

LECTURE 4

1. INTRODUCTION TO RIGIDITY

A Schubert variety in the Grassmannian $G(k, n)$ is smooth if and only if it is a linearly embedded sub-Grassmannian ([LS]). Even when a Schubert variety is singular, there may be smooth subvarieties of $G(k, n)$ representing the same cohomology class. For example, the Schubert variety representing σ_1 in $G(2, 5)$ is a singular hyperplane section of $G(2, 5)$ in the Plücker embedding. However, Bertini's Theorem ([Ha], II.8.18) guarantees that a general hyperplane section of $G(2, 5)$ is a smooth variety representing σ_1 . In contrast, σ_2 cannot be represented by a smooth subvariety of $G(2, 5)$. In fact, any proper subvariety of $G(2, 5)$ with cohomology class σ_2 is a Schubert variety. Nevertheless, there are many Schubert classes, such as $\sigma_{3,2,0}$ in $G(3, 7)$, that admit non-trivial deformations but cannot be represented by a smooth, proper subvariety of $G(k, n)$.

Definition 1.1. A Schubert class σ_λ is called *rigid* if the only proper subvarieties of $G(k, n)$ representing σ_λ are Schubert varieties.

Definition 1.2 (Rigid partition). We will call a partition $\lambda = (\mu_1^{i_1}, \dots, \mu_j^{i_j})$ a *rigid partition* for $G(k, n)$ if there does not exist an index $1 \leq s < j$ with $i_s = 1$ and $n - k > \mu_s = \mu_{s+1} + 1$.

For example, $(4, 4, 2, 0)$, $(3, 3, 2, 2)$, $(4, 3, 1, 1)$ and $(2, 2, 2, 0)$ are rigid partitions for $G(4, 8)$, whereas $(3, 2, 0, 0)$, $(4, 4, 1, 0)$ are not rigid partitions for $G(4, 8)$.

In recent years, the Schur rigidity of Schubert varieties have been extensively studied by differential geometers (see [Br], [Ho1], [Ho2] and [W]). Recall that a Schubert class is *Schur rigid* if the only integral varieties of the corresponding Schur differential system (see Definition 40 of [W] or 2.8.1 of [Br]) are Schubert varieties. In [Ho2], Hong proves that a large class of Schubert varieties are Schur rigid. In this lecture, we will discuss the algebro-geometric theory of rigidity instead and prove the following theorem.

Theorem 1.3. *A Schubert class σ_λ in $G(k, n)$ is rigid if and only if λ is a rigid partition for $G(k, n)$.*

Example 1.4. Theorem 1.3 implies that the Schubert class $\sigma_{3,2,0}$ in $G(3, 7)$ can be represented by proper subvarieties of $G(3, 7)$ that are not Schubert varieties. A deformation of the Schubert variety $\Sigma_{3,2,0}$ can be obtained as follows. Fix a four dimensional subspace V' of V . Let Y be a general hyperplane section of the Grassmannian $G(2, V')$ in its Plücker embedding. Let X be the subvariety of $G(3, V)$ parameterizing three-dimensional linear subspaces of V that contain a two-dimensional subspace parameterized by Y . Then X represents the class $\sigma_{3,2,0}$ in $G(3, 7)$, but is not isomorphic to a Schubert variety (for instance, the singular locus of X is irreducible, where as the singular locus of $\Sigma_{3,2,0}$ has two irreducible

components). Nevertheless, the class $\sigma_{3,2,0}$ cannot be represented by a smooth subvariety of $G(3,7)$. This example raises the problem of characterizing Schubert classes that can be represented by smooth subvarieties of $G(k,n)$.

Definition 1.5 (Non-smoothable partition). We will call a partition $\lambda = (\mu_1^{i_1}, \dots, \mu_j^{i_j})$ a *non-smoothable partition* for $G(k,n)$ if either there exists an index $1 \leq s < j$ such that $i_s \neq 1$ and $n - k > \mu_s$; or there exists an index $1 \leq s < j$ such that $n - k > \mu_s \neq \mu_{s+1} + 1$.

For example, $(3, 3, 1, 0)$, $(3, 2, 2, 1)$, $(4, 2, 0, 0)$ are non-smoothable partitions for $G(4,8)$, whereas $(3, 2, 1, 0)$, $(2, 1, 0, 0)$, $(4, 4, 2, 1)$ are not non-smoothable partitions for $G(4,8)$.

