LECTURE 1

The purpose of these lectures is to discuss some aspects of the geometry of homogeneous varieties. Let G be a complex, simple Lie group. For our purposes, G will usually be $GL(n)$ and occasionally be $SO(n)$ or $SP(n)$. If time permits, we might briefly discuss some homogeneous varieties for E_6 or E_7. A *rational homogeneous variety* is a projective variety which is a quotient of G by a parabolic subgroup. The most important examples include Grassmannians $G(k,n)$ and partial flag varieties $F(k_1,\dots,k_r;n)$ parameterizing partial flags $(V_1 \subset \cdots \subset V_r)$, where V_i is a k_i-dimensional subspace of a fixed n-dimensional vector space. These varieties are quotients of $GL(n)$ by the parabolic subgroup that stabilizes a fixed partial flag $F_{k_1} \subset F_{k_2} \subset \cdots \subset F_{k_r}$. We will spend the first week discussing various aspects of the geometry of Grassmannians. In the second week, we will discuss Grassmannians for other groups and partial flag varieties.

1. The Grassmannian

Grassmannians are the prototypical examples of homogeneous varieties and parameter spaces. Many of the constructions in the theory are motivated by analogous constructions for Grassmannians, hence we will develop the theory for the Grassmannian in detail. In this section, we begin by reviewing the basic facts about Grassmannians. The reader may refer to the books [H, Lectures 6 and 16], [GH, Chapter I.5] Chapter I.5, and the papers [Kl2] and [KL] for additional information. Although much of the theory works over arbitrary algebraically closed fields, for simplicity, we will always work over the complex numbers.

Let V be an n-dimensional vector space. The Grassmannian $G(k,n)$ parameterizes k-dimensional linear subspaces of V. We will shortly prove that it is a smooth, projective variety of dimension $k(n-k)$. It is often convenient to think of $G(k,n)$ as the parameter space of $(k-1)$-dimensional projective linear spaces in $\mathbb{P}^{n-1}$. When using this point of view, it is customary to denote the Grassmannian by $\mathbb{G}(k-1,n-1)$.

1.1. **The Grassmannian as a complex manifold.**

We will now give $G(k,n)$ the structure of an abstract variety. Given a k-dimensional subspace Ω of V, we can represent it by a $k \times n$ matrix. Choose a basis $v_1,\dots,v_k$ for Ω and form a matrix with $v_1,\dots v_k$ as the row vectors

$$M = \begin{pmatrix} \vec{v_1} \\ \dots \\ \vec{v_k} \end{pmatrix}.$$

The general linear group $GL(k)$ acts on the set of $k \times n$ matrices by left multiplication. This action corresponds to changing the basis of Ω. Two $k \times n$ matrices represent the same linear space if and only if they are in the same orbit of the action of $GL(k)$. Since the k vectors span Ω, the matrix M has rank k. Hence, M has a non-vanishing $k \times k$ minor. Consider the Zariski open set of matrices that have a fixed non-vanishing $k \times k$ minor. By multiplying on the left by the inverse of this $k \times k$ submatrix, we can normalize M so that this submatrix is the identity matrix. For example, if the $k \times k$ minor consists of the first k columns, the normalized matrix has the form

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & * & \cdots & * \\ 0 & 1 & \cdots & 0 & * & \cdots & * \\ & & & \cdots & & & \\ 0 & 0 & \cdots & 1 & * & \cdots & * \end{pmatrix}.$$

A normalized matrix gives a unique representation for the vector space Ω. The space of normalized matrices such that a fixed $k \times k$ minor is the identity is affine space $\mathbb{A}^{k(n-k)}$. When a matrix has two non-zero $k \times k$ minors corresponding to submatrices M_1 and M_2, the transition from one representation to another is given by left multiplication by $M_2^{-1}M_1$. By Cramer's rule, the inverse of a matrix is given by rational functions in the entries of a matrix. Consequently, the transition functions are algebraic functions. We thus endow $G(k,n)$ with the structure of a $k(n-k)$ dimensional abstract variety. Moreover, this construction endows $G(k,n)$ with the structure of a complex manifold of dimension $k(n-k)$.

