STABILITY OF NORMAL BUNDLES OF SPACE CURVES

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Abstract. In this paper, we prove that the normal bundle of a general Brill-Noether space curve of degree d and genus $g \geq 2$ is stable if and only if $(d, g) \notin \{(5,2),(6,4)\}$. When $g \leq 1$ and the characteristic of the ground field is zero, it is classical that the normal bundle is strictly semistable. We show that this fails in characteristic 2 for all rational curves of even degree.

1. Introduction

Let C be a smooth connected curve defined over an algebraically closed field k (of arbitrary characteristic). The normal bundle $N_{C/\mathbb{P}^r}$ of a smooth curve controls the deformations of the curve in $\mathbb{P}^r$ and plays a crucial role in many problems of geometry, arithmetic and commutative algebra. In this paper, we show that the normal bundle of a general Brill-Noether space curve of degree d and genus g is stable if and only if $g \geq 2$ and $(d, g) \notin \{(5,2),(6,4)\}$.

Let E be a vector bundle on a smooth curve C. Let the slope $\mu(E)$ be

$$\mu(E) := \frac{\deg(E)}{\mathrm{rk}(E)}.$$

Then E is called (semi)stable if every proper subbundle F of smaller rank satisfies

$$\mu(F) \underset{(-)}{<} \mu(E).$$

The bundle is called unstable if it is not semistable and strictly semistable if it is semistable but not stable.

By the Brill-Noether Theorem (see [KL72, GrH80, Gi82, ACGH85, O11, JP14, CLT18]), a general curve of genus g admits a nondegenerate, degree d map to $\mathbb{P}^r$ if and only if the Brill-Noether number $\rho(g, r, d)$ satisfies

$$\rho(g, r, d) := g - (r+1)(g - d + r) \geq 0.$$

When $r \geq 3$, there is a unique component of the Hilbert scheme that dominates the moduli space $\overline{M}_g$ and whose general member parameterizes a smooth, nondegenerate curve of degree d and genus g in $\mathbb{P}^r$. We call a member of this component a Brill–Noether curve. When $r = 3$, we call such a curve a Brill-Noether space curve. With this terminology, our main theorem is the following.

Theorem 1. *Let $C \subseteq \mathbb{P}^3$ be a general Brill–Noether space curve of degree d and genus g over an algebraically closed field k.*

(1) N_C is stable if and only if $g \geq 2$ and $(d, g) \notin \{(5,2),(6,4)\}$.
(2) N_C is strictly semistable if and only if $g < 2$ and one of the following holds: $\mathrm{char}(k) \neq 2$, *$g = 1$, or d is odd.*
(3) N_C is unstable if and only if $(d, g) \in \{(5,2),(6,4)\}$, or all of the following hold: $\mathrm{char}(k) = 2$, *$g = 0$, and d is even.*

The normal bundles of curves in projective space have been studied by many authors (for example, see [ALY19, BE84, CR18, EiL92, E83, ElH84, ElL81, N83, R07, S80, S82, S83]). Our results complete and unify these results for Brill-Noether space curves.

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If $(d,g) \in \{(5,2),(6,4)\}$, then C lies on a unique quadric Q and $N_{C/Q} \subset N_C$ gives a destabilizing subbundle. We will describe the geometry in these two cases more explicitly in §3.

Every bundle on $\mathbb{P}^1$ splits as a direct sum of line bundles. Hence, the normal bundle of a smooth rational curve can be written as $N_C = \bigoplus_{i=1}^{r-1} \mathcal{O}(a_i)$ for some integers $a_1, \dots, a_{r-1}$ with

$$\sum_{i=1}^{r-1} a_i = (r+1)d - 2.$$

If C is a general rational curve of degree at least $d \geq r$ in $\mathbb{P}^r$, and the characteristic of the ground field is not 2, then $N_{C/\mathbb{P}^r}$ splits as equally as possible, i.e. $|a_i - a_j| \leq 1$ (see [S80, R07, CR18, ALY19]). Hence, $N_{C/\mathbb{P}^r}$ is strictly semistable when $r-1$ divides $2d-2$ and is unstable otherwise. When $r=3$ and $\mathrm{char}(k) \neq 2$, since the quantity $2d-2$ is always even, the normal bundle of a general rational curve of degree $d \geq 3$ is strictly semistable. If the characteristic is 2, we show in Lemma 3.2 that all $a_i \equiv d \bmod 2$; this obstructs semistability for rational curves with d even.

Similarly, normal bundles of genus one curves have been studied extensively (see [EiL92, ElH84, ElL81]). By [ElH84], the normal bundle of a general nondegenerate genus one space curve is semistable. On the other hand, on a genus one curve, there are no stable rank 2 bundles of degree $4d$. Hence, the normal bundle of a general genus one space curve of degree $d \geq 4$ is strictly semistable. Our techniques will provide short arguments reproving the $g=0$ and 1 cases.

In higher genus, the previously known results were more sporadic. The stability of the normal bundle was proved for $(d,g)=(6,2)$ by Sacchiero [S83], for $(d,g)=(9,9)$ by Newstead [N83], for $(d,g)=(6,3)$ by Ellia [E83], and for $(d,g)=(7,5)$ by Ballico and Ellia [BE84]. Many of these cases will be important for our inductive arguments. For completeness, we will reprove these cases using our techniques or briefly recall the arguments. More generally, in [ElH84], Ellingsrud and Hirschowitz announced a proof of stability of normal bundles in an asymptotic range of degrees and genera; however, their results do not cover many of the most challenging cases of small degree.

We prove Theorem 1 by specialization. We use three basic specializations: (1) we specialize to a curve of degree $(d-1,g)$ union a 1-secant line; (2) we specialize to a curve of degree $(d-1,g-1)$ union a 2-secant line; and finally (3) we specialize to a curve of degree $(d-2,g-3)$ union a 4-secant conic. These degenerations reduce Theorem 1 to a finite set of base cases. The most challenging part of the paper is to verify these base cases.

We expect our techniques and results to generalize to $\mathbb{P}^r$ for $r \geq 3$ and hopefully settle the following conjecture.

Conjecture 1.1. *The normal bundle of a general Brill-Noether curve of genus at least 2 in $\mathbb{P}^r$ is stable except for finitely many triples (d,g,r).*

Conjecture 1.1 is closely related to several conjectures in the literature. For example, Aprodu, Farkas and Ortega have conjectured that the normal bundle of a general canonical curve of $g \geq 7$ is stable [AFO16, Conjecture 0.4] (see also [Br17]).

Organization of the paper. In §2, we will recall basic facts about normal bundles on nodal curves and elementary modifications. In §3, we will elaborate on the two cases $(d,g) \in \{(5,2),(6,4)\}$ as well as the obstruction to stability for rational curves in characteristic 2. In §4, we will introduce several basic degenerations to reduce the theorem to a small set of initial cases. For the rest of the paper, we will analyze these initial cases.

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2. Preliminaries

In this section, we collect basic facts on normal bundles of curves, stability of vector bundles, elementary modifications, and on certain reducible Brill–Noether curves. For more details, we refer the reader to [ALY19, L16a, L17]; when necessary, we provide a characteristic-independent proof here.

The normal bundle of a space curve. Let $C \subset \mathbb{P}^r$ be a smooth Brill–Noether curve of degree d and genus g. The normal bundle N_C is a rank $r-1$ vector bundle that is presented as a quotient

$$0 \to T_C \to T_{\mathbb{P}^r}|_C \to N_C \to 0,$$

of the restricted tangent bundle of $\mathbb{P}^r$ by the tangent bundle of C. The restricted tangent bundle is itself naturally a quotient in the Euler exact sequence

$$0 \to \mathcal{O}_C \to \mathcal{O}_C(1)^{\oplus(r+1)} \to T_{\mathbb{P}^r}|_C \to 0. \tag{1}$$

From this we see that $\deg(N_C) = (r+1)d + 2g - 2$. Specializing to $r = 3$, we have that

$$\mu(N_C) = 2d + g - 1,$$

and therefore N_C is stable if and only if all line subbundles $L \subseteq N_C$ have slope at most $2d + g - 2$.

Stability of vector bundles on nodal curves. In the course of our inductive argument, we will specialize a smooth Brill–Noether curve to a reducible nodal curve. In this section, we generalize the definition of stability of vector bundles to allow C to be a connected nodal curve. We will write

$$\nu : \tilde{C} \to C$$

for the normalization of C. For any node p of C, write $\tilde{p}_1$ and $\tilde{p}_2$ for the two points of $\tilde{C}$ over p.

Given a vector bundle E on C, the fibers of the pullback ν^*E to $\tilde{C}$ over $\tilde{p}_1$ and $\tilde{p}_2$ are naturally identified. Given a subbundle $F \subseteq \nu^*E$, it therefore makes sense to compare $F|_{\tilde{p}_1}$ and $F|_{\tilde{p}_2}$ inside $\nu^*E|_{\tilde{p}_1} \simeq \nu^*E|_{\tilde{p}_2}$.

Definition 2.1. Let E be a vector bundle on a connected nodal curve C. For a subbundle $F \subset \nu^*E$, define the adjusted slope μ_C^{adj} by

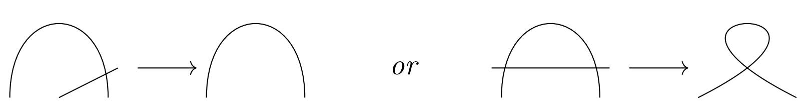
$$\mu_C^{\mathrm{adj}}(F) := \mu(F) - \frac{1}{\operatorname{rk} F} \sum_{p \in C_{\mathrm{sing}}} \operatorname{codim}_F \left(F|_{\tilde{p}_1} \cap F|_{\tilde{p}_2} \right),$$

where $\operatorname{codim}_F \left(F|_{\tilde{p}_1} \cap F|_{\tilde{p}_2} \right)$ refers to the codimension of the intersection in either $F|_{\tilde{p}_1}$ or $F|_{\tilde{p}_2}$ (which are equal since $\dim F|_{\tilde{p}_1} = \dim F|_{\tilde{p}_2}$). When the curve C is unambiguous, we will omit it from our notation and write simply $\mu^{\mathrm{adj}}(F)$. Note that if F is pulled back from C, then $\mu_C^{\mathrm{adj}}(F) = \mu(F)$. We say that E is (semi)stable if for all subbundles $F \subset \nu^*E$,

$$\mu^{\mathrm{adj}}(F) \underset{(-)}{<} \mu(\nu^*E) = \mu(E).$$

With this definition, stability is an open condition in families of connected nodal curves. To show this, we will need the following lemma.

Lemma 2.2. *Let $\beta : C' \to C$ be a map obtained by contracting a 1- or 2- secant $\mathbb{P}^1$:*



If E is a (semi)stable vector bundle on C, then β^*E is also (semi)stable.

Proof. Write $\nu\colon \tilde{C} \to C$ and $\nu'\colon \tilde{C}' \to C'$ for the normalization maps.

First consider the 1-secant case. Write x for the point of attachment (so that $C' = C \cup_x \mathbb{P}^1$). Let E be a (semi)stable vector bundle on C, and let $F \subset \nu'^* \beta^* E$ be any subbundle. Since $\nu'^*\beta^* E|_{\mathbb{P}^1}$ is trivial, we have

$$\mu(F|_{\mathbb{P}^1}) \leq 0. \tag{2}$$

Thus,

$$\mu_{C'}^{\text{adj}}(F) = \mu_{C}^{\text{adj}}(F|_{\tilde{C}}) + \mu(F|_{\mathbb{P}^1}) - \frac{\operatorname{codim}_F(F|_{\tilde{x}_1} \cap F|_{\tilde{x}_2})}{\operatorname{rk} F} \leq \mu_C^{\text{adj}}(F|_{\tilde{C}}) \underset{(-)}{<} \mu(E),$$

hence $\beta^* E$ is (semi)stable.

