cap X$ not in P. This may be seen by noting that $H^0(E)$ generates a subsheaf of E that is locally free except at y. One gets

$$M(4) \cong X.$$

Furthermore, the subsheaf considered above is a torsion-free sheaf F whose double-dual is E, with $c_2(F) = 5$. Thus F is a point in $\overline{M}(5,4) \subset \overline{M}^{\mathrm{tf}}(5)$.

We can obtain the identification

$$\overline{M(5)} \cong \mathbb{P}^3$$

with $X = M(4)$ corresponding to the set of torsion-free sheaves F considered above, and $M(5) = \mathbb{P}^3 - X$.

To do this, we identify $\overline{M(5)}$ with the set of restrictions $R(y)|_X$ where $R(y)$ is the syzygy reflexive sheaf

$$0 \to R(y) \to \mathcal{O}_{\mathbb{P}^3}(1)^{\oplus 3} \to J_{y/\mathbb{P}^3}(2) \to 0.$$

If $y \notin X$ then $R(y)|_X$ is a bundle and we obtain a point of $M(5)$. Whereas if $y \in X$ then $R(y)$ is torsion-free with a length 1 singularity at y, and $(R(y)|_X)^{**}$ is a point of $M(4)$.

Recall that all of $\overline{M}(5,4)$ is a fiber bundle over $M(4)$ whose fiber over E is a 3-dimensional space consisting of pairs (z, A) where $z \in X$ and $A \subset E_z$ is a line. We have a section from $M(4) = X$ into here sending $x \in X$ to the triple (E^x, x, A_x) where E^x is the bundle corresponding to x, and A_x is the line in $(E^x)_x$ generated by $H^0(E^x)$. This copy of X is the intersection of the two irreducible components $\overline{M(5)}$ and $\overline{M}(5,4)$ of dimensions 3 and 5 respectively.

Let's look more at the intersection of $\overline{M}(10,4)$ with $\overline{M(10)}$. From the dimensions, these are the two irreducible components of $\overline{M}(10)$. The first observation is that the intersection is nonempty.

This may be seen by noting that the other strata in the boundary stratification, such as $\overline{M}(10,5)$, need to be in the closure $\overline{M(10)}$ since $\overline{M}(10,4)$ is the lowest stratum so it is closed. However, $\overline{M}(10,5)$ lying over $M(5)$ contains degenerations over the degeneration from $M(5)$ into $\overline{M}(5,4)$. Thus, the closure of $\overline{M}(10,5)$ meets $\overline{M}(10,4)$.

We can then notice that the intersection $\overline{M}(10,4) \cap \overline{M(10)}$ has to be contained in the singular locus of $\overline{M}(10)$. A calculation for torsion-free sheaves in $\overline{M}(10,4)$ shows that their spaces of co-obstructions are nonzero, in codimension 1, along a specific divisor.

This divisor parametrizes $(E, s_1, \ldots, s_6)$ where s_i are distinct length 1 quotients of E, such that they are compatible with the spectral variety of a generic co-obstruction i.e. Higgs field $\phi : E \to E \otimes K_X$.

For each $E \in M(4)$, the space of such Higgs fields (up to scalar) has dimension 5, and given (E, ϕ) the space of 6-tuples has dimension 12. Putting these together gives dimension $2 + 5 + 12 = 19$ and we get a divisor (noting that ϕ is uniquely determined once $s_1, \ldots, s_6$ are fixed).

Dimension counts for other possible configurations yielding obstructed points show that this is the only divisor. Hence, it must be the intersection $\overline{M}(10, 4) \cap \overline{M(10)}$.

On a hypersurface of degree 6

Let's move on to looking at a next case, where the irreducibility property doesn't hold.

Theorem

If $X \subset \mathbb{P}^3$ is a very general hypersurface of degree 6, then $M_X(r = 2, c_1 = 1, c_2 = 11)$ has at least two irreducible components.