1.2. $G(k,n)$ **is a compact homogeneous space.** Given a k-dimensional vector space Ω and an element A of $GL(n)$, $A\Omega$ is another k-dimensional vector space. Furthermore, given two k-dimensional vector spaces Ω and Ω', there exists an element of $GL(n)$ that maps one to the other. Consequently, the group $GL(n)$ acts on $G(k,n)$ transitively. In fact, we can fix a Hermitian inner product on V and, by the Gram-Schmidt orthogonalization process, choose orthonormal bases for Ω and Ω'. We can then require the matrix transforming Ω to Ω' to be unitary. Hence, the unitary group $U(n)$, which is compact, maps continuously onto $G(k,n)$. We conclude that $G(k,n)$ is a connected, compact complex manifold homogeneous under the action of $GL(n)$.

1.3. $G(k,n)$ **is a projective variety.** So far we have treated the Grassmannian simply as an abstract variety. However, we can endow it with the structure of a smooth, projective variety via the Plücker embedding of $G(k,n)$ into $\mathbb{P}(\bigwedge^k V)$. Given a k-plane Ω, choose a basis for it $v_1, \dots, v_k$. The Plücker map $\mathrm{Pl} : G(k,n) \to \mathbb{P}(\bigwedge^k V)$ is defined by sending the k-plane Ω to $v_1 \wedge \cdots \wedge v_k$. If we pick a different basis $w_1, \dots, w_k$ for Ω, then

$$w_1 \wedge \cdots \wedge w_k = \det(M) v_1 \wedge \cdots \wedge v_k,$$

where M is the matrix giving the change of basis of Ω from $v_1, \dots, v_k$ to $w_1, \dots, w_k$. Hence, the map Pl is a well-defined map independent of the chosen basis for Ω.

The map Pl is injective since we can recover Ω from its image $p = [v_1 \wedge \cdots \wedge v_k] \in \mathbb{P}(\bigwedge^k V)$ as the set of all vectors $v \in V$ such that $v \wedge v_1 \wedge \cdots \wedge v_k = 0$. We say that a vector in $\bigwedge^k V$ is *completely decomposable* if it can be expressed as $v_1 \wedge v_2 \wedge \cdots \wedge v_k$ for k vectors $v_1, \dots, v_k \in V$.

Exercise 1.1. When $1 < k < \dim V$, most vectors in $\bigwedge^k V$ are not completely decomposable. Show, for example, that $e_1 \wedge e_2 + e_3 \wedge e_4 \in \bigwedge^2 V$ is not completely decomposable if e_1, e_2, e_3, e_4 is a basis for V.

A point of $\mathbb{P}(\bigwedge^k V)$ is in the image of the map Pl if and only if the representative $\sum p_{i_1,\ldots,i_k} e_1 \wedge \cdots \wedge e_{i_k}$ is completely decomposable. It is not hard to characterize the subvariety of $\mathbb{P}(\bigwedge^k V)$ corresponding to completely decomposable elements. Given a vector $u \in V^*$, we can define a contraction

$$u \lrcorner : \bigwedge^k V \to \bigwedge^{k-1} V$$

by setting

$$u \lrcorner (v_1 \wedge v_2 \wedge \cdots \wedge v_k) = \sum_{i=1}^{k} (-1)^{i-1} u(v_i)\ v_1 \wedge \cdots \wedge \hat{v}_i \wedge \cdots \wedge v_k$$

and extending linearly. The contraction map extends naturally to $u \in \bigwedge^j V^*$. An element $x \in \bigwedge^k V$ is completely decomposable if and only if $(u \lrcorner x) \wedge x = 0$ for every $u \in \bigwedge^{k-1} V^*$. We can express these conditions in coordinates. Choose a basis $e_1, \ldots, e_n$ for V and let $u_1, \ldots, u_n$ be the dual basis for V^*. Expressing the condition $(u \lrcorner x) \wedge x = 0$ in these coordinates, for every distinct set of $k-1$ indices $i_1, \ldots, i_{k-1}$ and a disjoint set of $k+1$ distinct indices $j_1, \ldots, j_{k+1}$, we obtain the Plücker relation

$$\sum_{t=1}^{k+1} (-1)^s p_{i_1,\ldots,i_{k-1},j_t} p_{j_1,\ldots,\hat{j}_t,\ldots j_{k+1}} = 0.$$