Similarly in the 2-secant case, write $C' = C'' \cup_{\{x,y\}} \mathbb{P}^1$. Denote by $\tilde{x}_1$ and $\tilde{y}_1$ (respectively $\tilde{x}_2$ and $\tilde{y}_2$) the corresponding points on $\mathbb{P}^1$ (respectively C''). Let $F \subset \nu'^*\beta^* E$. Since $\nu'^*\beta^*E|_{\mathbb{P}^1}$ is trivial, we can identify the fiber of E at $\tilde{x}_1$ with the fiber of E at $\tilde{y}_1$, and we have

$$\mu(F|_{\mathbb{P}^1}) \leq -\frac{1}{\operatorname{rk} F} \cdot \operatorname{codim}_F(F|_{\tilde{x}_1} \cap F|_{\tilde{y}_1}).$$

Thus, for any subbundle $F \subset \nu'^*\beta^* E$,

$$\begin{aligned}
\mu_{C'}^{\text{adj}}(F) &= \mu_{C''}^{\text{adj}}(F|_{\tilde{C}}) + \mu(F|_{\mathbb{P}^1}) - \frac{1}{\operatorname{rk} F} \cdot \left(\operatorname{codim}_F(F|_{\tilde{x}_1} \cap F|_{\tilde{x}_2}) + \operatorname{codim}_F(F|_{\tilde{y}_1} \cap F|_{\tilde{y}_2}) \right) \\
&\leq \mu_{C''}^{\text{adj}}(F|_{\tilde{C}}) - \frac{1}{\operatorname{rk} F} \cdot \left(\operatorname{codim}_F(F|_{\tilde{x}_1} \cap F|_{\tilde{y}_1}) + \operatorname{codim}_F(F|_{\tilde{x}_1} \cap F|_{\tilde{x}_2}) + \operatorname{codim}_F(F|_{\tilde{y}_1} \cap F|_{\tilde{y}_2}) \right)
\end{aligned}$$

Twice applying the "triangle inequality" $\operatorname{codim}(X \cap Y) + \operatorname{codim}(Y \cap Z) \geq \operatorname{codim}(X \cap Z)$,

$$\begin{aligned}
&\leq \mu_{C''}^{\text{adj}}(F|_{\tilde{C}}) - \frac{1}{\operatorname{rk} F} \cdot \operatorname{codim}_F(F|_{\tilde{x}_2} \cap F|_{\tilde{y}_2}) \\
&= \mu_C^{\text{adj}}(F|_{\tilde{C}}) \\
&\underset{(-)}{<} \mu(E).
\end{aligned}$$

□

Proposition 2.3. *Let $\mathscr{C} \to \Delta$ be a family of connected nodal curves over the spectrum of a discrete valuation ring, and $\mathscr{E}$ be a vector bundle on $\mathscr{C}$.*

(1) If the special fiber $\mathscr{E}_0 = \mathscr{E}|_0$ is (semi)stable, then the general fiber $\mathscr{E}^* = \mathscr{E}|_{\Delta^*}$ is also (semi)stable.

(2) If $\mathscr{C} \to \Delta$ is smooth, and $\mathscr{E}_0$ is semistable, then any subbundle $\mathscr{F}^* \subset \mathscr{E}^*$ with $\mu(\mathscr{F}^*) = \mu(\mathscr{E}^*)$ extends to a subbundle $\mathscr{F} \subset \mathscr{E}$.

Proof. Write $\nu\colon \tilde{\mathscr{C}} \to \mathscr{C}$ for the normalization.

For part (1), after possibly making a base change, let $\mathscr{F}^* \subset \nu^*\mathscr{E}^*$ be a subbundle with $\mu(\mathscr{F}^*)$ maximal. Since μ is constant in flat families and $\operatorname{codim}(X \cap Y)$ is lower semicontinuous, μ^{adj} is upper semicontinuous in flat families. Therefore, if $\mathscr{F}^*$ extends to a subbundle $\mathscr{F} \subset \nu^*\mathscr{E}$, then

$$\mu^{\text{adj}}(\mathscr{F}^*) \leq \mu^{\text{adj}}(\mathscr{F}_0) \underset{(-)}{<} \mu(\mathscr{E}_0) = \mu(\mathscr{E}^*). \tag{3}$$

Otherwise, we make a blowup $\tilde{\beta}\colon \tilde{\mathscr{C}}' \to \tilde{\mathscr{C}}$ in order to extend $\mathscr{F}^* \subset \nu^*\mathscr{E}^*$ to a subbundle $\mathscr{F} \subset \tilde{\beta}^*\nu^*\mathscr{E}$. By semistable reduction, we may ensure that the central fiber remains reduced. By gluing along sections identified under ν, the blowup $\tilde{\beta}$ induces a map $\beta\colon \mathscr{C}' \to \mathscr{C}$, which is an isomorphism away from the central fiber, and on the central fiber consists of replacing nodes by 1- and 2-secant $\mathbb{P}^1$'s. Applying Lemma 2.2, $\beta^*\mathscr{E}_0$ is (semi)stable. Therefore (3) holds for $\beta^*\mathscr{E}$.

For part (2), we imitate the above argument to extend $\mathscr{F}^*$ to a subbundle of $\beta^*\mathscr{E}$. Since $\mathscr{C} \to \Delta$ is smooth, β can be obtained by iteratively contracting 1-secant $\mathbb{P}^1$s. Since $\mu(\mathscr{F}^*) = \mu(\mathscr{E}^*)$ and $\beta^*\mathscr{E}_0$ is semistable, we must in particular have equality in equation (2) from the proof of Lemma

2.2 for every such contraction; thus, $\mathscr{F}$ is trivial along every exceptional divisor of β. In particular, $\mathscr{F}^*$ already extends to a subbundle $\mathscr{F} \subset \mathscr{E}$ without blowing up. □

Elementary modifications of vector bundles. Let E be a vector bundle on a scheme X and let $F \subset E$ be a subbundle. For any effective Cartier divisor $D \subset X$, we define the elementary modification of E at D towards F to be the kernel of the natural evaluation map

$$E[D \to F] := \ker\left(E \to (E/F)|_D\right).$$

For an exposition on this construction, see [ALY19, §2–3]. In particular, by [ALY19, Proposition 2.6], $E[D \to F]$ is a vector bundle.

Let $q \in \mathbb{P}^r$ be a point. In this paper we will be primarily concerned with modifications of the normal bundle $N_{C/\mathbb{P}^r}$ towards pointing bundles $N_{C\to q}$, which we now recall. For a more detailed exposition see [ALY19, §5–6]. Write

$$U_{C,q} = \{p \in C : T_pC \cap q = \varnothing\},$$

and let $\pi_q \colon U_{C,q} \to \mathbb{P}^{r-1}$ denote the projection map from q; note that π_q is unramified by construction. If $U_{C,q}$ is dense in C and contains the singular locus of C, then we may define $N_{C\to q}$ to be the unique extension to all of C of the bundle

$$N_{C\to q}|_{U_{C,q}} := \ker\left(N_C \to N_{\pi_q}\right),$$

where N_{π_q} denotes the normal sheaf of π_q. Our notation $N_{C\to q}$ is intended to suggest the geometry of sections: they point towards q in $\mathbb{P}^r$. By convention, we will write

$$N_C[p \to q] := N_C[p \to N_{C\to q}].$$

The following foundational result of Hartshorne-Hirschowitz underpins our degenerative approach.

Lemma 2.4 ([HH85, Corollary 3.2])**.** *Let $X \cup Y$ be a connected nodal curve in $\mathbb{P}^r$. Write $\{p_1, \ldots, p_n\} = X \cap Y$ and let $q_i \in T_{p_i}Y$ be a choice of point. Then*

$$N_{X\cap Y}|_X \simeq N_X(p_1 + \cdots + p_n)[p_1 \to q_1]\cdots[p_n \to q_n].$$

In the course of our degenerations, we will make use of the following lemma.

Lemma 2.5. *Let D be a (smooth) curve of type (a,b) on a smooth quadric surface Q. If q is a general point of D, then inside $\mathbb{P}N_D$, the two sections coming from the line subbundles $N_{D\to q}$ and $N_{D/Q}$ meet transversely at $a+b-2$ points.*

Proof. The fibers of $N_{D\to q}$ and $N_{D/Q}$ agree at p if and only if q is contained in T_pQ. This occurs exactly at the points p where the two lines through q in Q meet D. Since D is of type (a,b) on Q, for q general this happens at $a+b-2$ points of D.

On the other hand, with multiplicity, the intersection number of these two sections is

$$c_1(N_D) - c_1(N_{D\to q}) - c_1(N_{D/Q}) = (2ab + 2a + 2b) - (a + b + 2) - (2ab) = a + b - 2.$$

Therefore, when q is general, these sections intersect transversely at exactly $a+b-2$ points. □

It is a classical fact that the normal bundle of a rational normal (i.e. $(d,g) = (3,0)$) or elliptic normal (i.e. $(d,g) = (4,1)$) curve is semistable, which we record in the following lemma:

Lemma 2.6. *Let C be a general Brill–Noether curve of degree d and genus g, where $(d,g) = (3,0)$ or $(4,1)$. Then N_C is semistable.*

Proof. For $(d,g) = (3,0)$, let p be a point on C, and write $\overline{C} \subset \mathbb{P}^2$ for the image of C under projection from p (which is a conic). Then the semistability of N_C follows from the exact sequence

$$0 \to [N_{C\to p} \simeq \mathcal{O}_{\mathbb{P}^1}(5)] \to N_C \to [N_{\overline{C}}(p) \simeq \mathcal{O}_{\mathbb{P}^1}(5)] \to 0.$$

For $(d,g) = (4,1)$, we note that C is the complete intersection of two quadrics; hence $N_C \simeq \mathcal{O}_C(2) \oplus \mathcal{O}_C(2)$ is semistable. □

Reducible Brill–Noether curves. In this section we show that the basic degenerations we will employ in the proof of Theorem 1 are in the Brill–Noether component of the Hilbert scheme.

We say that two curves X and Y meet quasi-transversely at a set of points $\Gamma \subset \mathbb{P}^r$ if for each $p \in \Gamma$, the tangent lines T_pX and T_pY meet only in the isolated point p. (If $r \geq 3$, two curves never meet transversely!) The following Lemma is a special case of results of [L16a], but we include a characteristic-independent proof of this special case.

Lemma 2.7. *Let C be a general Brill–Noether curve of degree d and genus g and let R be one of the following*

(i) a 1-secant line meeting C quasi-transversely at p,
(ii) a 2-secant line meeting C quasi-transversely at p and q,
(iii) a 4-secant conic meeting C quasi-transversely at four coplanar points $p_1, \ldots, p_4$.

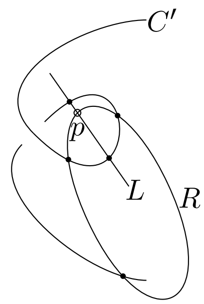
Then $C \cup R$ is a Brill–Noether curve of degree and genus (i) $(d+1, g)$, (ii) $(d+1, g+1)$, (iii) $(d+2, g+3)$.

Proof. By deformation theory, it suffices to show that $H^1(T_{\mathbb{P}^3}|_{C\cup R}) = 0$, so that the map $C \cup R \to \mathbb{P}^3$ may be lifted as $C \cup R$ is deformed to a general curve. Moreover, if C is general, then $H^1(T_{\mathbb{P}^3}|_C) = 0$ by the Gieseker-Petri Theorem. We have an exact sequence

$$0 \to T_{\mathbb{P}^3}|_R(-R \cap C) \to T_{\mathbb{P}^3}|_{C\cup R} \to T_{\mathbb{P}^3}|_C \to 0. \tag{4}$$

In cases (i) and (ii), $T_{\mathbb{P}^3}|_R \simeq \mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus 2}$. Hence $H^1(T_{\mathbb{P}^3}|_R(-p)) = 0$, respectively $H^1(T_{\mathbb{P}^3}|_R(-p-q)) = 0$, and therefore, by (4) and the Gieseker-Petri Theorem for C, we have that $H^1(T_{\mathbb{P}^3}|_{C\cup R}) = 0$.

For part (iii), by part (ii) we may specialize C to the union of a Brill–Noether curve C' of degree $d-1$ and genus $g-1$ and a 2-secant line L, such that R meets C' at three points and meets L at one point p. Let $\Gamma := (L \cup R) \cap C'$, denoted by solid dots below.



First, we show that (a) $C' \cup L \cup R$ is a smooth point of the Hilbert scheme and (b) we can smooth $L \cup R$ to a twisted cubic R' that continues to pass through the 5-points Γ. Let N be the subsheaf of $N_{L\cup R}(-\Gamma)$ whose sections fail to smooth the node at p. Restriction to L gives an exact sequence

$$0 \to [N_{L\cup R}|_R(-p-\Gamma) \simeq \mathcal{O} \oplus \mathcal{O}(-1)] \to N \to [N|_L \simeq \mathcal{O}(-1)^{\oplus 2}] \to 0; \tag{5}$$

hence, by the long exact sequence associated to (5), we have $H^1(N) = 0$. By deformation theory, statement (b) follows directly from $H^1(N) = 0$; this vanishing also implies $H^1(N_{C'\cup L\cup R}) = 0$ (and hence, by deformation theory, statement (a)).