Theorem 1.6. *Let λ be a non-smoothable partition for $G(k,n)$. Then σ_λ cannot be represented by a smooth subvariety of $G(k,n)$.*

Theorem 1.6 is nearly sharp in a sense that we now make precise. Suppose $V' \subset V$ is a subspace of dimension m . If $s \geq \max(0, k + m - n)$, then the Grassmannian $G(s,m)$ can be embedded into $G(k,n)$ as follows. Let U be a subspace of V of dimension $k - s$ such that $U \cap V' = 0$. Given $[W'] \in G(s,m)$, the span of W' and U is a k -dimensional subspace of V . This induces an embedding of $G(s,m)$ into $G(k,n)$ compatible with the Plücker embedding. We will call a sub-Grassmannian of $G(k,n)$ obtained as the image of such a morphism a *linearly embedded sub-Grassmannian*. The Gysin image of the fundamental class of $G(s,m)$ is σ_λ , where

$$\lambda = ((n - k)^{k-s}, (n - m + s - k)^s),$$

and the Gysin image of the fundamental class of a hyperplane section in $G(s,m)$ is $\sigma_{\lambda'}$, where

$$\lambda' = ((n - k)^{k-s}, n - m + s - k + 1, (n - m + s - k)^{s-1})$$

in the cohomology of $G(k,n)$. In particular, note that the Schubert classes σ_λ with

$$\lambda = ((n - k)^{k-s}, p^s) \quad \text{or} \quad ((n - k)^{k-s-1}, p + 1, p^s),$$

can be represented by smooth subvarieties of the Grassmannian $G(k,n)$ for any $0 \leq s \leq k$ and $0 \leq p \leq n - k - 1$.

More generally, the space of Schubert varieties of class $\sigma_{n-r,n-r}$ contained in $G(2,n)$ is parameterized by the Grassmannian $G(r,n)$. Consider the subvariety of $G(r,n)$ parameterizing the locus of subvarieties of $G(2,n)$ with class $\sigma_{n-r,n-r}$ that are contained in a general hyperplane section of $G(2,n)$ in the Plücker embedding. Consider the incidence correspondence

$$I := \{(X, H) \mid X \subset G(2,n) \cap H \text{ with } [X] = \sigma_{n-r,n-r}\} \subset G(r,n) \times (\mathbb{P}^{\binom{n}{2}-1})^*$$

parameterizing pairs of a Schubert subvariety of $G(2,n)$ with class $\sigma_{n-r,n-r}$ and a hyperplane section of $G(2,n)$ in the Plücker embedding containing it. The first projection exhibits I as a projective space bundle over $G(r,n)$ with fibers isomorphic to projective spaces of dimension $\binom{n}{2} - \binom{r}{2} - 1$. Consequently, I is smooth. By generic smoothness ([Ha], III.10.7), a general fiber of the second projection is a smooth subvariety of $G(r,n)$. This variety is the zero locus of a section of the vector bundle $\bigwedge^2 S^*$ over $G(r,n)$, where S denotes the tautological bundle of $G(r,n)$, hence

has cohomology class $\sigma_{r-1, r-2, r-3, \dots, 3, 2, 1, 0}$. Using the construction in the previous paragraph, it follows that the Schubert classes σ_λ in $G(k, n)$ with

$$\lambda = ((n-k)^s, p, p-1, p-2, \dots, p-k-s+2, p-k+s+1)$$

with $k \geq s \geq 0$ and $p-k+s+1 \geq 0$ can be represented by smooth subvarieties of $G(k, n)$. Consequently, Theorem 1.6 is sharp in the following sense. Let λ be a partition with $i_t = 1$ and $\mu_t = \mu_{t+1} + 1$ for all $1 \leq t < j$, then σ_λ can be represented by a smooth subvariety of $G(k, k+3+\mu_j)$. However, this still leaves the following problem open.

Problem 1.7. Let $\lambda = (\mu_1^{i_1}, \dots, \mu_j^{i_j})$ be a partition with $i_t = 1$ and $\mu_t = \mu_{t+1} + 1$ for every $1 \leq t < j$ and $\sum_{t=1}^j i_t = k$. For which n does there exist a smooth variety in $G(k, n)$ representing the class σ_λ ?