From now on $X \subset \mathbb{P}^3$ will denote a hypersurface of degree 6.

We have $K_X = \mathcal{O}_X(2)$. A bundle E comes from the Serre construction in an exact sequence

$$0 \to \mathcal{O}_X \to E \to J_{P/X}(1) \to 0$$

if and only if P satisfies the Cayley-Bacharach condition for the line bundle $K_X \otimes \mathcal{O}_X(1) = \mathcal{O}_X(3)$.

In other words, we now look for subschemes P satisfying $CB(3)$. To get $c_2 = 11$ we should choose a subscheme of length $|P| = 11$

First construction

The first construction is to suppose that P consists of 11 general points on $Y := X \cap H$ where $H \subset \mathbb{P}^3$ is a hyperplane.

Lemma

Such P satisfy $CB(3)$. The space of bundles coming from this construction has dimension 13, hence it must be contained in an irreducible component of the moduli space of dimension ≥ 13.

Proof.

Here Y is a plane curve of degree 6 so it has genus 10. The bundle $\mathcal{O}_{\mathbb{P}^3}(3)|_Y$ has degree 18. Vanishing of a section on 10 general points means that we get a section of a general line bundle of degree 8 so the section vanishes on Y. (One can alternatively see that it actually vanishes on the plane H.) This shows $CB(3)$.

For the dimension estimate notice that $h^1(E) = 1$ for these bundles so the choice of section is a space of dimension 1. The choice of extension class is generically unique. The space of pairs $(E, \text{section})$ is equal to the space of P's that has dimension 3 for the plane plus 11 for the points on Y giving 14. Subtracting 1 for the choice of section, we get a 13 dimensional space. □

The proof doesn't tell, however, whether this family constitutes an irreducible component itself.

Second construction

The second construction is to suppose that P consists of 11 of the 18 points of intersection of X with a rational normal cubic curve C.

Let us first note that such a subscheme P satisfies the $CB(3)$ condition.

Indeed, if we choose any 10 of the 11 points, then forms of degree 3 restrict to forms of degree 9 on $C \cong \mathbb{P}^1$, so vanishing at 10 points implies vanishing along all of C.

Lemma

The space of bundles coming from P in a rational normal cubic, has dimension 12. The tangent space of $M_X(11)$ at a general point in the family has dimension 12 so these families lie in 12-dimensional irreducible components.

Proof.

Generally speaking the main calculation that enters into the tangent space in the case of the Serre construction is the space of sections of the square of the ideal. In this case, we claim that

$$H^0(J_{2P/X}(3)) = 0.$$

Various calculations then yield the answer of dimension 12.

The claim is shown by first considering a smooth quadric surface Q containing C. Given a section of $J_{2P/X}(3)$ it vanishes on C, then as a section of bidegree $(2,1)$ on Q it vanishes at 11 points on the rational normal curve, making that one also vanish on C and hence on Q. We are left with a section of $\mathcal{O}_{\mathbb{P}^3}(1)$ vanishing again on the 11 points so it vanishes. □

Here, the proof doesn't tell whether the space of choices of 11 out of the 18 points is itself irreducible or not.

These two constructions give irreducible components of the moduli space $M_X(11)$ having different dimensions, so they must be different. This proves the reducibility theorem in degree 6.

Conjecture

The above constructions for bundles with $c_2 = 11$ *on a hypersurface* $X \subset \mathbb{P}^3$ *give exactly two components of* $M_X(11)$ *of dimensions* 12 *and* 13 *respectively.*

Further questions and directions

Several questions are naturally posed.

- What about bundles of degree 0? Strictly semistable points can introduce other singularities and complicate the argument. What can we say about the existence of universal families in the even degree case?
- What about hypersurfaces which are not generic, for example with Picard number > 1?
- Non-generic quintic hypersurfaces also have a new deformation direction: *Horikawa surfaces*. What do the moduli spaces of rank 2 bundles on general Horikawa surfaces look like?
- For degrees $d \geq 6$ can we say something general about the moduli spaces, for certain ranges of values of c_2?