To complete the proof, $T_{\mathbb{P}^3}|_{R'}(-R' \cap C') \simeq \mathcal{O}(-1)^{\oplus 4}$ has no higher cohomology and so (4) and the Gieseker-Petri Theorem for C' show that $H^1(T_{\mathbb{P}^3}|_{C'\cup R'}) = 0$. Therefore $C' \cup R'$ is in the Brill–Noether component. Since $C' \cup L \cup R$ is a smooth point of the Hilbert scheme and both $C' \cup R'$ and $C \cup R$ are deformations of this, they are in the same component; in particular, $C \cup R$ is in the Brill–Noether component. □

3. The Unstable Cases

Arbitrary characteristic. In two cases — $(d,g) \in \{(5,2),(6,4)\}$ — Theorem 1 asserts that, over a field of any characteristic, N_C is unstable. In both of these cases, C lies on a quadric Q, and from the normal bundle exact sequence,

$$0 \to [N_{C/Q} \simeq K_C(2)] \to N_C \to [N_Q|_C \simeq \mathcal{O}_C(2)] \to 0, \tag{6}$$

we have that N_C has a subbundle $N_{C/Q}$ of slope $2d+2g-2$. If $(d,g)=(5,2)$ (respectively $(6,4)$) then $\mu(N_{C/Q}) = 12$ (respectively 18), which is strictly more than $\mu(N_C) = 11$ (respectively 15).

In fact, we can say more. Note that $\mathrm{Ext}^1(\mathcal{O}_C(2), K_C(2)) \simeq H^1(K_C)$ is 1-dimensional; therefore there are only two such extensions up to isomorphism (the split extension, and a unique nontrivial extension).

When $(d,g)=(6,4)$, such curves C are the complete intersection of a quadric and cubic surface, and so (6) is split. When $(d,g)=(5,2)$, the following lemma is equivalent to the assertion that (6) is nonsplit:

Lemma 3.1. *Let D be a Brill–Noether curve of degree 5 and genus 2 and let Q be the unique quadric containing it. The inclusion $K_D \simeq N_{D/Q}(-2) \subseteq N_D(-2)$ induces an isomorphism on global sections*

$$H^0(K_D) \simeq H^0(N_D(-2)).$$

Proof. As $H^0(K_D) \hookrightarrow H^0(N_D(-2))$, it suffices to show that $h^0(N_D(-2)) = 2$. We will prove this by degenerating the curve D to the union of an elliptic normal curve E of degree 4 and genus 1 and a general 2-secant line L meeting E quasi-transversely at p and q, which is a Brill–Noether curve by Lemma 2.7(ii).

Recall that $N_L \simeq \mathcal{O}_L(1)^{\oplus 2}$. Therefore by Lemma 2.4, $N_{E\cup L}(-2)|_L \simeq \mathcal{O}_L \oplus \mathcal{O}_L$ has 2 global sections. Furthermore, since $H^0(N_{E\cup L}(-2)|_L(-p-q)) = 0$, we have that

$$H^0(N_{E\cup L}(-2)) \hookrightarrow H^0(N_{E\cup L}(-2)|_E). \tag{7}$$

The curve E is the complete intersection of 2 quadrics Q_1 and Q_2 in $\mathbb{P}^3$. Since it is one condition for a quadric containing E to contain L as well, we may assume that Q_1 contains L. Then the normal bundle restricted to E

$$N_{E\cup L}(-2)|_E \simeq N_{E/Q_1}(-2)(p+q) \oplus N_{E/Q_2}(-2) \simeq \mathcal{O}_E(p+q) \oplus \mathcal{O}_E,$$

has 3 global sections. It remains to show that one of these sections is not in the image of (7).

We claim that the unique (up to scaling) section of $\mathcal{O}_E$ is not in the image of (7). Indeed, since L is transverse to Q_2, this section fails to smooth both nodes; if it extended across L, it must extend to a section in $H^0(N_L(-2)) \subset H^0(N_{E\cup L}|_L(-2))$. But $N_L(-2) \simeq \mathcal{O}_L(-1) \oplus \mathcal{O}_L(-1)$ has no global sections, so any extension across would have to vanish identically along L, and in particular at p and q (which this section does not). □

Characteristic 2. Theorem 1 asserts that, in characteristic 2, there are infinitely many pairs $(d,g) = (2k,0)$ for which the normal bundle of a general Brill–Noether curve is unstable. This is the first case of a more general phenomena occurring only in characteristic 2.

Let $C \subset \mathbb{P}^r$ be a Brill–Noether curve. In any characteristic, the Euler sequence (1) shows that the bundle $N_C^\vee(1)$ sits in an exact sequence

$$0 \to N_C^\vee(1) \to \mathcal{O}_C^{\oplus r+1} \to \mathscr{P}^1(\mathcal{O}_C(1)) \to 0, \tag{8}$$

where $\mathscr{P}^1(\mathcal{O}_C(1))$ is the first bundle of principal parts of the line bundle $\mathcal{O}_C(1)$.

Now assume that $\mathrm{char}(k) = 2$ and let $\pi\colon C \to C^{(2)}$ denote the (relative) Frobenius morphism. Given a reduced point $c \in C$, the fiber of π containing c is the nonreduced point $2c$. Therefore

$$\mathscr{P}^1(\mathcal{O}_C(1)) \simeq \pi^*\pi_*\mathcal{O}_C(1).$$

Thus $N_C^\vee(1) \simeq \pi^* K$ is isomorphic to the pullback of a vector bundle K under Frobenius. Using this, we have the following.

Lemma 3.2. *Assume that* $\operatorname{char}(k) = 2$ *and let* $C \simeq \mathbb{P}^1$ *be a rational curve of degree* d *in* $\mathbb{P}^r$ *over* k*. Then the normal bundle splits as*

$$N_C \simeq \bigoplus_i \mathcal{O}_{\mathbb{P}^1}(a_i),$$

for integers $a_i \equiv d \pmod 2$*.*

Proof. If $\operatorname{char}(k) = 2$, then $N_C^\vee(1) \simeq \pi^* K$ for some vector bundle K on $\mathbb{P}^1$. Write $K \simeq \bigoplus \mathcal{O}_{\mathbb{P}^1}(k_i)$. Since $\pi^*\mathcal{O}_{\mathbb{P}^1}(a) \simeq \mathcal{O}_{\mathbb{P}^1}(2a)$, we have $N_C \simeq \bigoplus \mathcal{O}_{\mathbb{P}^1}(d - 2k_i)$ as desired. □

Corollary 3.3. *Let* C *be a general rational curve in* $\mathbb{P}^r$ *of degree* $d \geq r$*. Then* N_C *is semistable only if* $2d \equiv 2 \pmod{r-1}$*; in characteristic* 2*, this can be strengthened to* $d \equiv 1 \pmod{r-1}$*.*

Proof. In any characteristic, N_C can only be semistable if $\mu(N_C) = d + \frac{2d-2}{r-1}$ is an integer. In characteristic 2, Lemma 3.2 implies that furthermore $\mu(N_C) - d$ must be an *even* integer. □

Remark 1. When $r = 3$, we prove in Section 6 that Corollary 3.3 gives the only obstruction to semistability for the normal bundle of a rational curve in characteristic 2. With a little more work, one can show the same in any projective space.

4. Stability and degeneration I

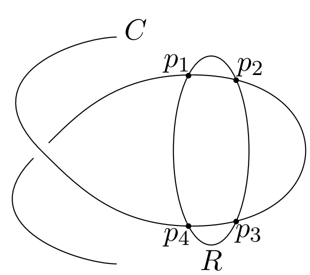
In this section, by specializing to the union of a general Brill–Noether curve and a 4-secant conic, we reduce Theorem 1 to the cases $g \leq 8$. Our main tool will be the following first basic lemma proving stability by degeneration.

Lemma 4.1. *Suppose that* $C = X \cup Y$ *is a reducible curve and* E *is a vector bundle on* C *such that* $E|_X$ *and* $E|_Y$ *are semistable. Then* E *is semistable. Furthermore, if one of* $E|_X$ *or* $E|_Y$ *is stable, then* E *is stable.*

Proof. Write $\nu \colon X \sqcup Y \to C$ for the normalization map. For any subbundle $F \subseteq \nu^* E$ we have

$$\mu^{\text{adj}}(F) \leq \mu(F|_X) + \mu(F|_Y) \underset{(-)}{<} \mu(E|_X) + \mu(E|_Y) = \mu(E). \qquad \square$$

4-secant conic degenerations. Let C be a Brill–Noether curve of degree $d \geq 4$ and genus g in $\mathbb{P}^3$. Let $H \subset \mathbb{P}^3$ be a 2-plane meeting C transversely; let $p_1, \ldots, p_4$ be four points in $C \cap H$. For $R \subset H$ a conic through $p_1, \ldots, p_4$, the union $C \cup R$ is a Brill–Noether curve of degree $d+2$ and genus $g+3$ by Lemma 2.7(iii).



Lemma 4.2. *In the above setup, if* C *is a general Brill–Noether curve with* $(d, g) \neq (3, 0)$ *or* $(4, 1)$*, then*

$$N_{C\cup R}|_R \simeq \mathcal{O}_{\mathbb{P}^1}(5) \oplus \mathcal{O}_{\mathbb{P}^1}(5)$$

is semistable.

Proof. We will prove this lemma by degeneration of C. If C admits a degeneration to $X \cup Y$, where $\deg X \geq 4$, then we may consider degenerations $X \cup Y \cup R$ of $C \cup R$ where the conic R meets X alone; this reduces the case of C to the case of X.

By repeatedly applying Lemma 2.7 to pull off 1-secant lines, 2-secant lines, or 4-secant conics, we thus reduce to the case where (d, g) satisfies

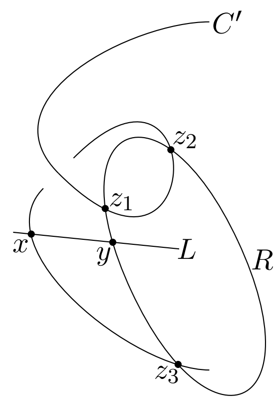
$$\rho(g, 3, d) \geq 0, \quad g \geq 0, \quad \text{and} \quad (d, g) \neq (3, 0), (4, 1), \tag{9}$$

but (d', g') fails to satisfy (9) for each of $(d', g') = (d-1, g), (d-1, g-1)$, and $(d-2, g-3)$.

By inspection, this is only possible if $(d, g) = (4, 0), (5, 2)$, or $(6, 4)$. (Indeed, if $g \geq 5$, then $(d', g') = (d-2, g-3)$ satisfies (9); if $g \leq 4$ and $d \geq 7$, then $(d', g') = (d-1, g)$ satisfies (9); the finitely many cases with $g \leq 4$ and $d \leq 6$ are easily verified.)

In these cases, C is of type $(3, d-3)$ on a quadric. Specializing C to the union of a curve of type $(3, 1)$ with $d-4$ lines of type $(0, 1)$, it thus remains only to consider the case $(d, g) = (4, 0)$.

When C is a rational quartic curve, we specialize C to $C' \cup L$ where C' is a rational normal curve and L is a 1-secant line meeting C' at a point x. Since C has degree 4, we must specialize R to meet L in one point y and C' in a set $\{z_1, z_2, z_3\}$ of three points:



Since $N_{C'} \simeq \mathcal{O}_{\mathbb{P}^1}(5) \oplus \mathcal{O}_{\mathbb{P}^1}(5)$, we may arrange for C' to have general tangent directions at the points z_i. Thus, $N_{C' \cup R}|_R \simeq \mathcal{O}_{\mathbb{P}^1}(5) \oplus \mathcal{O}_{\mathbb{P}^1}(4)$. In particular, we have a distinguished subspace of $N_R|_y$ given by the positive subbundle $\mathcal{O}_{\mathbb{P}^1}(5)|_y \subset N_{C' \cup R}|_y \simeq N_R|_y$ — or equivalently, a distinguished plane $\Lambda \supset T_y R$. Since $x \in C'$ is general, we have $x \notin \Lambda$. Thus

$$N_{C' \cup L \cup R}|_R \simeq N_{C' \cup R}|_R(y)[y \to x] \simeq \mathcal{O}_{\mathbb{P}^1}(5) \oplus \mathcal{O}_{\mathbb{P}^1}(5). \qquad \square$$

Remark 2. For $(d, g) = (4, 1)$, the conclusion of Lemma 4.2 is false: For any R, the curve C lies on a quadric Q containing R, and $N_{(C \cup R)/Q}|_R$ is destabilizing.

Let p_i' be a point on $T_{p_i} R \smallsetminus p_i$. Then by Lemma 4.2 combined with Lemma 4.1, stability for

$$N_C[p_1 \to p_1'][p_2 \to p_2'][p_3 \to p_3'][p_4 \to p_4']$$

implies stability for $N_{C \cup R}$, and hence for the normal bundle of a general Brill–Noether space curve of degree $d+2$ and genus $g+3$.