Let us explain the strategy of the proof of Theorem 1.6 in the case of $G(2, n)$. Theorem 1.6 asserts that the only classes $\sigma_{a,b}$ that can be represented by smooth subvarieties of $G(2, n)$ have either $a = n-2$ or $a-b \leq 1$. Let $\sigma_{a,b}$ be a Schubert class in $G(2, n)$ such that $a < n-2$ and $a-b > 1$. We need to show that $\sigma_{a,b}$ cannot be represented by a smooth subvariety of $G(2, n)$. Suppose to the contrary that X is a smooth, proper subvariety of $G(2, n)$ representing $\sigma_{a,b}$. Then by intersecting X with a general Schubert variety $\Sigma_{n-3-a, n-3-a}$ (which is smooth), we obtain a smooth, proper variety that represents the class $\sigma_{n-3, n-3+b-a}$. Hence, to prove Theorem 1.6 for $G(2, n)$, it suffices to show that a variety Y representing the class $\sigma_{n-3, n-3-c}$ for $c > 1$ cannot be smooth.

The dimension of Y is $c+2$. By Pieri's formula, the degree of Y in the Plücker embedding of $G(2, n)$ is $c+1$. Any subvariety of $G(2, n)$ representing a Schubert class σ_λ can be specialized to a scheme whose support is a Schubert variety Σ_λ . In particular, the dimension of the linear span of Y in the Plücker embedding has to be greater than or equal to $2c+2$, the dimension of the linear span of a Schubert variety $\Sigma_{n-3, n-3-c}$. We conclude that the Plücker embedding of Y has to be a minimal variety in projective space of dimension $c+2$ and degree $c+1$.

Recall that the degree d of an n -dimensional, non-degenerate, irreducible variety in \mathbb{P}^r is bounded by $d \geq r+1-n$. The varieties for which equality holds are called *minimal varieties* and have been classified by Del Pezzo [D] and Bertini [Be]. A modern proof can be found in [EH].

Classification of varieties of minimal degree. [EH, Theorem 1] *If X is an irreducible, non-degenerate variety of dimension r and degree $n+1-r$ in \mathbb{P}^n , then X is either a quadric hypersurface, a rational normal scroll, the Veronese surface in \mathbb{P}^5 or a cone over any of these varieties.*

Since $c > 1$, the degree of Y is at least three. The dimension of a smooth rational scroll or a Veronese surface is less than or equal to its degree. Since the degree of Y is less than its dimension, by the classification of the varieties of minimal degree, Y is a cone. Therefore, Y is singular. In fact, it is easy to see that Y is a cone over a rational normal scroll. This suffices to prove Theorem 1.6 for $G(2, n)$. The proof in the general case follows the same strategy, however, the base cases that need to be treated are not necessarily minimal varieties. Fortunately, the base cases are all cones over Segre varieties and do not admit any non-trivial infinitesimal deformations.

Theorem 1.3 is then proved by analyzing the singular locus of a variety representing a Schubert class σ_λ . For example, any proper variety X representing the class $\sigma_{n-3,0}$ in $G(2, n)$ has to be a cone over a rational normal scroll. The cone point p of X determines a two-dimensional subspace L of V . Since the Zariski tangent space to X at p has dimension $2n - 4$, we conclude that the span of X has to be the projectivized tangent space to $G(2, n)$ at p . Hence, X is contained in the Schubert variety of two-dimensional subspaces of V that intersect L . Since both varieties are irreducible and of the same dimension, X is equal to the Schubert variety. The proof in the general case proceeds by showing that the singular locus determines the partial flag defining the Schubert variety of the same class.

There are many variants and generalizations of these problems.

Problem 1.8. Which non-negative linear combinations of Schubert classes in $G(k, n)$ can be represented by irreducible/smooth proper subvarieties?

Hong's results in [Ho2] imply that positive, integral multiples of many Schubert classes cannot be represented by smooth, proper subvarieties of the Grassmannian. On the other hand, it is possible to find many linear combinations that can be represented by smooth subvarieties by simply exhibiting smooth subvarieties of the Grassmannian. The following proposition gives some simple constructions of smooth subvarieties of Grassmannians. We thus obtain a large collection of cohomology classes that can be represented by smooth subvarieties.