Example 1.2. The simplest and everyone's favorite example is $G(2,4)$. In this case, there is a unique Plücker relation

$$p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23} = 0.$$

The Plücker map embeds $G(2,4)$ in $\mathbb{P}^5$ as a smooth quadric hypersurface.

Exercise 1.3. Write down all the Plücker relations for $G(2,5)$ and $G(3,6)$.

In fact, the Plücker relations generate the ideal of the Grassmannian.

Exercise 1.4.
(1) Show that $GL(n)$ acts transitively on the zero locus of the Plücker relations.
(2) Calculate the Jacobian matrix at $(1,0,\ldots,0)$ to show that the scheme cut out by the Plücker relations is smooth of dimension $k(n-k)$. Conclude that the Plücker relations cut out the Grassmannian scheme theoretically.
(3) Show that the set of Plücker relations generates the ideal of the Grassmannian. The easiest way of doing this requires some representation theory.

We can summarize our discussion in the following theorem.

Theorem 1.5. *The Grassmannian $G(k,n)$ is a smooth, irreducible, rational, projective variety of dimension $k(n-k)$.*

1.4. **The Grassmannian is projectively normal.** A smooth, projective variety $X \subset \mathbb{P}^n$ is *projectively normal* if the restriction map $H^0(\mathcal{O}_{\mathbb{P}^n}(k)) \to H^0(\mathcal{O}_X(k))$ is surjective for every $k \geq 0$. The Borel-Bott-Weil Theorem implies that given a nef line bundle L on a homogeneous variety $X = G/P$, the action of G on $H^0(X,L)$ is an irreducible representation. Consequently, the restriction map $H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(k)) \to H^0(X, \mathcal{O}_X(k))$ is surjective for all $k \geq 1$ when X is embedded in $\mathbb{P}^N$ by a complete linear system. In particular, Grassmannians in their Plücker embedding are projectively normal. It would be very interesting to understand the syzygies of homogeneous varieties under embeddings by complete linear systems. Unfortunately, even the case of projective space is currently open.

1.5. **The cohomology ring of $G(k,n)$.** The cohomology ring of the complex Grassmannian (and more generally, the Chow ring of the Grassmannian) can be very explicitly described. Fix a flag

$$F_\bullet : 0 = F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_n = V,$$

where F_i is an i-dimensional subspace of V. Let λ be a partition with k parts satisfying the conditions

$$n-k \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 0.$$

We will call partitions satisfying these properties *admissible partitions*. Given a flag $F_\bullet$ and an admissible partition λ, we can define a subvariety of the Grassmannian called the *Schubert variety* $\Sigma_{\lambda_1,\ldots,\lambda_k}(F_\bullet)$ of type λ with respect to the flag $F_\bullet$ to be

$$\Sigma_{\lambda_1,\ldots,\lambda_k}(F_\bullet) := \{\ [\Omega] \in G(k,n)\ :\ \dim(\Omega \cap F_{n-k+i-\lambda_i}) \geq i\ \}.$$

The homology and cohomology classes of a Schubert variety depend only on the partition λ and do not depend on the choice of flag. For each partition λ, we get a homology class and a cohomology class (Poincaré dual to the homology class). When writing partitions, it is customary to omit the parts that are equal to zero. We will follow this custom and write, for example, Σ_1 instead of $\Sigma_{1,0}$.