We note that S. Pal and D. Battacharya are working on the case of hypersurfaces of degree 6, trying to get a fuller picture of the structure of irreducible components and when the moduli space is good. They obtain a partial result towards the above conjecture for components when $c_2 = 11$.

We expect the moduli space to be reducible for very general hypersurfaces of any degree $d \geq 6$.

One can ask, what is the interval of values of c_2 where the moduli space is reducible, going to look like? Coskun and Huizenga show that for high enough degrees the number of components can be arbitrarily large.

Smoothing a torsion-free sheaf

We can give a preliminary discussion of the question of obstructions for smoothing a torsion-free sheaf, at least to first order. Thanks very much to Marcos Jardim for posing that question yesterday.

A reference is:
I.V. Artamkin. Deforming torsion-free sheaves on an algebraic surface. *Mathematics of the USSR-Izvestiya*, 36(3), (1991), 449.

Suppose X is a smooth projective surface and we are looking at rank 2 torsion-free sheaves. Suppose F is a torsion-free sheaf, and $E = F^{**}$ its double-dual fitting into the exact sequence

$$0 \to F \to E \to S \to 0.$$

We make the following hypothesis: that

$$S = \bigoplus S_i$$

with S_i being skyscraper sheaves of length 1 supported at distinct points x_i.

Locally near x_i we can write $E \cong L \oplus M$ and $F \cong L \oplus G$ where

$$0 \to G \to M \to S \to 0$$

with S being a length 1 skyscraper sheaf at a point $x \in X$. We have the Koszul resolutions (local)

$$\left(\mathcal{O} \to \mathcal{O}^{\oplus 2} \to \mathcal{O}\right) \sim S \quad \text{and} \quad \left(\mathcal{O} \to \mathcal{O}^{\oplus 2}\right) \sim G.$$

In particular, G and F have resolutions of length 1. For S, we define

$$S^* := \underline{Ext}^2(S, \mathcal{O}),$$

a direct sum of skyscraper sheaves whose values are $(S_x)^* \otimes \omega_x^{-1}$.

After some manipulations we get an exact sequence of sheaves:

$$\underline{Ext}^0(F,F) \to \underline{Ext}^0(E,E) \to (L^* \otimes S) \oplus T_x(X)$$

$$\to \underline{Ext}^1(F,F) \to S^* \otimes L \to 0.$$

This yields a map of complexes of sheaves

$$\underline{Ext}^{\cdot}(F,F) \to S^* \otimes L[-1]$$

that may be seen as the map governing local smoothing of the non-locally free part of F.

Notice that the smoothing part only has a coefficient in morphisms from S_x to L_x (the latter being the kernel of $E_x \to S_x$). This amounts to saying that creating an instanton requires an extra direction, for example we can never smooth a rank 1 torsion-free sheaf.

Let A be the kernel of the above map of complexes. We have two exact triangles

$$A \to \underline{Ext}^{\cdot}(F, F) \to S^* \otimes L[-1].$$

and

$$A \to \underline{Hom}(E, E) \to (L^* \otimes S) \oplus T_x(X).$$

Remark that $H^2(A) \cong H^2(\mathrm{End}(E))$. The first triangle gives a long exact sequence

$$0 \to H^1(A) \to \mathrm{Ext}^1(F,F) \to H^0(S^* \otimes L) \to H^2(\mathrm{End}(E)) \to \mathrm{Ext}^2(F,F) \to 0.$$

Whereas the second triangle gives

$$(L^* \otimes S) \oplus T_x(X) \to H^1(A) \to H^1(\mathrm{End}(E)) \to 0$$

with the map on the left being injective in case E is stable so its only endomorphisms are scalars.