Deformations of r-secant rational curves. To use the above degeneration to reduce to a finite list of genera, we must know that such a conic can be suitably deformed while preserving the incidence conditions with D.

In greater generality, let D be a Brill–Noether curve, and R be a rational curve meeting D at distinct points $p_1, p_2, \ldots, p_r$. The following key assumption generalizes the conclusion of Lemma 4.2:

Assumption 4.3. The restricted normal bundle $N_{D\cup R}|_R$ is perfectly balanced with slope

$$\mu(N_{D\cup R}|_R) \geq r + 1.$$

Lemma 4.4. *Under assumption 4.3, there exists a deformation $R(t)$ of R, and $p_i(t)$ of p_i, such that the rational curve $R(t)$ meets D quasi-transversely in $p_1(t), p_2(t), \ldots, p_r(t)$, and $p_i(t)$ has nonzero derivative at $t = 0$ for all i.*

Proof. For any i, let N_i denote the vector bundle on R obtained by gluing the vector bundles $N_{R\cup D}|_{R\smallsetminus p_i}$ and $N_R|_{R\smallsetminus\{p_1,\ldots,\hat{p_i},\ldots,p_r\}}$ along the natural isomorphism $N_{R\cup D}|_{R\smallsetminus\{p_1,\ldots,p_r\}} \simeq N_R|_{R\smallsetminus\{p_1,\ldots,p_r\}}$. Then obstructions to lifting deformations of p_i to deformations of R that preserve the incidence conditions with D at the p_j lie in $H^1(N_i(-p_1 - \cdots - p_r))$; it thus suffices to show

$$H^1(N_i(-p_1 - \cdots - p_r)) = 0.$$

But $N_i(-p_1 - \cdots - p_r)$ fits in an exact sequence

$$0 \to N_{R\cup D}|_R(-p_1 - \cdots - p_{i-1} - 2p_i - p_{i+1} - \cdots - p_r) \to N_i(-p_1 - \cdots - p_r) \to \mathcal{O}_{p_i} \to 0,$$

and so it remains to note that our assumption that $N_{D\cup R}|_R$ is perfectly balanced with slope $\mu(N_{D\cup R}|_R) \geq r + 1$ implies

$$H^1(N_{R\cup D}|_R(-p_1 - \cdots - p_{i-1} - 2p_i - p_{i+1} - \cdots - p_r)) = 0. \qquad \square$$

Reduction to a finite list of genera.

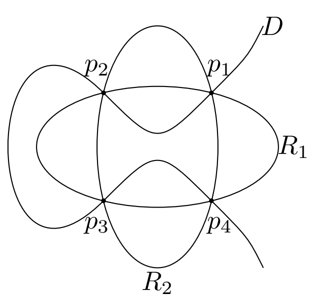
Lemma 4.5. *Suppose that Theorem 1 is true for all $g \leq 8$. Then it is true for all g.*

Proof. If $\rho(g, 3, d) \geq 0$ and $g \geq 9$, then

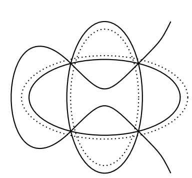
$$\rho(g-6, 3, d-4) = \rho(g,3,d) + 2 \geq 0 \quad \text{and} \quad g - 6 \geq 2 \quad \text{and} \quad (d-4, g-6) \notin \{(5,2),(6,4)\}, \tag{10}$$

$$\text{and} \qquad d - 4 \geq 4. \tag{11}$$

By (10), a general Brill–Noether curve D of degree $d-4$ and genus $g-6$ has N_D stable by induction. Let H be a general hyperplane; by (11), we may let $R_1 \subseteq H$ and $R_2 \subseteq H$ be general 4-secant conics, both of which meet D at $p_1, \ldots, p_4$:



By Lemma 4.4, we may deform R_i to 4-secant conics $R_i(t)$ meeting D at $p_{i1}(t)$, $p_{i2}(t)$, $p_{i3}(t)$, and $p_{i4}(t)$, such that $p_{1j}(t)$ and $p_{2j}(t)$ have distinct derivatives:



Combining lemmas 4.1 and 4.2, it remains to show the stability of $N_C[p_{ij}(t) \to p'_{ij}(t)]$ for $t \in \Delta$ general, where $p'_{ij}(t)$ denotes a point on $T_{p_{ij}(t)}C \smallsetminus p_{ij}(t)$. By the discussion in Remark 3.4 of [ALY19], these vector bundles fit together to form a vector bundle over $D \times \Delta$ whose fiber over $0 \in \Delta$ is the bundle $N_D(-p_1 - p_2 - p_3 - p_4)$ — which is stable since we have already seen that N_D is stable by induction. □

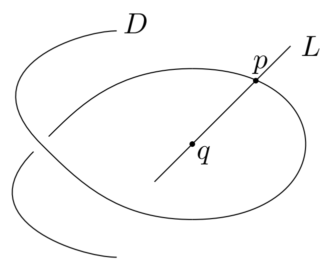
5. Stability and Degeneration II: Gluing Data

In order to settle the base cases $g \leq 8$, we will need to use degenerations of C to reducible curves $X \cup Y$ where neither $N_{X\cup Y}|_X$ nor $N_{X\cup Y}|_Y$ are necessarily stable. The basic idea is to compare destabilizing subbundles of $N_{X\cup Y}|_X$ and $N_{X\cup Y}|_Y$, and show that they cannot agree sufficiently over $X \cap Y$.

1-secant degenerations. In some cases, we can construct a *modification* of the restriction $N_{X\cup Y}|_X$ whose stability rules out a destabilizing subbundle of $N_{X\cup Y}|_X$ that could agree sufficiently with a destabilizing subbundle of $N_{X\cup Y}|_Y$. This technique works well when we can understand the geometry of Y explicitly. Here we apply this technique when $Y = L$ is a 1-secant line.

Let D be a smooth Brill–Noether curve and L a quasi-transverse 1-secant line meeting D at p. Although $N_{D\cup L}|_L$ is not semistable, so we cannot apply Lemma 4.1, we can identify the unique destabilizing subbundle of $N_{D\cup L}|_L$, and construct a modification of $N_{D\cup L}|_D$ as described above.

For inductive arguments it will be more useful to consider a slightly more general setup: Let $N'_{D\cup L}$ be any vector bundle equipped with an isomorphism with $N_{D\cup L}$ over an open set U of $D \cup L$ containing L, and write N'_D for the bundle obtained by gluing $N_D|_U$ to $N'_{D\cup L}|_{D\smallsetminus p}$ along the isomorphism $N_D|_{U\smallsetminus p} \simeq N_{D\cup L}|_{U\smallsetminus p} \simeq N'_{D\cup L}|_{U\smallsetminus p}$. To state the lemma, let $q \in L \smallsetminus p$.



Lemma 5.1. *In the above setup, if $N'_D[p \to q][p \to q] \simeq N'_D[2p \to q]$ is (semi)stable, then $N'_{D\cup L}$ is also (semi)stable.*

Proof. Write $\nu\colon D \sqcup L \to D \cup L$ for the normalization map, and $\tilde{p}_1$ and $\tilde{p}_2$ for the points above p on L and D respectively. Suppose that $F \subseteq \nu^* N'_{D\cup L}$ is a line subbundle.

First, we consider the restriction of F to L. Let x be a point on T_pD and let Λ be the plane spanned by x and L. Let H be another plane such that $L = \Lambda \cap H$. Then by Lemma 2.4,

$$N'_{D\cup L}|_L \simeq N_L(p)[p \to x] \simeq N_{L/H} \oplus N_{L/\Lambda}(p) \simeq \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(2).$$

Consequently,

$$\mu(F|_L) \leq \begin{cases} 2 & \text{if } F|_{\tilde{p}_1} = N_{L/\Lambda}(p)|_{\tilde{p}_1}; \\ 1 & \text{otherwise.} \end{cases} \tag{12}$$

Second, we consider the restriction of F to D. If $F|_{\tilde{p}_2} = N_{D\to q}(p)|_{\tilde{p}_2}$, then $F|_D$ is a subbundle of $N'_{D\cup L}|_D[p \to q] \simeq N'_D(p)[2p \to q]$; otherwise $F|_D(-\tilde{p}_2)$ is a subbundle of $N'_D(p)[2p \to q]$. Because $N'_D[2p \to q]$ is (semi)stable by assumption and of slope $\mu(N'_D) - 1$, it follows that $N'_D(p)[2p \to q]$ is (semi)stable of slope $\mu(N'_D)$. Consequently,

$$\mu(F|_D) \underset{(-)}{<} \begin{cases} \mu(N'_D) + 1 & \text{if } F|_{\tilde{p}_2} \neq N_{D\to q}(p)|_{\tilde{p}_2}; \\ \mu(N'_D) & \text{otherwise.} \end{cases} \tag{13}$$

Finally, by [ALY19, Lemma 8.5], the subspace $N_{L/\Lambda}(p)|_{\tilde{p}_1}$ glues to the subspace $N_{D\to q}(p)|_{\tilde{p}_2}$. Consequently,

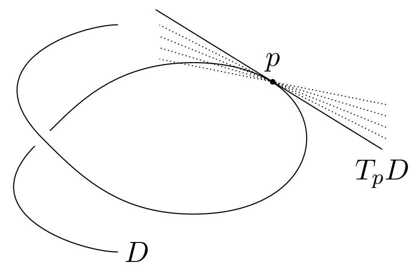
$$\operatorname{codim}_F(F|_{\tilde{p}_1} \cap F|_{\tilde{p}_2}) \geq \begin{cases} 1 & \text{if } F|_{\tilde{p}_1} = N_{L/\Lambda}(p)|_{\tilde{p}_1} \text{ and } F|_{\tilde{p}_2} \neq N_{D\to q}(p)|_{\tilde{p}_2}; \\ 0 & \text{otherwise.} \end{cases} \tag{14}$$

To finish the proof, we simply combine (12), (13), and (14), to obtain

$$\mu^{\text{adj}}(F) = \mu(F|_L) + \mu(F|_D) - \operatorname{codim}_F(F|_{\tilde{p}_1} \cap F|_{\tilde{p}_2}) \underset{(-)}{<} \mu(N'_D) + 2 = \mu(N'_{D\cup L}). \qquad \square$$

Lemma 5.2. *Assume that the characteristic of the ground field is not 2. Suppose that N_D is (semi)stable. If $q \in \mathbb{P}^3$ is a general point and $p \in D$ has ordinary ramification, then the elementary modification $N_D[2p \to q]$ is (semi)stable.*

Proof. Let $\Lambda \subset \mathbb{P}^3$ be a 2-plane containing T_pD that is not the osculating 2-plane to D at p. For parameter $s \in \mathbb{P}^1$, let L_s be the pencil of lines through p in Λ specializing to T_pD when $s = 0$ and let $q(s)$ be a choice of point on $L_s \smallsetminus p$.



As (semi)stability is open, and $N_D(-p)$ is (semi)stable by assumption, it suffices to show that the modifications $N_D[2p \to q(s)]$ for $s \neq 0$ fit together into a flat family specializing to $N_D(-p)$ when $L_s = T_pD$. To do this, we first observe that, for $s \neq 0$,

$$N_D[2p \to q(s)] \coloneqq \ker\left(N_D \to \frac{N_D|_{2p}}{N_{D\to q(s)}|_{2p}}\right)$$

is determined by the 2-dimensional subspace $N_{D\to q(s)}|_{2p}$ of the 4-dimensional space $N_D|_{2p}$. As the Grassmanian $\mathrm{Gr}(2,4)$ is separated and proper, there is a unique limit of these spaces as $s \to 0$. It suffices to prove, by a calculation in local coordinates, that this subspace is $N_D(-p)|_{2p} \subseteq N_D|_{2p}$.

Choose an affine neighborhood $\mathbb{A}^3_{xyz} \subseteq \mathbb{P}^3$ and coordinates such that $p = (0,0,0)$, the tangent line T_pD is $y = z = 0$, the osculating two-plane is $z = 0$, and Λ is $y = 0$. Let $q(s) = (1,0,s)$ so that L_s is the line through $(1,0,s)$ and $(0,0,0)$.