The Schubert classes give an additive basis for the cohomology ring of the Grassmannian. In order to prove this, it is useful to introduce a stratification of $G(k,n)$. Pick an ordered basis $e_1, e_2, \ldots, e_n$ of V and let $F_\bullet$ be the standard flag for V defined by setting $F_i =< e_1, \ldots, e_i >$. The Schubert cell $\Sigma^c_{\lambda_1,\ldots,\lambda_k}(F_\bullet)$ is defined as $\{[\Omega] \in G(k,n) \mid$

$$\dim(\Omega \cap F_j) = \left\{ \begin{array}{lc} 0 & \text{for } j < n-k+1-\lambda_1 \\ i & \text{for } n-k+i-\lambda_i \leq j < n-k+i+1-\lambda_{i+1} \\ k & \text{for } n-\lambda_k \leq j \end{array} \right\}.$$

Given a partition λ, define the *weight* of the partition to be

$$|\lambda| = \sum_{i=1}^{k} \lambda_i.$$

The Schubert cell $\Sigma^c_{\lambda_1,\ldots,\lambda_k}(F_\bullet)$ is isomorphic to $\mathbb{A}^{k(n-k)-|\lambda|}$. For $\Omega \in \Sigma^c_{\lambda_1,\ldots,\lambda_k}(F_\bullet)$ we can uniquely choose a distinguished basis so that the matrix having as rows this basis has the form

$$\begin{pmatrix} * & \cdots & * & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 \\ * & \cdots & * & 0 & * & \cdots & * & 1 & \cdots & 0 & \cdots & 0 \\ & & & & \cdots & & & & & & & \\ * & \cdots & * & 0 & * & \cdots & * & 0 & \cdots & 1 & \cdots & 0 \end{pmatrix},$$

where the only non-zero entry in the $(n-k+i-\lambda_i)$-th column is a 1 in row i and all the (i,j) entries are 0 if $j > n-k+i-\lambda_i$. Thus, we see that the Schubert cell is isomorphic to $\mathbb{A}^{k(n-k)-|\lambda|}$. We can express the Grassmannian as a disjoint union of these Schubert cells

$$G(k,n) = \bigsqcup_{\lambda \text{ admissible}} \Sigma^c_\lambda(F_\bullet).$$

Let $\sigma_{\lambda_1,\dots,\lambda_k}$ denote the cohomology class Poincaré dual to the fundamental class of the Schubert variety $\Sigma_{\lambda_1,\dots,\lambda_k}$. Since the Grassmannian has a cellular decomposition where all the cells have even real dimension, we conclude the following theorem.

Theorem 1.6. *The integral cohomology ring $H^*(G(k,n),\mathbb{Z})$ is torsion free. The classes of Schubert varieties σ_λ as λ varies over admissible partitions give an additive basis of $H^*(G(k,n),\mathbb{Z})$.*

Exercise 1.7. Deduce as a corollary of Theorem 1.6 that the Euler characteristic of $G(k,n)$ is $\binom{n}{k}$. Compute the Betti numbers of $G(k,n)$.

Exercise 1.8. Using the fact that $G(k,n)$ has a stratification by affine spaces, prove that the Schubert cycles give an additive basis of the Chow ring of $G(k,n)$. Show that the cycle map from the Chow ring of $G(k,n)$ to the cohomology ring is an isomorphism.

Example 1.9. To make the previous discussion more concrete, let us describe the Schubert varieties in $G(2,4) = \mathbb{G}(1,3)$. For drawing pictures, it is more convenient to use the projective viewpoint of lines in $\mathbb{P}^3$. The possible admissible partitions are $(1),(1,1),(2),(2,1),(2,2)$ and the empty partition. A flag in $\mathbb{P}^3$ corresponds to a choice of point q contained in a line l contained in a plane P contained in $\mathbb{P}^3$.

(1) The codimension 1 Schubert variety Σ_1 parameterizes lines that intersect the line l.
(2) The codimension 2 Schubert variety $\Sigma_{1,1}$ parameterizes lines that are contained in the plane P.
(3) The codimension 2 Schubert variety Σ_2 parameterizes lines that pass through the point p.
(4) The codimension 3 Schubert variety $\Sigma_{2,1}$ parameterizes lines in the plane P and that pass through the point p.
(5) The codimension 4 Schubert variety $\Sigma_{2,2}$ is a point corresponding to the line l.