Proposition 1.9. *The following are examples of classes that can be represented by smooth subvarieties of the Grassmannian.*

- (1) (Bertini) The cohomology class $r\sigma_1^m$ can be represented by a smooth subvariety of $G(k, n)$ for every $r > 0$ and $0 < m < k(n - k)$.
- (2) The cohomology class $\sum_\nu \alpha_{\lambda, \mu}^\nu \sigma_\nu$, where $\alpha_{\lambda, \mu}^\nu$ is the Littlewood-Richardson coefficient, can be represented by a smooth subvariety of $G(k, n)$ provided that λ and μ have the form $(b_1^{i_1}, \dots, b_j^{i_j})$ and $(c_1^{i_1}, \dots, c_j^{i_1})$, respectively, with the property that $b_i + c_{j-i} \geq n - k$.
- (3) The cohomology classes $r\sigma_{((n-k)^{k-1}, a)}$, with $r > 0$ and $a < n - k$, and $r\sigma_{((n-k)^s, (n-k-1)^{k-s})}$, with $r > 0$ and $k > s > 0$, can be represented by smooth subvarieties of $G(k, n)$.
- (4) The space of projective linear spaces of dimension k contained in a general complete intersection in \mathbb{P}^n is either empty or a smooth subvariety in $G(k + 1, n + 1)$. Irreducible components of the spaces of linear spaces contained in Grassmannians are smooth.
- (5) Let $Y_i \in G(k_i, n_i)$, $1 \leq i \leq j$, be smooth subvarieties with classes $c_i = \sum c_\lambda^i \sigma_\lambda$. Let $\tilde{c}_i = \sum c_{\tilde{\lambda}}^i \sigma_{\tilde{\lambda}}$, where $\tilde{\lambda}$ is the partition obtained by adding $n - n_i - k + k_i$ to each of the parts of λ and a tail of $k - k_i$ zeros. Then, the product $Y_1 \times \dots \times Y_j$ can be embedded as a smooth subvariety of $G(\sum_{i=1}^j k_i, \sum_{i=1}^j n_i)$ representing the class $\prod \tilde{c}_i$.

The following problem is a closely related variant of Problem 1.8:

Problem 1.10. Which cohomology classes in the Grassmannian can be expressed as a linear combination (allowing negative signs!) of the classes of smooth proper subvarieties?

Example 1.11. While σ_2 in $G(2, 5)$ cannot be represented by a smooth, proper subvariety, it can be expressed as the difference of the classes of two smooth subvarieties:

$$\sigma_2 = (\sigma_1)^2 - \sigma_{1,1}.$$

The Schubert variety $\Sigma_{1,1}$ is smooth and by Bertini's Theorem a general codimension two linear section of the Plücker embedding of $G(2, 5)$ is a smooth representative of $(\sigma_1)^2$. In [K1], Kleiman proves that if c is the cohomology class of a variety of dimension $d \leq \frac{k(n-k)}{2}$ in $G(k, n)$, then $q! c$, where $q = k(n-k) - d - 1$, can be written as a linear combination of classes of smooth subvarieties. In [KL], Kleiman and Landolfi conjectured that in $G(3, 6)$ the codimension two Schubert cycles cannot be deformed into smooth cycles by rational equivalence. In [HRT], R. Hartshorne, E. Rees and E. Thomas have shown the stronger fact that σ_2 in $G(3, 6)$ cannot even be written as a linear combination of the classes of smooth closed submanifolds of $G(3, 6)$. The complete characterization of classes that can be expressed as a linear combination of smooth subvarieties currently seems out of reach.

2. SINGULARITIES OF SCHUBERT VARIETIES

There is a wealth of information about the singularities of Schubert varieties: Schubert varieties are normal, Cohen-Macaulay with rational singularities and admit natural resolutions such as the Bott-Samelson resolution. The singular loci of Schubert varieties and the multiplicity along each singular locus can be explicitly described. We refer the reader to the excellent book by Billey and Lakshmibai for further information, detailed history and extensive references [BL].

Schubert varieties in the Grassmannian admit a natural resolution $f : \tilde{\Sigma} \rightarrow \Sigma$ such that the exceptional locus of f has codimension at least two. Let $\lambda = (\mu_1^{i_1}, \mu_2^{i_2}, \dots, \mu_j^{i_j})$ be a partition for $G(k, n)$ and let $a_s = \sum_{l=1}^s i_l$. Let $\tilde{\Sigma}_\lambda$ be the Schubert variety in the flag variety $F(a_1, a_2, \dots, a_j; n)$ defined by

$$\tilde{\Sigma}_\lambda := \{(V_1, \dots, V_j) \in F(a_1, a_2, \dots, a_j; n) \mid V_s \subset F_{n-k+a_s-\lambda_{a_s}}, 1 \leq s \leq j\}.$$

Since $\tilde{\Sigma}_\lambda$ is an iterated tower of Grassmannian bundles, it is smooth. Furthermore, $\tilde{\Sigma}_\lambda$ maps onto Σ_λ by the natural projection

$$\pi : F(a_1, a_2, \dots, a_j; n) \rightarrow G(k, n).$$

By Zariski's Main Theorem, the map is an isomorphism over the locus of k -planes W parameterized by Σ_λ with the property that $\dim(W \cap F_{n-k+a_s-\lambda_{a_s}}) = a_s$ for $1 \leq s \leq j$. The map $\pi|_{\tilde{\Sigma}_\lambda}$ has positive dimensional fibers over the locus of k -planes W with the property that $\dim(W \cap F_{n-k+a_s-\lambda_{a_s}}) > a_s$ for some $1 \leq s \leq j$.

