LECTURE 2

1. Pieri and Giambelli Rules

We now give two presentations for the cohomology ring of the Grassmannian. These presentations are useful for theoretical computations. However, we will soon develop Littlewood-Richardson rules, positive combinatorial rules for computing Littlewood-Richardson coefficients, that are much more effective in computing and understanding the structure of the cohomology ring of $G(k, n)$.

A partition λ with $\lambda_2 = \cdots = \lambda_k = 0$ is called a *special partition.* A Schubert cycle defined with respect to a special partition is called a *Pieri cycle.* Pieri's rule is a formula for multiplying an arbitrary Schubert cycle with a Pieri cycle.

Theorem 1.1 (Pieri's formula). *Let σ_λ be a Pieri cycle. Suppose σ_μ is any Schubert cycle with parts $\mu_1, \dots, \mu_k$. Then*

$$\sigma_\lambda \cdot \sigma_\mu = \sum_{\substack{\mu_i \leq \nu_i \leq \mu_{i-1} \\ |\nu| = |\lambda| + |\mu|}} \sigma_\nu \tag{1}$$

Proof. By codimension considerations, we must have $|\nu| = |\lambda| + |\mu|$. We need to compute the coefficient of each of the cycles σ_ν satisfying the codimension condition. By the orthogonality relations, to compute the coefficient, we need to calculate $\sigma_\lambda \cdot \sigma_\mu \cdot \sigma_{\nu^*}$. First, let's calculate $\sigma_\mu \cdot \sigma_{\nu^*}$. We already know that if $\nu^* \not\subset \mu^*$, this product vanishes. Hence, we need $\mu \subset \nu$, which gives us the conditions $\mu_i \leq \nu_i$ for every $1 \leq i \leq k$. From now on we can suppose that $\mu_i \leq \nu_i$. Set $a_i = n - k + i - \mu_i$ and $b_{k-i+1} = n - k + k - i + 1 - \nu^*_{k-i+1} = k - i + 1 + \nu_i$. We conclude that $A_i = F_{a_i} \cap G_{b_{k-i+1}}$ is a vector space of dimension $\nu_i - \lambda_i + 1$ spanned by the basis vectors $e_{n-b_{k-i+1}+1}, \dots, e_{a_i}$. If $\nu_i \leq \mu_{i-1}$ notice that $F_{a_{i-1}} \cap G_{b_{k-i+1}} = 0$ since this dimension is $\max(0, \nu_i - \mu_{i-1})$. In that case, we learn that any linear space in the intersection of $\Sigma_\mu(F_\bullet) \cap \Sigma_{\nu^*}(G_\bullet)$ must be spanned by one vector from each vector space A_i. Furthermore, the vector spaces A_i are independent and span a linear space of dimension $|\lambda| + k$. On the other hand, the linear spaces are also required to intersect a general linear space $H_{n-k+1-\lambda}$ of dimension $n - k + 1 - \lambda$. This linear space intersects the span of the A_i in precisely a one dimensional space, say spanned by a vector v. Let $B_i = \operatorname{Span}_{j \neq i} A_j$ Then the k-plane in the triple intersection is uniquely determined by the k-one dimensional subspaces $A_i \cap \operatorname{Span}(B_i, v)$. Conversely, suppose that $\nu_i > \mu_{i-1}$. Then A_i and A_{i-1} have at least a one-dimensional intersection. Hence, the span of all the subspaces A_i is at most of dimension $|\lambda| + k - 1$. However, this subspace needs to still contain all the linear spaces in the intersection of $\Sigma_\mu(F_\bullet) \cap \Sigma_{\nu^*}(G_\bullet)$. Since this subspace is now disjoint from $H_{n-k+1-\lambda}$, there cannot be any linear spaces in the triple intersection. Hence, there is one subspace in the intersection precisely when $\mu_i \leq \nu_i \leq \mu_{i-1}$ and otherwise the triple intersection is empty. This proves Pieri's formula. $\square$

Exercise 1.2. Show that the locus where a Plücker coordinate vanishes corresponds to a Schubert variety Σ_1. Observe that the class of Σ_1 generates the second homology of the Grassmannian. In particular, the Picard group is isomorphic to $\mathbb{Z}$. Conclude that $\mathcal{O}_{G(k,n)}(\Sigma_1)$ is the very ample generator of the Picard group and it gives rise to the Plücker embedding.