A pictorial representation of these Schubert varieties is given in the next figure.

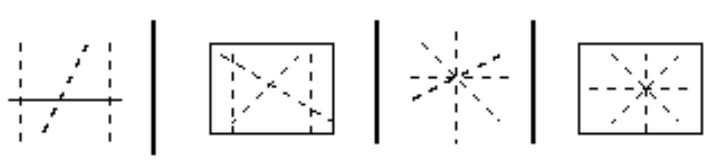

FIGURE 1. Pictorial representations of $\Sigma_1, \Sigma_{1,1}, \Sigma_2$ and $\Sigma_{2,1}$, respectively.

Exercise 1.10. Following the previous example, work out the explicit geometric description of all the Schubert varieties in $G(2,5)$ and $G(3,6)$.

Since the cohomology of Grassmannians is generated by Schubert cycles, given two Schubert cycles σ_λ and σ_μ, their product in the cohomology ring can be expressed as a linear combination of Schubert cycles.

$$\sigma_\lambda \cdot \sigma_\mu = \sum_\nu c^\nu_{\lambda,\mu}\ \sigma_\nu$$

The structure constants $c^\nu_{\lambda,\mu}$ of the cohomology ring with respect to the Schubert basis are known as Littlewood - Richardson coefficients.

We will describe several methods for computing the Littlewood-Richardson coefficients. It is crucial to know when two varieties intersect transversely. Kleiman's Transversality Theorem provides a very useful criterion for ascertaining that the intersection of two varieties in a homogeneous space is transverse. Here we will recall the statement and sketch a proof. For a more detailed treatment see [Kl1] or [Ha] Theorem III.10.8.

Theorem 1.11. *(Kleiman) Let k be an algebraically closed field. Let G be an integral algebraic group scheme over k and let X be an integral algebraic scheme with a transitive G action. Let $f : Y \to X$ and $f' : Z \to X$ be two maps of integral algebraic schemes. For each rational element of $g \in G$, denote by gY the X-scheme given by $y \mapsto gf(y)$.*

(1) *Then there exists a dense open subset U of G such that for every rational element $g \in U$, the fiber product $(gY) \times_X Z$ is either empty or equidimensional of the expected dimension*

$$\dim(Y) + \dim(Z) - \dim(X).$$

(2) *If the characteristic of k is zero and Y and Z are regular, then there exists an open, dense subset U' of G such that for $g \in U'$, the fiber product $(gY) \times_X Z$ is regular.*

Proof. The theorem follows from the following lemma.

Lemma 1.12. *Suppose all the schemes in the following diagram are integral over an algebraically closed field k.*

$$\begin{array}{ccccc} & W & & Z & \\ p \swarrow & & q \searrow & & r \swarrow \\ S & & & X & \end{array}$$

If q is flat, then there exists a dense open subset of S such that $p^{-1}(s) \times_X Z$ is empty or equidimensional of dimension

$$\dim(p^{-1}(s)) + \dim(Z) - \dim(X).$$

If in addition, the characteristic of k is zero, Z is regular and q has regular fibers, then $p^{-1}(s) \times_X Z$ is regular for a dense open subset of S.

The theorem follows by taking $S = G$, $W = G \times Y$ and $q : G \times Y \to X$ given by $q(g, y) = gf(y)$. The lemma follows by flatness and generic smoothness. More precisely, since q is flat, the fibers of q are equidimensional of dimension

$\dim(W) - \dim(X)$. By base change the induced map $W \times_X Z \to Z$ is also flat, hence the fibers have dimension $\dim(W \times_X Z) - \dim(Z)$. Consequently,

$$\dim(W \times_X Z) = \dim(W) + \dim(Z) - \dim(X).$$

There is an open subset $U_1 \subset S$ over which p is flat, so the fibers are either empty or equidimensional with dimension $\dim(W) - \dim(S)$. Similarly there is an open subset $U_2 \subset S$, where the fibers of $p \circ pr_W : X \times_X Z \to S$ is either empty or equidimensional of dimension $\dim(X \times_X Z) - \dim(S)$. The first part of the lemma follows by taking $U = U_1 \cap U_2$ and combining these dimension statements. The second statement follows by generic smoothness. This is where we use the assumption that the characteristic is zero. □