Let t be an étale local coordinate at p for D. Then in an étale neighborhood of p, the curve D is given parametrically by

$$D(t) = \begin{pmatrix} t \\ t^2 + a_3t^3 + \cdots \\ b_3t^3 + \cdots \end{pmatrix}.$$

We trivialize N_D in a neighborhood of p by $\partial/\partial y$ and $\partial/\partial z$. A section of N_D is then given by

$$(m_0 + m_1t + m_2t^2 + \cdots)\frac{\partial}{\partial y} + (n_0 + n_1t + n_2t^2 + \cdots)\frac{\partial}{\partial z}. \tag{15}$$

We must determine the conditions on the m_i and n_i such that this section points towards $q(s)$ to second order in t. The vector from $D(t)$ on D to $q(s)$

$$D(t) - q(s) = \begin{pmatrix} t - 1 \\ t^2 + a_3t^3 + \cdots \\ b_3t^3 + \cdots - s \end{pmatrix}$$

is equivalent as a section of N_D to its translate by a tangent vector

$$D(t) - q(s) - (t-1)D'(t) = \begin{pmatrix} t-1 \\ t^2 + a_3t^3 + \cdots \\ b_3t^3 + \cdots - s \end{pmatrix} - \begin{pmatrix} t-1 \\ (t-1)(2t + 3a_3t^2 + \cdots) \\ (t-1)(3b_3t^2 + \cdots) \end{pmatrix} = \begin{pmatrix} 0 \\ 2t + (3a_3 - 1)t^2 + \cdots \\ -s - 3b_3t^2 + \cdots \end{pmatrix}.$$

This normal vector now corresponds to the section

$$(2t + (3a_3 - 1)t^2 + \cdots)\frac{\partial}{\partial y} + (-s - 3b_3t^2 + \cdots)\frac{\partial}{\partial z}$$

under our choosen trivialization. The condition on the m_i and n_i for a section as in (15) to point towards $q(s)$ at $2p$ is that

$$\det\begin{pmatrix} 2t + \cdots & m_0 + m_1t + \cdots \\ -s + \cdots & n_0 + n_1t + \cdots \end{pmatrix} = -sm_0 + (2n_0 + sm_1)t + \cdots$$

vanish to second order in t. When $s \neq 0$, this cuts out the 2-dimensional subspace $m_0 = 2n_0 + sm_1 = 0$ in the four dimensional vector space with coordinates m_0, m_1, n_0, n_1.

In characteristic distinct from 2, the limit as $s \to 0$ of this subspace is simply $m_0 = n_0 = 0$, i.e. the subspace $N_D(-p)|_{2p} \subset N_D|_{2p}$ as claimed. □

Corollary 5.3. *Suppose that N_D is (semi)stable for D a general Brill–Noether curve of degree d and genus g in $\mathbb{P}^3$. Then N_C is (semi)stable for C a general Brill–Noether curve of degree $d + \epsilon$ and genus g in $\mathbb{P}^3$, where*

$$\epsilon = \begin{cases} 1 & \text{if } \operatorname{char}(k) \neq 2; \\ 2 & \text{if } \operatorname{char}(k) = 2. \end{cases}$$

Proof. We specialize C to the union of a general Brill–Noether curve D with ϵ one-secant lines. Applying Lemma 5.1, it suffices to show that $N_D[2p \to q]$ (respectively $N_D[2p_1 \to q_1][2p_2 \to q_2]$) is (semi)stable, where the p_i denote general points on D, and the q_i denote general points in $\mathbb{P}^3$.

As we limit p_1 and p_2 together to a common point p, the vector bundles $N_D[2p_1 \to q_1][2p_2 \to q_2]$ fit together to form a vector bundle with central fiber $N_D(-2p)$ (c.f. the discussion in Remark 3.4 of [ALY19]) — which is (semi)stable by assumption.

In characteristic distinct from 2, we apply Lemma 5.2 to conclude that $N_D[2p \to q]$ is (semi)stable as desired. □

6. Reduction to a finite list of (d, g)

In this section we combine the results of the previous section to reduce the proof of Theorem 1 to a finite list of base cases.

Proposition 6.1. *Suppose that Theorem 1 holds for curves of degree d and genus g with*

$$(16)\quad (d,g) \in \{(3,0),(4,1),(5,1),(6,2),(7,2),(6,3),(7,3),\\ (7,4),(8,4),(7,5),(8,5),(8,6),(9,6),(9,7),(10,7),(9,8),(10,8)\}.$$

Then Theorem 1 holds in all cases. If the characteristic of the ground field is not 2*, then it suffices to replace list* (16) *with*

$$(17)\quad (d,g) \in \{(3,0),(4,1),(6,2),(6,3),(7,4),(7,5),(8,6),(9,7),(9,8)\}.$$

Proof. We will prove this by induction on d and g. By Lemma 4.5, it suffices to prove this when $g \leq 8$. If the characteristic is not equal to 2, then by Corollary 5.3, it suffices to check (semi)stability for the smallest degree in each genus for which Theorem 1 asserts that the normal bundle is (semi)stable. Similarly, if the characteristic is 2, it suffices to check (semi)stability for the two smallest degrees.

Note that, for rational curves of even degrees in characteristic 2, we have already established that the normal bundles are unstable. Thus we do not need to include $(4, 0)$ in our list (16). □

Remark 3. By Lemma 2.6, we already know semistability for $(d,g) = (3,0)$ and $(4,1)$. This establishes Theorem 1 for curves of genus 0 in any characteristic, and for curves of genus 1 in characteristic distinct from 2.

Remark 4. The reason that the cases $(6,2)$ and $(7,4)$ appeared in our list (16) of remaining cases is that the cases $(5,2)$ and $(6,4)$ were exceptions to Theorem 1, and so our induction on the degree broke down. In fact, one cannot degenerate such curves to the union of a Brill–Noether curve D of degree $d-1$ and genus g with a 1-secant line and apply Lemma 5.1 (even without applying Lemma 5.2); in both cases, $N_D[2p \to q]$ is unstable (if Q denotes the unique quadric containing D then $N_{D/Q}(-2p) \subset N_D[2p \to q]$ is destabilizing).

7. Base Cases: Applications of Gluing Data

In this section, we establish those base cases appearing in Proposition 6.1 which can be studied using the techniques of Section 5.

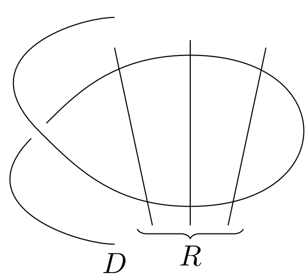
The case $(d,g) = (5,1)$. We degenerate to the union of an elliptic normal curve C with a 1-secant line. By Lemma 5.1, it suffices to show $N_C[2u \to v]$ is semistable, where $u \in C$ and $v \in \mathbb{P}^3$ are general. Fix a quadric Q containing C, and specialize v to a general point on C. By Lemma 2.5, there are exactly two points on C at which the fibers of $N_{C\to v}$ and $N_{C/Q}$ coincide; specialize u to one of them. Then $N_C[2u \to v]$ fits in an exact sequence

$$0 \to [N_{C/Q}(-u) \simeq \mathcal{O}_C(2)(-u)] \to N_C[2u \to v] \to [N_Q|_C(-u) \simeq \mathcal{O}_C(2)(-u)] \to 0,$$

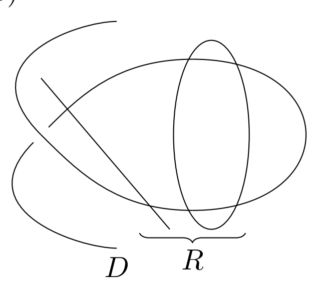
so is semistable as desired.

The cases $(d,g) = (9,7), (10,7), (9,8)$, and $(10,8)$. When $(d,g) = (10,7)$ respectively $(10,8)$, we first degenerate the curve to the union of a general Brill–Noether curve C of degree 9 and genus 7 respectively 8, and general 1-secant line M, meeting C at u. Choose a point $v \in M \smallsetminus u$ so $M = \overline{uv}$. By Lemma 5.1, it suffices to show that $N_C(u)[2u \to v]$ is stable.

Therefore, in order to deal with all of our cases $(d,g) \in \{(9,7),(10,7),(9,8),(10,8)\}$, we begin with a curve C of degree 9 and genus 7 or 8. We will degenerate C to the union of a general canonical curve D (of degree 6 and genus 4) and a union R of rational curves meeting D quasi-transversely at a set Γ of 6 points (three general 2-secant lines when $g = 7$, respectively a general 2-secant line and a general 4-secant conic when $g = 8$).



$(d,g) = (9,7)$



$(d,g) = (9,8)$

Write Q for the unique quadric containing D. In both cases, the tangent lines to R at Γ are transverse to Q, and so the restricted normal bundle $N_{D\cup R}|_D$ fits into a balanced exact sequence:

$$0 \to [N_{D/Q} \simeq \mathcal{O}_D(3)] \to N_{D\cup R}|_D \to [N_Q|_D(\Gamma) \simeq \mathcal{O}_D(2)(\Gamma)] \to 0. \tag{18}$$

In particular, $N_{D\cup R}|_D$ is strictly semistable, and $N_{D/Q}$ gives a destabilizing line bundle.

Similarly, after specializing v to a point on D, Lemma 2.5 asserts that there are 4 points u on D where the fibers $N_{D\to v}|_u$ and $N_{D/Q}|_u$ coincide to first order. Specializing u to one of these points, we again have a balanced exact sequence

$$0 \to N_{D/Q} \to N_{D\cup R}|_D(u)[2u \to v] \to N_Q|_D(\Gamma) \to 0. \tag{19}$$

In particular, $N_{D\cup R}|_D(u)[2u \to v]$ is strictly semistable, and $N_{D/Q}$ gives a destabilizing line bundle.

Let L be a line component of R, meeting D at p_1 and p_2 with $p_i' \in T_{p_i}D \smallsetminus p_i$, and denote by Λ_i the plane spanned by p_i' and L. Then

$$N_{D\cup R}|_L \simeq N_{L/\Lambda_1}(p_1) \oplus N_{L/\Lambda_2}(p_2) \simeq \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(2).$$

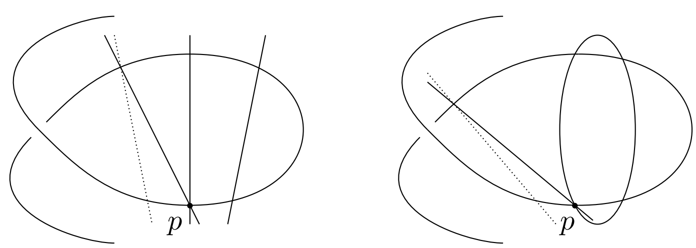
Combining this with Lemma 4.2, the restriction of $N_{D\cup R}$ (resp. $N_{D\cup R}(u)[2u \to v]$) to each of the components of R is also strictly semistable.

In particular, writing $\nu\colon D \sqcup R \to D \cup R$ for the normalization, any destabilizing subbundle $F \subset \nu^* N_{D\cup R}$ (resp. $F \subset \nu^* N_{D\cup R}(u)[2u \to v]$) must be destabilizing on every component and agree at the points lying over the nodes $D \cap R$. The key observation is that, because $N_{D/Q}$ is a subbundle of N_D as well, its fiber at each of the points of Γ is exactly the subspace that does not smooth that node. On the other hand, if L denotes a component of R which is a line, then any destabilizing $\mathcal{O}(2)$ has a fiber at one or more of the nodes that fails to smooth it (otherwise it would be a subbundle of $N_L \simeq \mathcal{O}(1) \oplus \mathcal{O}(1)$). It thus remains to show that $N_{D/Q}$ is the *unique* destabilizing subbundle of $N_{D\cup R}|_D$ (resp. $N_{D\cup R}|_D(u)[2u \to v]$), or equivalently:

Lemma 7.1. *The sequences* (18) *and* (19) *are nonsplit, i.e.*

$$H^0(N_{D\cup R}|_D(-2)(-\Gamma)) = 0 \qquad \textit{and} \qquad H^0(N_{D\cup R}|_D(-2)(-\Gamma)(u)[2u \to v]) = 0.$$

Proof. To show the desired vanishing, we degenerate two points of Γ together to a common point p on D:



Let N denote the bundle obtained by gluing $N_{D\cup R}|_{D\smallsetminus p}$ to $N_D(p)|_{D\smallsetminus(\Gamma\smallsetminus p)}$ along the natural isomorphism $N_{D\cup R}|_{D\smallsetminus\Gamma} \simeq N_D(p)|_{D\smallsetminus\Gamma}$. By the discussion in Remark 3.4 of [ALY19], the bundles $N_{D\cup R}|_D$ (resp. $N_{D\cup R}|_D(u)[2u \to v]$) fit together to form a bundle whose central fiber is the bundle N (resp. $N(u)[2u \to v]$). It thus remains to show

$$H^0(N(-2)(-\Gamma)) = 0 \qquad \text{and} \qquad H^0(N(u)[2u \to v](-2)(-\Gamma)) = 0.$$

To do this, we use the exact sequence

$$0 \to [N_{D/Q}(p) \simeq \mathcal{O}_D(3)(p)] \to [N \text{ or } N(u)[2u \to v]] \to [N_Q|_D(\Gamma - p) \simeq \mathcal{O}_D(2)(\Gamma - p)] \to 0;$$

twisting this sequence by $\mathcal{O}_D(-2)(-\Gamma)$ and taking global sections, it remains to check that

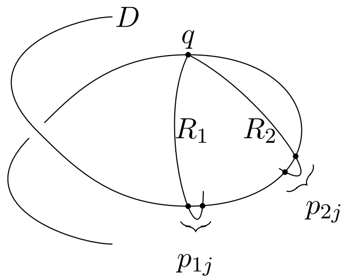
$$H^0(\mathcal{O}_D(1)(-(\Gamma - p))) = H^0(\mathcal{O}_D(-p)) = 0.$$

But this is clear since the five points of $\Gamma - p = \Gamma^{\text{red}}$ are in linear general position. □

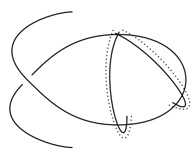
8. Stability and degeneration III: Limits of Gluing Data

As in the previous section, we again want to degenerate to reducible curves $X \cup Y$ where neither $N_{X\cup Y}|_X$ nor $N_{X\cup Y}|_Y$ are necessarily stable, but the destabilizing subbundles on each component do not agree at $X \cap Y$. The fundamental difficulty we address in this section is that it is often difficult to compute the destabilizing subbundles on each component without further degeneration. We therefore study the agreement conditions at $X \cap Y$ as the points of $X \cap Y$ come together.