Exercise 1.3. Compute the degree of the Grassmannian $G(k,n)$ under the Plücker embedding. The answer is provided by $\sigma_1^{k(n-k)}$. When $k=2$, this computation is relatively easy to carry out. By Pieri's formula σ_1 times any cycle in $G(2,n)$ either increases the first index of the cycle or it increases the second index provided that it is less than the first index. This means that the degree of the Grassmannian $G(2,n)$ is the number of ways of walking from one corner of an $(n-2)\times(n-2)$ to the opposite corner without crossing the diagonal. This is well-known to be the Catalan number

$$\frac{(2(n-2))!}{(n-2)!(n-1)!}.$$

The general formula is more involved. The degree of $G(k,n)$ is given by

$$(k(n-k))!\prod_{i=1}^{k}\frac{(i-1)!}{(n-k+i-1)!}.$$

The special Schubert cycles generate the cohomology ring of the Grassmannian. In order to prove this we have to express every Schubert cycle $\sigma_{\lambda_1,\dots,\lambda_k}$ as a linear combination of products of special Schubert cycles. Consider the following example

$$\sigma_{4,3,2} = \sigma_2\cdot\sigma_{4,3} - \sigma_4\cdot\sigma_{4,1} + \sigma_6\cdot\sigma_{2,1}.$$

To check this equality, using Pieri's rule expand the products.

$$\sigma_2\cdot\sigma_{4,3} = \sigma_{4,3,2}+\sigma_{4,4,1}+\sigma_{5,3,1}+\sigma_{5,4}+\sigma_{6,3}$$

$$\sigma_4\cdot\sigma_{4,1} = \sigma_{4,4,1}+\sigma_{5,3,1}+\sigma_{6,2,1}+\sigma_{7,1,1}+\sigma_{5,4}+\sigma_{6,3}+\sigma_{7,2}+\sigma_{8,1}$$

$$\sigma_6\cdot\sigma_{2,1} = \sigma_{7,1,1}+\sigma_{6,2,1}+\sigma_{7,2}+\sigma_{8,1}$$

Note the following features of this calculation. The class $\sigma_{4,3,2}$ only occurs in the first product. All other products occur twice with different signs.

Exercise 1.4. Using Pieri's formula generalize the preceding example to prove the following identity

$$(-1)^k\sigma_{\lambda_1,\dots,\lambda_k} = \sum_{j=1}^{k}(-1)^j\sigma_{\lambda_1,\dots,\lambda_{j-1},\lambda_{j+1}-1,\dots,\lambda_k-1}\cdot\sigma_{\lambda_j+k-j}$$

Theorem 1.5 (Giambelli's formula)**.** *Any Schubert cycle may be expressed as a linear combination of products of special Schubert cycles as follows*

$$\sigma_{\lambda_1,\dots,\lambda_k} = \begin{vmatrix} \sigma_{\lambda_1} & \sigma_{\lambda_1+1} & \sigma_{\lambda_1+2} & \cdots & \sigma_{\lambda_1+k-1} \\ \sigma_{\lambda_2-1} & \sigma_{\lambda_2} & \sigma_{\lambda_2+1} & \cdots & \sigma_{\lambda_2+k-2} \\ & \cdots & \cdots & \cdots & \\ \sigma_{\lambda_k-k+1} & \sigma_{\lambda_k-k+2} & \sigma_{\lambda_k-k+3} & \cdots & \sigma_{\lambda_k} \end{vmatrix}$$

Exercise 1.6. Expand the determinant by the last column and use the previous exercise to prove Giambelli's formula by induction.

Exercise 1.7. Use Giambelli's formula to express $\sigma_{3,2,1}$ in $G(4,8)$ in terms of special Schubert cycles. Using Pieri's rule find the class of its square.

2. universal bundles on the Grassmannian

Pieri's formula and Giambelli's formula together give an algorithm for computing the cup product of any two Schubert cycles. Unfortunately, in practice this algorithm is tedious to use. We will rectify this problem shortly.

One extremely useful way comes from considering the universal exact sequence of bundles on $G(k,n)$. Let T denote the tautological bundle over $G(k,n)$. Recall that the fiber of T over a point $[\Omega]$ is the vector subspace Ω of V. There is a natural inclusion

$$0 \to T \to \underline{V} \to Q \to 0$$

with quotient bundle Q.

Theorem 2.1. *As a ring the cohomology ring of $G(k,n)$ is isomorphic to*

$$\mathbb{R}[c_1(T), \dots, c_k(T), c_1(Q), \dots, c_{n-k}(Q)]/(c(T)c(Q) = 1).$$

Moreover, the chern classes of the Quotient bundle generate the cohomology ring.

The Chern classes of the tautological bundle and the quotient bundle are easy to see in terms of Schubert cycles. As an exercise prove the following proposition:

Proposition 2.2. *The chern classes of the tautological bundle are given as follows:*

$$c_i(T) = (-1)^i \sigma_{1,\dots,1}$$

where there are i ones. The chern classes of the quotient bundle are given by

$$c_i(Q) = \sigma_i.$$

Exercise 2.3. Calculate the number of lines on a general cubic hypersurface in $\mathbb{P}^3$. More generally, calculate the class of the variety of lines contained in a general cubic hypersurface in $\mathbb{P}^n$.