The Grassmannians $G(k, n)$ are homogeneous under the action of $GL(n)$. Taking $f : Y \to G(k, n)$ and $f' : Z \to G(k, n)$ to be the inclusion of two subvarieties in Kleiman's transversality theorem, we conclude that $gY \cap Z$ is either empty or a proper intersection for a general $g \in GL(n)$. Furthermore, if the characteristic is zero and Y and Z are smooth, then $gY \cap Z$ is smooth. In particular, the intersection is transverse. Hence, Kleiman's Theorem is an extremely powerful tool for computing products in the cohomology.

Example 1.9 continued. Let us work out the Littlewood - Richardson coefficients of $G(2, 4) = \mathbb{G}(1, 3)$. It is simplest to work dually with the intersection of Schubert varieties. Suppose we wanted to calculate $\Sigma_2 \cap \Sigma_2$. Σ_2 is the class of lines that pass through a point. If we take two distinct points, there will be a unique line containing them both. We conclude that $\Sigma_2 \cap \Sigma_2 = \Sigma_{2,2}$. By Kleiman's transversality theorem, we know that the intersection is transverse. Therefore, this equality is a scheme theoretic equality. Similarly, $\Sigma_{1,1} \cap \Sigma_{1,1} = \Sigma_{2,2}$, because there is a unique line contained in two distinct planes in $\mathbb{P}^3$. On the other hand $\Sigma_{1,1} \cap \Sigma_2 = 0$ since there will not be a line contained in a plane and passing through a point not contained in the plane.

The hardest class to compute is $\Sigma_1 \cap \Sigma_1$. Since Schubert classes give an additive basis of the cohomology, we know that $\Sigma_1 \cap \Sigma_1$ is expressible as a linear combination of $\Sigma_{1,1}$ and Σ_2. Suppose

$$\Sigma_1 \cap \Sigma_1 = a\Sigma_{1,1} + b\Sigma_2$$

We just computed that both $\Sigma_{1,1}$ and Σ_2 are self-dual cycles. In order to compute the coefficient we can calculate the triple intersection. $\Sigma_1 \cap \Sigma_1 \cap \Sigma_2$ is the set of lines that meet two lines l_1, l_2 and contain a point q. There is a unique such line given by $\overline{ql_1} \cap \overline{ql_2}$. The other coefficient can be similarly computed to see $\sigma_1^2 = \sigma_{1,1} + \sigma_2$.

Exercise 1.13. Work out the multiplicative structure of the cohomology ring of $G(2, 4) = \mathbb{G}(1, 3)$, $G(2, 5) = \mathbb{G}(1, 4)$ and $G(3, 6) = \mathbb{G}(2, 5)$.

In the calculations for $G(2, 4)$, it was important to find a dual basis to the Schubert cycles in $H^4(G(2, 4), \mathbb{Z})$. Given an admissible partition λ, we define a dual partition λ^* by setting $\lambda_i^* = n - k - \lambda_{k-i+1}$. Pictorially, if the partition λ is represented by a Young diagram inside a $k \times (n - k)$ box, the dual partition λ^* is the partition complementary to λ in the $k \times (n - k)$ box.

Exercise 1.14. Show that the dual of the Schubert cycle $\sigma_{\lambda_1,\ldots,\lambda_k}$ is the Schubert cycle $\sigma_{n-k-\lambda_k,\ldots,n-k-\lambda_1}$. Conclude that the Littlewood - Richardson coefficient $c^{\nu}_{\lambda,\mu}$ may be computed as the triple product $\sigma_\lambda \cdot \sigma_\mu \cdot \sigma_{\nu^*}$.

The method of undetermined coefficients we just employed is a powerful technique for calculating the classes of subvarieties of the Grassmannian. Let us do an example to show another use of the technique.