Let D be a Brill–Noether curve. Fix distinct points $q, p_{11}, \ldots, p_{1r_1}, p_{21}, \ldots, p_{2r_2} \in D$. Let R_i be a rational curve meeting D quasi-transversely exactly at $q, p_{i1}, \ldots, p_{ir_i}$, such that the tangent directions at q to D, R_1, and R_2 span $\mathbb{P}^3$.



Assume that both R_i satisfy Assumption 4.3. Using this assumption we may apply Lemma 4.4 to show that there exists an étale neighborhood $\Delta = q_i(t)$ of $q \in D$, which we normalize so $q_1(t)$ and $q_2(t)$ have distinct derivatives at $t = 0$, and deformations $R_i(t)$ of R_i, and $p_{ij}(t)$ of p_{ij}, such that for $t \in \Delta$, the rational curve $R_i(t)$ meets D quasi-transversely in $q_i(t), p_{i1}(t), \ldots p_{ir_i}(t)$.



Suppose that, for $t \in \Delta^* := \Delta \smallsetminus 0$, the normal bundle $N_{D\cup R_1(t)\cup R_2(t)}$ is not stable. These bundles fit together to form a vector bundle $\hat{\mathcal{N}}$ over Δ^*. However, since $D \cup R_1 \cup R_2$ is not lci, its normal sheaf is not a vector bundle; there is therefore no obvious way to extend $\hat{\mathcal{N}}$ over Δ. Thus, extracting information at the central fiber is subtle.

By the discussion in Remark 3.4 of [ALY19], we may nevertheless extend the *restriction* $\hat{\mathcal{N}}|_D$ to a bundle $\mathcal{N}$ on $D \times \Delta$ whose fiber $N := \mathcal{N}|_0$ over $0 \in \Delta$ is obtained from gluing $N_{D\cup R_1\cup R_2}|_{D\smallsetminus q}$ to $N_D(q)|_{D\smallsetminus\{p_{11},\ldots,p_{1r_1},p_{21},\ldots,p_{2r_2}\}}$ along the natural isomorphism

$$N_{D\cup R_1\cup R_2}|_{D\smallsetminus\{q,p_{11},\ldots,p_{1r_1},p_{21},\ldots,p_{2r_2}\}} \simeq N_D|_{D\smallsetminus\{q,p_{11},\ldots,p_{1r_1},p_{21},\ldots,p_{2r_2}\}} \simeq N_D(q)|_{D\smallsetminus\{q,p_{11},\ldots,p_{1r_1},p_{21},\ldots,p_{2r_2}\}}.$$

Write $\nu\colon D \sqcup R_1(t) \sqcup R_2(t) \to D \cup R_1(t) \cup R_2(t)$ for the normalization map. Let $\hat{\mathcal{L}} \subset \nu^*\hat{\mathcal{N}}$ be a destabilizing line bundle, i.e. which satisfies $\mu^{\text{adj}}(\hat{\mathcal{L}}) \geq \mu(\hat{\mathcal{N}})$. Let ℓ_D, ℓ_1, and ℓ_2 denote the slopes of the restriction of $\hat{\mathcal{L}}$ to D, $R_1(t)$, and $R_2(t)$, and c denote the number of nodes of $D \cup R_1(t) \cup R_2(t)$ above which the fibers of $\hat{\mathcal{L}}$ do not coincide (for $t \in \Delta^*$). Since being perfectly balanced is open, Condition 4.3 implies that the $\hat{\mathcal{N}}|_{R_i(t)}$ are perfectly balanced. We therefore have

$$\ell_i \leq \mu(\hat{\mathcal{N}}|_{R_i(t)}) \quad \text{and} \quad c \geq 0, \tag{20}$$

but

$$\mu^{\text{adj}}(\hat{\mathcal{L}}) = \ell_1 + \ell_2 + \ell_D - c \geq \mu(\hat{\mathcal{N}}|_{R_1(t)}) + \mu(\hat{\mathcal{N}}|_{R_2(t)}) + \mu(\hat{\mathcal{N}}|_D).$$

If $\ell_D > \mu(\hat{\mathcal{N}}|_D)$, i.e. $\mathcal{N}^* = \hat{\mathcal{N}}|_D$ is unstable, then N is unstable by Proposition 2.3. Thus either:

(i) N is unstable, or

(ii) (20) is an equality — i.e. $\ell_i = \mu(\hat{\mathcal{N}}|_{R_i(t)})$ and $c = 0$ — and $\ell_D = \mu(N)$.

In case (ii), our first task is to translate the condition that (20) is an equality to information about the restriction $\mathcal{L}^* = \hat{\mathcal{L}}|_D$. (The condition that $\ell_D = \mu(N)$ already concerns $\mathcal{L}^*$.) To do this, observe that since the $\hat{\mathcal{N}}|_{R_i(t)}$ are perfectly balanced, we have a canonical isomorphism

$$\varphi_{ij}^*\colon \mathbb{P}\mathcal{N}^*|_{q_i(t)} \xrightarrow{\sim} \mathbb{P}\mathcal{N}^*|_{p_{ij}(t)} \quad \text{for } t \in \Delta^*.$$

Writing $\mathcal{L}^* = \hat{\mathcal{L}}|_D$, the condition that (20) is an equality then implies that

$$\mathcal{L}^*|_{p_{ij}(t)} = \varphi_{ij}^*(\mathcal{L}^*|_{q_i(t)}) \quad \text{for } t \in \Delta^*. \tag{21}$$

By Proposition 2.3, we can extend $\mathcal{L}^*$ across the central fiber to a subbundle $\mathcal{L} \subset \mathcal{N}$, and consider the restriction $L := \mathcal{L}|_0 \subset N$ to the central fiber. Our second task is to figure out what (21) implies for L. (Figuring out what $\ell_D = \mu(N)$ implies for L is easy: Since μ is constant in flat families, it implies $\mu(L) = \mu(N)$.)

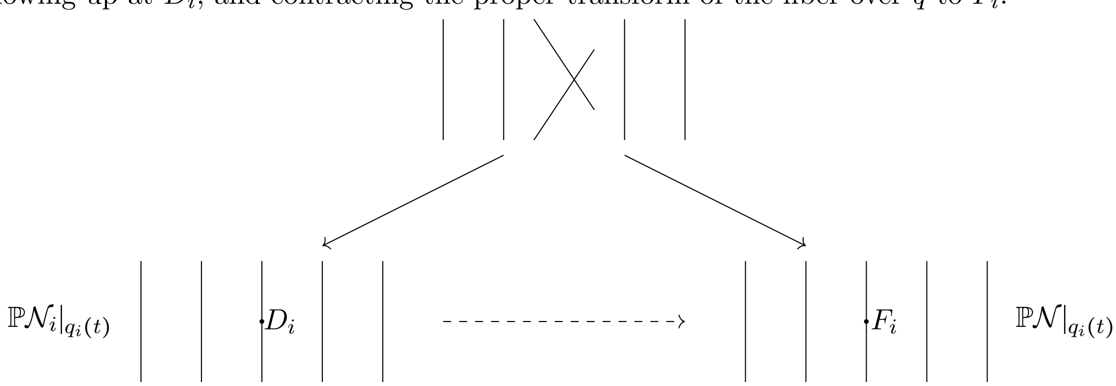
To do this, we observe that the bundles $N_{D \cup R_i(t)}$ fit together to form bundles $\hat{\mathcal{N}}_i$ over Δ (including over $t = 0$). Writing $\mathcal{N}_i = \hat{\mathcal{N}}_i|_D$, there are natural inclusions $\mathcal{N}_i \subset \mathcal{N}$, which are isomorphisms away from $R_{\bar{i}}(t) \cap D$ (here $\bar{i} = 3 - i$ denotes the other index) — so in particular at $q_i(t)$ for $t \neq 0$, and at $p_{ij}(t)$ for all t. This inclusion induces a birational isomorphism on projectivizations $\mathbb{P}\mathcal{N}_i \dashrightarrow \mathbb{P}\mathcal{N}$. The advantage to working with $\mathcal{N}_i$ is that $\hat{\mathcal{N}}_i|_{R_i(t)}$ is perfectly balanced, so we obtain regular maps *defined over Δ (in particular for $t = 0$)*:

$$\varphi_{ij}\colon \mathbb{P}\mathcal{N}_i|_{q_i(t)} \xrightarrow{\sim} \mathbb{P}\mathcal{N}_i|_{p_{ij}(t)} \quad \text{for } t \in \Delta,$$

that are compatible with the φ_{ij}^* in the sense that the following diagram commutes:

$$\begin{array}{ccc} \mathbb{P}\mathcal{N}_i|_{q_i(t)} & \xrightarrow{\varphi_{ij}} & \mathbb{P}\mathcal{N}_i|_{p_{ij}(t)} \\ \downarrow & & \| \\ \mathbb{P}\mathcal{N}|_{q_i(t)} & \overset{\varphi_{ij}^*}{\dashrightarrow} & \mathbb{P}\mathcal{N}|_{p_{ij}(t)} \end{array}$$

We now restrict to the graph of $q_i(t)$. Then the map $\mathcal{N}_i \subset \mathcal{N}$ drops rank exactly over $t = 0$. Its kernel at $t = 0$ is the one-dimensional subspace $D_i \subset N_{D \cup R_i}|_q$ corresponding to sections that fail to smooth the node at q, and its image is given by the one-dimensional subspace of $F_i \subset N|_q$ corresponding to the tangent direction of R_i at q. The rational map $\mathbb{P}\mathcal{N}_i \dashrightarrow \mathbb{P}\mathcal{N}$ is thus obtained by blowing up at D_i, and contracting the proper transform of the fiber over q to F_i:



The line subbundle $\mathcal{L}|_{q_i(t)} \subset \mathcal{N}|_{q_i(t)}$ defines a section of $\mathbb{P}\mathcal{N}|_{q_i(t)}$ and (by curve-to-projective extension) of $\mathbb{P}\mathcal{N}_i|_{q_i(t)}$; if the first of these sections does not pass through F_i, then the second *must* pass through D_i. Combining this with (21), when we pass to the central fiber, the fibers of L at the p_{ij} can sometimes be described in terms of

$$D_{ij} := \varphi_{ij}(D_i).$$

Namely, by our assumption that the tangent directions to D, R_1, and R_2 span $\mathbb{P}^3$, the subspaces F_1 and F_2 are disjoint. The fiber $L|_q \subset N|_q$ thus either:

(a) Coincides with neither F_1 nor F_2: In this case, $L|_{p_{ij}} = D_{ij}$.
(b) Coincides with F_1 but not F_2: In this case, $L|_{p_{2j}} = D_{2j}$ and $L|_q = F_1$.
(c) Coincides with F_2 but not F_1: In this case, $L|_{p_{1j}} = D_{1j}$ and $L|_q = F_2$.

The upshot of this is the following lemma.