The term $H^1(A)$ is the term of deformations of the pair consisting of the bundle E and the quotient S. The inclusion of $(L^* \otimes S) \oplus T_x(X)$ represents, for the first part, modifying the quotients at each point, and for the second part modifying the locations of the points.

The map $\mathrm{Ext}^1(F, F) \to H^0(S^* \otimes L)$ tells us when a deformation is going to smooth each of the local quotients.

Artamkin points out that we can use trace-free endomorphisms everywhere.

The obstruction to smoothing is therefore the following thing: we notice that $H^0(S^* \otimes L)$ is a direct sum of pieces at each of the points. There will be a smoothing to first order if there is a collection (sum) in here, nonzero at each point, that maps to zero in the obstruction space $H^2(\mathrm{End}^0(E))$.

This can be measured using the dual

$$H^0(\mathrm{End}^0(E) \otimes K_X) \to \bigoplus L_x^* \otimes S_x \otimes (K_X)_x$$

The new space of co-obstructions is going to be the kernel. Heuristically, adding non-locally free points to go up in the boundary stratification cancels out obstructions.

Brief explanation for Cayley-Bacharach

I'm putting here some slides giving an explanation of the Cayley-Bacharach condition.

Suppose given two line bundles L, M and an ideal sheaf $J_{P/X}$. We look for extension classes

$$0 \to L \to E \to J_{P/X} \otimes M \to 0$$

such that E is locally free. The space of extensions is

$$\mathrm{Ext}^1(J_{P/X} \otimes M, L) \cong H^0(J_{P/X} \otimes L^* \otimes M \otimes K_X)^*.$$

Brief explanation for Cayley-Bacharach

We note that E is non-locally free if and only if there exists an inclusion of torsion-free sheaves $E \hookrightarrow E'$ of co-length 1. Having such an inclusion corresponds to having an exact sequence with a map from the previous one:

$$0 \to L \to E' \to J_{P'/X} \otimes M \to 0$$

where $P' \subset P$ is defined by an ideal $J_{P'/P}$ of length 1.

Brief explanation for Cayley-Bacharach

In turn, this means that there exists P' such that our extension class is in the image of the map

$$\mathrm{Ext}^1(J_{P'/X} \otimes M, L) \to \mathrm{Ext}^1(J_{P/X} \otimes M, L).$$

Existence of a locally free extension is therefore equivalent to the condition that for all possible P' (there are only finitely many), the above map is not surjective.

Non-surjectivity is equivalent to saying that the maps on Serre duals

$$H^1(J_{P/X} \otimes L^* \otimes M \otimes K_X) \to H^1(J_{P'/X} \otimes L^* \otimes M \otimes K_X)$$

are not injective.

Brief explanation for Cayley-Bacharach

We have

$$H^1(J_{P/X} \otimes L^* \otimes M \otimes K_X) \cong \frac{H^0(\mathcal{O}_P \otimes L^* \otimes M \otimes K_X)}{H^0(L^* \otimes M \otimes K_X)}$$

and same for P'.

In our case line bundles themselves don't have H^1 but if they did, those terms would be the same for P and P'.

Thus, non-injectivity means that there should be an element of $H^0(\mathcal{O}_P \otimes L^* \otimes M \otimes K_X)$ that restricts to 0 on P' but doesn't come from a global section.

Brief explanation for Cayley-Bacharach

In other words, the condition for having a locally free extension is that at all subschemes P', the 1-dimensional space

$$H^0(J_{P'/P} \otimes L^* \otimes M \otimes K_X)$$

doesn't come from $H^0(J_{P'/X} \otimes L^* \otimes M \otimes K_X)$. In other words, sections of the line bundle that vanish over P' are not supposed to be non-vanishing on P.

This is the Cayley-Bacharach condition: in order for there to exist a locally free extension, for every $P' \subset P$ defined by an ideal of length 1, any section of $L^* \otimes M \otimes K_X$ that vanishes on P' should also vanish on P.