Example 1.15. How many lines are contained in the intersection of two general quadric hypersurfaces in $\mathbb{P}^4$? In order to work out this problem we can calculate the class of lines contained in a quadric hypersurface in $\mathbb{P}^4$ and square the class. The dimension of the space of lines on a quadric hypersurface is 3. The classes of dimension 3 in $\mathbb{G}(1,4)$ are given by σ_3 and $\sigma_{2,1}$. We can, therefore, write this class as $a\sigma_3 + b\sigma_{2,1}$. The coefficient of σ_3 is zero because σ_3 is self-dual and corresponds to lines that pass through a point. As long as the quadric hypersurface does not contain the point, the intersection will be zero. On the other hand, $b = 4$. $\Sigma_{2,1}$ parameterizes lines in $\mathbb{P}^4$ that intersect a $\mathbb{P}^1$ and are contained in a $\mathbb{P}^3$ containing the $\mathbb{P}^1$. The intersection of the quadric hypersurface with the $\mathbb{P}^3$ is a quadric surface. The lines have to be contained in this surface and must pass through the two points of intersection of the $\mathbb{P}^1$ with the quadric surface. There are four such lines. We conclude that there are 16 lines that are contained in the intersection of two general quadric hypersurfaces in $\mathbb{P}^4$.

Exercise 1.16. Another way to verify that there are 16 lines in the intersection of two general quadric hypersurfaces in $\mathbb{P}^4$ is to observe that such an intersection is a quartic Del Pezzo surface D_4. Such a surface is the blow-up of $\mathbb{P}^2$ at 5 general points embedded by its anti-canonical linear system. Check that the lines in this embedding correspond to the (-1)-curves on the surface and show that the number of (-1)-curves on this surface is 16 (see for example [Ha] Chapter 5).

Exercise 1.17. Let C be a smooth, complex, irreducible, non-degenerate curve of degree d and genus g in $\mathbb{P}^3$. Compute the class of the variety of lines that are secant to C.

References

[Be] E. Bertini. *Introduzione alla geometria proiettiva degli iperspazi* Enrico Spoerri, Pisa, 1907, MR1548022, JFM 49.0484.08.

[BL] S. Billey and V. Lakshmibai. *Singular loci of Schubert varieties.* Birkhäuser, Boston, MA, 2000, MR1782635, Zbl 0959.14032.

[BH] A. Borel and A. Haefliger, La classe d'homologie fondamental d'un espace analytique, *Bull. Soc. Math. France*, **89** (1961), 461–513.

[Br] R. Bryant. *Rigidity and quasi-rigidity of extremal cycles in Hermitian symmetric spaces.* Princeton University Press Ann. Math. Studies AM–153, 2005.

[C1] I. Coskun, Rigid and non-smoothable Schubert classes, *J. Differential Geom.*, **87** no. 3 (2011), 493–514.

[C2] I. Coskun, Rigidity of Schubert classes in orthogonal Grassmannians, to appear Israel J. Math.

[C3] I. Coskun, Restriction varieties and geometric branching rules, *Adv. Math.*, **228** no. 4 (2011), 2441–2502.

[C4] I. Coskun, Symplectic restriction varieties and geometric branching rules, to appear in Clay Proceedings.

[C5] I. Coskun, Degenerations of surface scrolls and Gromov-Witten invariants of Grassmannians, *J. Algebraic Geom.*, **15** (2006), 223–284.

[D] Del Pezzo. Sulle superficie di ordine n immerse nello spazio di $n+1$ dimensioni, *Nap. rend.*, **XXIV**, 1885, 212–216, JFM 19.0841.01.

[EH] D. Eisenbud, and J. Harris. On varieties of minimal degree (a centennial account) Algebraic geometry, Bowdoin 1985 *Proc. Sympos. Pure Math.* **46** no. 1, Amer. Math. Soc., Providence, RI, 1987, MR0927946, Zbl 0646.14036.