Lemma 8.1. *With the above notation, if*

every sub-line-bundle of...	*has slope...*
N	$\leq \mu(N)$
$N[p_{ij} \to D_{ij}]$	$< \mu(N)$
$N[q \to F_1][p_{2j} \to D_{2j}]$	$< \mu(N)$
$N[q \to F_2][p_{1j} \to D_{1j}]$	$< \mu(N)$,

then $N_{D \cup R_1(t) \cup R_2(t)}$ is stable, for $t \in \Delta$ generic. In particular, if these four vector bundles are merely semistable, *then $N_{D \cup R_1(t) \cup R_2(t)}$ is* stable *for $t \in \Delta$ generic.*

Now suppose that R_i is a 2-secant line (meeting D at q and p_{i1}), and write $q' \in T_qD \smallsetminus q$ and $p'_{i1} \in T_{p_{i1}}D \smallsetminus p_{i1}$ for points on the tangent lines to D at q and p_{i1} respectively. Then we have the explicit decomposition

$$N_{D \cup R_i}|_{R_i} \simeq N_{R_i \to q'}(q) \oplus N_{R_i \to p'_{i1}}(p_{i1}) \simeq \mathcal{O}_{\mathbb{P}^1}(2)^{\oplus 2}. \tag{22}$$

In particular, we see that Assumption 4.3 is satisfied. Moreover, we may use this decomposition to compute the subspace D_{i1}: In terms of (22),

$$D_i = N_{R_i \to p'_{i1}}(p_{i1})|_q \quad \Rightarrow \quad D_{i1} = N_{R_i \to p'_{i1}}(p_{i1})|_{p_{i1}}.$$

To describe this in a way that is compatible with the isomorphism

$$N_{D \cup R_i}|_D \simeq N_D(q + p_{i1})[q \to p_{i1}][p_{i1} \to q],$$

we apply Lemma 8.4 of [ALY19], which states that under this isomorphism we have

$$D_{i1} = N_{D \to q}(p_{i1})|_{p_{i1}} \subset N_D(q + p_{i1})[q \to p_{i1}][p_{i1} \to q]|_{p_{i1}}.$$

When both R_1 and R_2 are 2-secant lines, Lemma 8.1 thus gives:

Corollary 8.2. *If R_1 and R_2 are 2-secant lines, and the bundles*
(a) $N_D[p_{11} \to q][p_{21} \to q]$,
(b) $N_D[2p_{11} \to q][2p_{21} \to q]$,
(c) $N_D[p_{11} \to q][q \to p_{11}][2p_{21} \to q]$, *and*
(d) $N_D[2p_{11} \to q][p_{21} \to q][q \to p_{21}]$.
are all semistable, *then $N_{D \cup R_1(t) \cup R_2(t)}$ is* stable *for $t \in \Delta$ generic.*

Remark 5. Since (d) is obtained from (c) by permuting p_{21} and p_{11}, it suffices to prove semistability of (a)–(c).

Now suppose only that R_1 is a 2-secant line. Applying Lemma 8.1, the stability of $N_{D \cup R_1(t) \cup R_2(t)}$ for $t \in \Delta$ generic follows from the assertions that:

every sub-line-bundle of...	has slope...
N	$\leq \mu(N)$
$N[p_{11} \to q][p_{2j} \to D_{2j}]$	$< \mu(N)$
$N[q \to p_{11}][p_{2j} \to D_{2j}]$	$< \mu(N)$
$N[q \to F_2][p_{11} \to q]$	$< \mu(N)$.

This follows in turn from the assertion that

$$N[p_{11} \to q] \quad \text{and} \quad N[q \to p_{11}]$$

are stable. We therefore have:

Corollary 8.3. *Suppose that R_1 is a 2-secant line, and write $p'_{2j} \in T_{p_{2j}} R_2 \smallsetminus p_{2j}$ for points on the tangnet lines to R_2 at the p_{2j}. If the bundles*

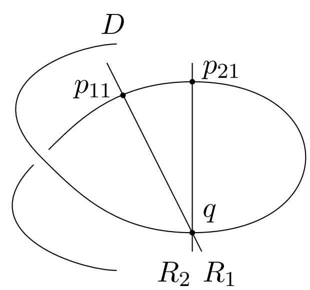
(a) $N_D[p_{2j} \to p'_{2j}][2p_{11} \to q]$ *and*
(b) $N_D[p_{2j} \to p'_{2j}][p_{11} \to q][q \to p_{11}]$

are both stable/semistable*, then $N_{D \cup R_1(t) \cup R_2(t)}$ is* stable *for $t \in \Delta$ generic.*

The bundles $N_D[p_{2j} \to p'_{2j}][2p_{11} \to q]$ and $N_D[p_{2j} \to p'_{2j}][p_{11} \to q][q \to p_{11}]$ appearing in Corollary 8.3 are rank 2 vector bundles of odd degree, and hence stability is equivalent to semistability.

9. Base Cases: Applications of Limits of Gluing Data

The cases $(d, g) = (7, 2), (6, 3), (7, 3), (7, 4), (8, 4)$, and $(8, 5)$. In these cases, we degenerate to the union of a general Brill–Noether D curve of degree $d - 2$ and genus $g - 2$, a 2-secant line R_1 through general points q and p_{11}, and a 2-secant line R_2 through q and another general point p_{21}.



Then R_1 and R_2 satisfy Assumption 4.3, and so by Lemma 4.4, the union $D \cup R_1 \cup R_2$ deforms to the union of D and two general 2-secant lines, which by Lemma 2.7(ii) is a Brill–Noether curve of degree d and genus g. By Corollary 8.2, it suffices to check that the three bundles 8.2(a)-(c) are semistable when D is a general curve of degree $d - 2$ and genus $g - 2$.

$(d, g) = (7, 2)$. Here D is of degree 5 and genus 0. We further degenerate D to the union of a general rational normal curve C (i.e., degree 3 and genus 0) and two general 1-secant lines $\overline{u_1, v_1}$ and $\overline{u_2, v_2}$ meeting C at u_1 and u_2 respectively. By Lemma 5.1, it therefore suffices to show that the bundles

(a) $N_C[p_{11} \to q][p_{21} \to q][2u_1 \to v_1][2u_2 \to v_2]$, and
(b) $N_C[2p_{11} \to q][2p_{21} \to q][2u_1 \to v_1][2u_2 \to v_2]$, and
(c) $N_C[p_{11} \to q][q \to p_{11}][2p_{21} \to q][2u_1 \to v_1][2u_2 \to v_2]$,

are semistable. Limiting u_1 to p_{11} and u_2 to p_{21} (c.f. the discussion in Remark 3.4 of [ALY19]), we obtain

(a) $N_C(-p_{11} - p_{21})[p_{11} \to v_1][p_{21} \to v_2]$
(b) $N_C(-2p_{11} - 2p_{21})$
(c) $N_C(-p_{11} - 2p_{21})[p_{11} \to v_1][q \to p_{11}]$

After further limiting p_{11} to p_{21} in (a) (resp. q to p_{11} in (c)), and using the fact that $N_{C \to v_1}|_{p_{11}}$ is a general subspace, these bundles all specialize to twists of N_C, and are therefore semistable.

$(d,g) = (6,3)$ and $(7,3)$. When $(d,g) = (6,3)$, then D is of degree 4 and genus 1. For uniformity of notation, we write $C = D$.

When $(d,g) = (7,3)$, then D is of degree 5 and genus 1. We further degenerate D to the union of a general Brill–Noether curve C of degree 4 and genus 1, with a general 1-secant line M meeting C at u. Write $v \in M \smallsetminus u$ for another point on M. By Lemma 5.1, in these cases it suffices to prove semistability of the bundles 8.2(a)-(c) with the extra modification $[2u \to v]$.

Combining these cases, it suffices to show that the following 6 bundles on C are semistable:

(a) $N_C[p_{11} \to q][p_{21} \to q]$ and $N_C[2u \to v][p_{11} \to q][p_{21} \to q]$,
(b) $N_C[2p_{11} \to q][2p_{21} \to q]$ and $N_C[2u \to v][2p_{11} \to q][2p_{21} \to q]$,
(c) $N_C[p_{11} \to q][q \to p_{11}][2p_{21} \to q]$ and $N_C[2u \to v][p_{11} \to q][q \to p_{11}][2p_{21} \to q]$.

Lemma 9.1. *Let C be an irreducible curve, and u, v, p_{11}, p_{21}, q be general points on C. Suppose that the bundles* (1) *and* (3) *below are semistable:*

(1) $N_C[2p_{11} \to q]$
(2) $N_C[2p_{11} \to q][2p_{21} \to q]$
(3) $N_C[p_{11} \to q][q \to p_{11}][2p_{21} \to q]$.

Then all of the following bundles are also semistable:

(a) $N_C[p_{11} \to q][p_{21} \to q]$
(b) $N_C[p_{11} \to q][p_{21} \to v]$
(c) $N_C[2u \to v][p_{11} \to q][p_{21} \to q]$
(d) $N_C[2u \to v][2p_{11} \to q][2p_{21} \to q]$
(e) $N_C[2u \to v][p_{11} \to q][q \to p_{11}][2p_{21} \to q]$
(f) $N_C[u \to v][v \to u][p_{11} \to q][p_{21} \to q]$
(g) $N_C[u \to v][v \to u][2p_{11} \to q][2p_{21} \to q]$
(h) $N_C[u \to v][v \to u][2p_{11} \to q][p_{21} \to q][q \to p_{21}]$.

Proof. We argue by specializing the various points on C, to reduce to twists of bundles that we already assumed or proved were semistable.

(a) Specialize p_{21} to p_{11}; the resulting bundle is $N_C[2p_{11} \to q]$, i.e. (1).
(b) Specialize v to q; the resulting bundle is $N_C[p_{11} \to q][p_{21} \to q]$, i.e. (a).
(c) Specialize u to p_{21}; the resulting bundle is $N_C[p_{11} \to q][p_{21} \to v](-p_{21})$, c.f. (b).
(d) Specialize u to p_{21}; the resulting bundle is $N_C[2p_{11} \to q](-2p_{21})$, c.f. (1).
(e) Specialize u to q; the resulting bundle is $N_C[p_{11} \to q][q \to v][2p_{21} \to q](-q)$.
Then specialize v to p_{11}; the resulting bundle is $N_C[p_{11} \to q][q \to p_{11}][2p_{21} \to q](-q)$, c.f. (3).
(f) Specialize u to p_{21}; the resulting bundle is $N_C[p_{11} \to q][v \to p_{21}](-p_{21})$.
Exchanging v and p_{21}, this is $N_C[p_{11} \to q][p_{21} \to v](-v)$, c.f. (b).
(g) Specialize v to p_{21}; the resulting bundle is $N_C[u \to p_{21}][2p_{11} \to q][p_{21} \to q](-p_{21})$.
Then specialize u to p_{11}; the resulting bundle is $N_C[p_{11} \to q][p_{21} \to q](-p_{11} - p_{21})$, c.f. (a).
(h) Specialize v to p_{11}; the resulting bundle is $N_C[u \to p_{11}][p_{11} \to q][p_{21} \to q][q \to p_{21}](-p_{11})$.
Then specialize u to q; the resulting bundle is $N_C[p_{11} \to q][p_{21} \to q](-p_{11} - q)$, c.f. (a). ☐

Applying Lemma 9.1(a)(c)(d)(e), and using (2) and (3) directly, it remains only to show that the three bundles (1)–(3) are semistable.

Let Q be a quadric containing C. In cases (1) and (3), specialize p_{11} to one of the two points guaranteed by Lemma 2.5 for the point $q \in C$; in case (2), specialize both p_{11} and p_{21} to the two points guaranteed by Lemma 2.5 for the point $q \in C$. After these specializations, the inclusion $C \subset Q$ induces normal bundle exact sequences for the modified bundles (1), (2), and (3):

$$0 \to N_{C/Q}(-p_{11}) \to N_C[2p_{11} \to q] \to N_Q|_C(-p_{11}) \to 0$$
$$0 \to N_{C/Q}(-p_{11} - p_{21}) \to N_C[2p_{11} \to q][2p_{21} \to q] \to N_Q|_C(-p_{11} - p_{21}) \to 0$$
$$0 \to N_{C/Q}(-2p_{21}) \to N_C[p_{11} \to q][q \to p_{11}][2p_{21} \to q] \to N_Q|_C(-p_{11} - q) \to 0.$$

These sequences are balanced because $\mu(N_{C/Q}) = 8 = \mu(N_Q|_C)$, so this establishes the semistability of the modified bundles in (1), (2), and (3) as desired.

$(d,g) = (7,4), (8,4)$, and $(8,5)$. When $(d,g) = (7,4)$, then D is of degree 5 and genus 2. For uniformity of notation, we write $C = D$.