Exercise 2.4. Calculate the number of lines on a general quintic threefold.

Exercise 2.5. Calculate the number of lines contained in a general pencil of quartic surfaces in $\mathbb{P}^3$. Carry out the same calculation for a general pencil of sextic hypersurfaces in $\mathbb{P}^4$.

3. The local structure of the Grassmannian

The tangent bundle of the Grassmannian has a simple intrinsic description in terms of the tautological bundle T and the quotient bundle Q. There is a natural identification of the tangent bundle of the Grassmannian with homomorphisms from T to Q, in other words

$$TG(k,n) = \mathrm{Hom}(T, Q).$$

In particular, the tangent space to the Grassmannian at a point $[\Omega]$ is given by $\mathrm{Hom}(\Omega, V/\Omega)$. One way to realize this identification is to note that the Grassmannian is a homogeneous space for $GL(n)$. The tangent space at a point may be naturally identified with quotient of the Lie algebra of $GL(n)$ by the Lie algebra of the stabilizer. The Lie algebra of $GL(n)$ is the endomorphisms of V. Those that stabilize Ω are those homomorphisms $\phi : V \to V$ such that $\phi(\Omega) \subset \Omega$. These homomorphisms are precisely homomorphisms $\mathrm{Hom}(\Omega, V/\Omega)$.

Exercise 3.1. Use the above description to obtain a description of the tangent space to the Schubert variety $\Sigma_{\lambda_1,\dots,\lambda_k}$ at a smooth point $[\Omega]$ of the variety.

We can use the description of the tangent space to check that the intersection of Schubert cycles in previous calculations were indeed transverse. For example, suppose we take the intersection of two Schubert varieties Σ_1 in $\mathbb{G}(1,3)$ defined with respect to two skew-lines. Then the intersection is a smooth variety. In vector space notation, we can assume that the conditions are imposed by two non-intersecting two-dimensional vector spaces V_1 and V_2. Suppose a 2-dimensional vector space Ω meets each in dimension 1. The tangent space to Ω at the intersection is given by

$$\phi \in \mathrm{Hom}(\Omega, V/\Omega) \ \text{ such that } \ \phi(\Omega \cap V_i) \subset [V_i] \in V/\Omega.$$

As long as V_1 and V_2 do not intersect, Ω has exactly a one-dimensional intersection with each of V_i and these span Ω. On the other hand, the quotient of V_i in V/Ω is one-dimensional. We conclude that the dimension of such homomorphisms is 2. Since this is equal to the dimension of the variety, we deduce that the variety is smooth.

Exercise 3.2. Carry out a similar analysis for the other examples we did above.

Using the description of the tangent bundle, we can calculate the canonical class of $G(k,n)$. We use the splitting principle for Chern classes. Let $\alpha_1,\dots,\alpha_k$ be the Chern roots of S^*. We then have the equation

$$c(S^*) = \prod_{i=1}^{k}(1+\alpha_i) = 1 + \sigma_1 + \sigma_{1,1} + \cdots + \sigma_{1,1,\dots,1}.$$

Similarly, let $\beta_1,\dots,\beta_{n-k}$ be the Chern roots of Q. We then have the equation

$$c(Q) = \prod_{j=1}^{n-k}(1+\beta_j) = 1 + \sigma_1 + \sigma_2 + \cdots + \sigma_{n-k}.$$

The Chern classes of the tangent bundle can be expressed as

$$c(TG(k,n)) = c(S^* \otimes Q) = \prod_{i=1}^{k}\prod_{j=1}^{n-k}(1+\alpha_i+\beta_j).$$

In particular, the first Chern class is equal to $n\sigma_1$. Since this class is n times the ample generator of the Picard group, we conclude the following theorem.

Theorem 3.3. *The canonical class of $G(k,n)$ is equal to $-n\sigma_1$. $G(k,n)$ is a Fano variety of Picard number one and index n.*

4. The Grassmannian as a functor

Definition 4.1. Let S be a scheme, E a vector bundle on S and k a natural number less than or equal to the rank of E. The functor

$$Gr(k,E) : \{\text{schemes over} S\} \to \{\text{sets}\}$$

associates to every S scheme X the set of rank k subvector bundles of $E \times_S X$.

Theorem 4.2. *The functor $Gr(k,E)$ is represented by a scheme $G_S(k,E)$ and a subvector bundle $U \subset E \times_S G_S(k,E)$ of rank k.*

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