[EH] David Eisenbud and Joe Harris. *The geometry of schemes*, volume 197 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2000.

[Ful1] W. Fulton. *Young tableaux*, volume 35 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 1997.

[Ful2] W. Fulton. *Intersection theory*, volume 2 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics*. Springer-Verlag, Berlin, second edition, 1998.

[GH] P. Griffiths and J. Harris. *Principles of Algebraic Geometry.* Wiley Interscience, 1978.

[H] J. Harris. *Algebraic geometry*, volume 133 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, 1995.

[HM] J. Harris and I. Morrison. *Moduli of curves.* Springer-Verlag, 1998.

[Ha] R. Hartshorne. *Algebraic geometry.* Springer-Verlag, New York, 1977. Graduate Texts in Mathematics, No. 52.

[HRT] R. Hartshorne, E. Rees, and E. Thomas. Nonsmoothing of algebraic cycles on Grassmann varieties. *Bull. Amer. Math. Soc.* **80** no. 5 (1974), 847–851, MR0357402, Zbl 0289.14011.

[Hi] H. Hironaka. A note on algebraic geometry over ground rings. The invariance of Hilbert characteristic functions under the specialization process. *Illinois J. Math.* **2** (1958), 355–366, MR0102519, Zbl 0099.15701.

[Ho1] J. Hong. Rigidity of smooth Schubert varieties in Hermitian symmetric spaces. *Trans. Amer. Math. Soc.* **359** (2007), 2361–2381, MR2276624, Zbl 1126.14010.

[Ho2] J. Hong. Rigidity of singular Schubert varieties in $Gr(m,n)$. *J. Differential Geom.* **71** no. 1 (2005), 1–22, MR2191767, Zbl 1137.14311.

[IM] A. Iliev, and L. Manivel, The Chow ring of the Cayley plane, *Compositio Math.* **141** (2005), 146–160.

[Kl1] S. L. Kleiman. The transversality of a general translate. *Compositio Math.* **28**(1974), 287–297.

[Kl2] S. L. Kleiman. Problem 15: Rigorous foundation of Schubert's enumerative calculus. In *Mathematical developments arising from Hilbert problems (Proc. Sympos. Pure Math., Northern Illinois Univ., De Kalb, Ill., 1974)*, pages 445–482. Proc. Sympos. Pure Math., Vol. XXVIII. Amer. Math. Soc.,
Providence, R. I., 1976.

[KL] S. L. Kleiman and D. Laksov. Schubert calculus. *Amer. Math. Monthly* **79**(1972), 1061–1082.

[KT] A. Knutson and T. Tao. Puzzles and (equivariant) cohomology of Grassmannians. *Duke Math. J.* **119**(2003), 221–260.

[K] J. Kollár. *Rational curves on algebraic varieties*, volume 32 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics*. Springer-Verlag, Berlin, 1996.

[Ko] J. Kollár. Flatness criteria. *J. Algebra* **175** (1995), 715–727, MR1339664, Zbl 0853.14008.

[LS] V. Lakshmibai and C.S. Seshadri. Singular locus of a Schubert variety. *Bull. Amer. Math. Soc.* vol. **11** no. 2 (1984), 363–366, MR0752799, Zbl 0549.14016.

[NS] S.I. Nikolenko, and N.S. Semenov, Chow ring structure made simple, arxiv: 0606335v1.pdf

[R] C. Robles, Schur flexibility of cominuscule Schubert varieties, preprint.

[RT] C. Robles and D. The, Rigid Schubert varieties in compact Hermitian symmetric spaces, *Selecta Math.*, **18** no. 2 (2012), 717–777.

[S] I.R. Shafarevich, *Basic Algebraic Geometry I*, Springer-Verlag, 1994.

[V1] R. Vakil. A geometric Littlewood-Richardson rule. *to appear Ann. of Math.*

[V2] R. Vakil. Schubert induction. *to appear Ann.of Math.*

[W] M. Walters. Geometry and uniqueness of some extreme subvarieties in complex Grassmannians. *Ph. D. thesis,* University of Michigan, 1997.