When $(d,g) = (8,4)$, then D is of degree 6 and genus 2. We further degenerate D to the union of a general Brill–Noether curve C of degree 5 and genus 2, with a general 1-secant line M meeting C at u. Write $v \in M \smallsetminus u$ for another point on M. By Lemma 5.1, in these cases it suffices to prove semistability of the bundles 8.2(a)-(c) with the extra modification $[2u \to v]$.

When $(d,g) = (8,5)$, then D is of degree 6 and genus 3. We further degenerate D to the union of a general Brill–Noether curve C of degree 5 and genus 2, with a general 2-secant line M meeting C at u and v. Since $N_{C\cup L}|_L \simeq \mathcal{O}_L(2) \oplus \mathcal{O}_L(2)$ is semistable, it suffices to show that each of the bundles (a)-(c) are semistable when restricted to C, i.e. it suffices to prove semistability of the bundles 8.2(a)-(c) with the extra modification $[u \to v][v \to u]$.

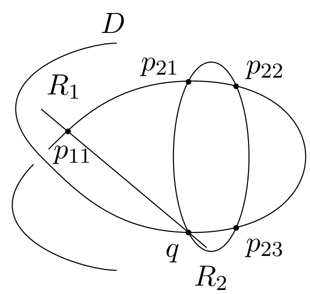
Combining these cases, we have to check the semistability of 9 modifications of N_C. Applying Lemma 9.1(a)(c)(d)(e)(f)(g)(h), and using (2) and (3) directly, it suffices to check that the three modifications (1), (2), and (3) are semistable for C a general curve of degree 5 and genus 2.

Let Q be the unique quadric containing C. In all cases, specialize p_{21} to one of the three points on C guaranteed by Lemma 2.5 for which $N_{C\to q}|_{p_{21}}$ and $N_{C/Q}|_{p_{21}}$ coincide to first order. Then after these specializations, the inclusion $C \subset Q$ induces the following normal bundle exact sequences for the modified bundles in (1), (2), and (3):

$$0 \to N_{C/Q}(-2p_{11}) \to N_C[2p_{11} \to q] \to N_Q|_C \to 0$$
$$0 \to N_{C/Q}(-2p_{11} - p_{21}) \to N_C[2p_{11} \to q][2p_{21} \to q] \to N_Q|_C(-p_{21}) \to 0$$
$$0 \to N_{C/Q}(-p_{11} - p_{21} - q) \to N_C[p_{11} \to q][q \to p_{11}][2p_{21} \to q] \to N_Q|_C(-p_{21}) \to 0.$$

These sequences are balanced because $\mu(N_{C/Q}) = 12$ and $\mu(N_Q|_C) = 10$, so this establishes the semistability of the modified bundles in (1), (2), and (3) as desired.

The cases $(d,g) = (8,6)$ and $(9,6)$. In these cases, we degenerate to the union of a general Brill–Noether curve D of degree $d-3$ and genus $g-4=2$, a general 2-secant line R_1, meeting D quasi-transversely precisely at q and p_{11}, a general 4-secant conic R_2, meeting D quasi-transversely precisely at q, p_{21}, p_{22}, and p_{23}.



Then R_1 and R_2 satisfy Assumption 4.3, and so by Lemma 4.4, the union $D \cup R_1 \cup R_2$ deforms to the union of D, a 2-secant line, and a 4-secant conic, which by Lemma 2.7(ii) and (iii) is a Brill–Noether curve of degree d and genus g. By Corollary 8.3, it suffices to check that the two bundles

(a) $N_D[p_{21} \to p'_{21}][p_{22} \to p'_{22}][p_{23} \to p'_{23}][2p_{11} \to q]$ and
(b) $N_D[p_{21} \to p'_{21}][p_{22} \to p'_{22}][p_{23} \to p'_{23}][p_{11} \to q][q \to p_{11}]$

are stable when D is a general curve of degree $d-3$ and genus 2. Limiting p_{11} to p_{21}, these bundles fit into families whose central fibers are

(a) $N_D[p_{22} \to p'_{22}][p_{23} \to p'_{23}][p_{21} \to q]$

(b) $N_D[p_{22} \to p'_{22}][p_{23} \to p'_{23}][q \to p_{21}]$

These bundles are symmetric under exchanging p_{21} and q, so it suffices to show the stability of the first bundle.

When $(d,g) = (8,6)$, then D is of degree 5 and genus 2; in this case, for uniformity of notation, we write $C = D$, so our problem is simply to show the stability of the bundle

$$N_C[p_{22} \to p'_{22}][p_{23} \to p'_{23}][p_{21} \to q]. \tag{23}$$

When $(d,g) = (9,6)$, then D is of degree 6 and genus 2. We further degenerate D to the union of a general Brill–Noether curve C of degree 5 and genus 2, with a general 1-secant line M meeting C at u. Write $v \in M \smallsetminus u$ for another point on M. By Lemma 5.1, in these cases it suffices to prove stability for the bundle

$$N_C[p_{22} \to p'_{22}][p_{23} \to p'_{23}][p_{21} \to q][2u \to v].$$

Limiting u to p_{21} reduces the stability of this bundle to the stability of

$$N_D[p_{22} \to p'_{22}][p_{23} \to p'_{23}][p_{21} \to v],$$

and subsequently limiting v to q reduces its stability to the stability of (23).

All that remains is thus to show that (23) is stable. The normal bundle exact sequence for the inclusion of C in the unique quadric Q containing it gives rise to the exact sequence

$$0 \to N_{C/Q}(-p_{21} - p_{22} - p_{23}) \to N_C[p_{22} \to p'_{22}][p_{23} \to p'_{23}][p_{21} \to q] \to \mathcal{O}_C(2) \to 0. \tag{24}$$

These bundles have slopes 9, 9.5, and 10, respectively; hence it suffices to show that this sequence is nonsplit, i.e. that

$$H^0(N_C(-2)[p_{22} \to p'_{22}][p_{23} \to p'_{23}][p_{21} \to q]) = 0.$$

By Lemma 3.1, all sections of $N_C(-2)$ come from $H^0(N_{C/Q}(-2))$, which has dimension 2. After imposing three negative modifications out of the quadric at general points, we therefore have no global sections as desired.

10. Curves of degree 6 and genus 2

This case was done by Sacchiero in [S83]. For completeness, we provide a characteristic-independent proof here. We shall need the following lemma:

Lemma 10.1. *Let E be a vector bundle on a smooth curve C sitting in an exact sequence*

$$0 \to L_1 \to E \to L_2 \to 0,$$

where L_1 and L_2 are line bundles. If $\mu(L_2) = \mu(L_1) + 2$, and

$$\operatorname{Hom}(L_2(-p), E) \simeq H^0(E \otimes L_2^\vee(p)) = 0$$

for all $p \in C$, then E is stable.

Proof. Let $\phi \colon F \hookrightarrow E$ be the inclusion of a line subbundle. Then either ϕ factors through $L_1 \hookrightarrow E$, in which case $F \simeq L_1$ is not destabilizing, or projection from E to L_2 gives a nonzero map $F \to L_2$.

In the second case, $F \simeq L_2(-p_1 - \cdots - p_n)$. Since $\operatorname{Hom}(L_2(-p), E) = 0$ for all $p \in C$ by assumption, but $\operatorname{Hom}(L_2(-p_1 - \cdots - p_n), E) \neq 0$, we must have $n \geq 2$. Therefore

$$\mu(F) = \mu(L_2) - n = \mu(E) - n + 1 < \mu(E).$$ □

Now let C be a general Brill–Noether curve of degree $d = 6$ and genus $g = 2$. Since $d > g + r$, our curve C is a projection of a general Brill–Noether curve $\tilde{C} \subset \mathbb{P}^4$; by Lemma 13.2 and the proof of Proposition 13.5 of [ALY19], $\tilde{C}$ is a quadric section of a cubic scroll. Thus, C lies on a cubic surface S singular along a line (the projection of the cubic scroll), and the normal bundle exact sequence for C in S gives

$$0 \to \mathcal{O}_C(2) \to N_{C/\mathbb{P}^3} \to K_C(2) \to 0. \tag{25}$$

We have $\mu(\mathcal{O}_C(2)) = 11$ and $\mu(K_C(2)) = 13$, so by Lemma 10.1, it suffices to show for any $p \in C$,

$$H^0(N_C(-2) \otimes K_C^\vee(p)) = 0.$$

Let $q \in C$ be conjugate to p under the hyperelliptic involution on C, so $K_C^\vee(p) \simeq \mathcal{O}_C(-q)$ and we must show $H^0(N_C(-2)(-q)) = 0$. As $N_{C/S}(-2) \simeq \mathcal{O}_C$ has one nowhere-vanishing section, it suffices to show $N_{C/S}(-2) \hookrightarrow N_C(-2)$ is surjective on global sections; i.e., that $h^0(N_C(-2)) = 1$.

We now prove this by degeneration. (We could not degenerate first, since our desired degeneration would break the exact sequence (25).) Namely, we degenerate C to the union $D \cup_u L$ of a general curve D of degree 5 and genus 2, and a general 1-secant line L meeting at the point u. Let v be a point on L away from u. By [ALY19, Lemma 8.5], it suffices to show $h^0(N_D(-2)(u)[2u \to v]) = 1$.

Let Q be the unique quadric containing D. By Lemma 3.1, $H^0(N_D(-2))$ is 2-dimensional. When we twist up by u, we have an exact sequence

$$0 \to N_{D/Q}(-2)(u) \to N_D(-2)(u) \to \mathcal{O}_D(u) \to 0.$$

As $N_{D/Q}(-2)(u) \simeq K_D(u)$ has exactly 2 global sections and vanishing H^1, the associated long exact sequence in cohomology gives $h^0(N_D(-2)(u)) = 3$. Consequently, the image of the evaluation map

$$H^0(N_D(-2)(u)) \to N_D(-2)(u)|_u$$

is a 1-dimensional subspace of the fiber at u. Since the line L is general, the fiber $N_{D\to v}|_u$ will not coincide with this 1-dimensional subspace. Therefore, the inclusion $N_D(-2) \subset N_D(-2)(u)[u \to v]$ induces an isomorphism on global sections. Combining this with Lemma 3.1, the inclusion

$$N_{D/Q}(-2) \subset N_D(-2)(u)[u \to v]$$

also induces an isomorphism on global sections. Modifying once more towards v, and noting that the generality of v guarantees that $N_{D\to v}$ and $N_{D/Q}$ are transverse at u, we conclude that $N_{D/Q}(-2)(-u) \subset N_D(-2)(u)[2u \to v]$ induces an isomorphism on global sections. Thus

$$h^0(N_D(-2)(u)[2u \to v]) = h^0(N_{D/Q}(-2)(-u)) = h^0(K_D(-u)) = 1.$$

11. Curves of degree 7 and genus 5

In this section, for completeness we recall Ballico and Ellia's argument [BE84] that shows that if C is a non-hyperelliptic and non-trigonal space curve of degree 7 and genus 5, then N_C is stable. Equivalently, they show that $N_C^\vee(3)$ is stable. The bundle $N_C^\vee(3)$ has degree 6, hence we need that it does not admit a line bundle of degree 3 or more. Let

$$0 \to L \to N_C^\vee(3) \to M \to 0$$

be a destabilizing sequence. An elementary Riemann-Roch calculation shows that $h^0(\mathcal{I}_C(3)) \geq 3$, where $\mathcal{I}_C$ denotes the ideal sheaf of C in $\mathbb{P}^3$. Since there cannot be a cubic surface double along a curve of degree 7, the long exact sequence associated to the exact sequence

$$0 \to \mathcal{I}_C^2(3) \to \mathcal{I}_C(3) \to N_C^\vee(3) \to 0$$

shows that the image of

$$h\colon H^0(\mathcal{I}_C(3)) \to H^0(N_C^\vee(3))$$

has dimension at least 3. Consequently,

$$\dim(H^0(L) \cap \operatorname{im}(h)) + \dim(H^0(M)) \geq 3.$$

If the degree of L is at least 3, then the degree of M is at most 3. Since the curve is not trigonal or hyperelliptic, we conclude that $h^0(M) \leq 1$. Hence, $\dim(H^0(L) \cap \operatorname{im}(h)) \geq 2$. Thus, there are two cubics in the ideal of C whose image in $N_C^\vee(3)$ lie in the same line subbundle L. Hence, these cubics are everywhere tangent along C. By Bezout's Theorem, these cubic surfaces intersect in a curve of degree 9 and cannot be tangent along a curve of degree 7. Consequently, $N_C^\vee(3)$ cannot have a line subbundle of degree 3 or more and is stable.

This completes the proof of Theorem 1.

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