Definition 2.1. Given a partition $\lambda = (\mu_1^{i_1}, \dots, \mu_j^{i_j})$ for $G(k, n)$, a *singular partition* λ_s associated to λ is $\lambda_s = (\mu_1^{i_1}, \dots, (\mu_s + 1)^{i_s+1}, \mu_{s+1}^{i_{s+1}-1}, \dots, \mu_j^{i_j})$ for some $1 \leq s < j$ provided that the resulting partition is admissible for $G(k, n)$.

Remark 2.2. In terms of Young diagrams, the singular partitions are obtain from λ by adding a hook to λ so that the resulting partition is still admissible for $G(k, n)$.

The exceptional locus of the map $\pi|_{\tilde{\Sigma}_\lambda}$ consists of the union of the inverse images of $\Sigma_{\lambda_s}(F_\bullet)$ for every singular partition associated to λ . The locus $\pi^{-1}(\Sigma_{\lambda_s}(F_\bullet))$ is irreducible and has codimension greater than one. (Σ_{λ_s} has codimension $\mu_s - \mu_{s+1} + i_s + 1$ in $\Sigma_\lambda(F_\bullet)$ and the general fiber dimension of π restricted to $\pi^{-1}(\Sigma_{\lambda_s}(F_\bullet))$ is i_s .) Furthermore, the exceptional locus of π is empty if and only if $j = 1$ or $j = 2$ and $\mu_1 = n - k$. The following lemma allows us to determine the singular locus of a Schubert variety.

Lemma 2.3. *Let $f : X \rightarrow Y$ be a birational morphism from a smooth, projective variety X onto a normal projective variety Y . Assume that f is an isomorphism in codimension one. Then $p \in Y$ is a singular point if and only if $f^{-1}(p)$ is positive dimensional.*

Proof. By Zariski's Connectedness Theorem ([Ha], III.11.4), if $f^{-1}(p)$ is not positive dimensional, then it is a point. Consider the open set U in Y consisting of the locus where $f^{-1}(p)$ is a single point. Then, by Zariski's Main Theorem, $f|_{f^{-1}(U)} : f^{-1}(U) \rightarrow U$ is an isomorphism. Since $f^{-1}(U)$ is a Zariski open subset of a smooth variety, $f^{-1}(U)$ and consequently U is smooth. Conversely, suppose p is a point such that $f^{-1}(p)$ is positive dimensional. If p is smooth, then to check that the map f is a local isomorphism, it suffices to check that the Jacobian does not vanish. Since the map f is an isomorphism in codimension one and the vanishing locus of the Jacobian of f is a divisor, we conclude that the Jacobian does not vanish. On the other hand, since f is not a local isomorphism around $f^{-1}(p)$, we conclude that p has to be a singular point. \square

We obtain the following two corollaries due to Lakshmibai and Seshadri ([LS]).

Corollary 2.4 ([LS]). *A Schubert variety Σ_λ in $G(k, n)$ is smooth if and only if it is a linearly embedded sub-Grassmannian, or equivalently, $\lambda = ((n - k)^s, p^{k-s})$.*

Corollary 2.5 ([LS]). *The singular locus of a Schubert variety $\Sigma_\lambda(F_\bullet)$ in $G(k, n)$ is the union of all the Schubert varieties $\Sigma_{\lambda_s}(F_\bullet)$, where λ_s is a singular partition associated to λ .*

Observation 2.6. In particular, by Kleiman's Transversality Theorem ([K11]), the intersection of a collection of general Schubert varieties $\Sigma_{\lambda^1}, \dots, \Sigma_{\lambda^j}$ is smooth and non-empty provided that $\sigma_{\lambda^1} \cdots \sigma_{\lambda^j} \neq 0$ and $\sigma_{\lambda_s^i} \cdot \prod_{l \neq i} \sigma_{\lambda^l} = 0$ for every $1 \leq i \leq j$ and every singular partition λ_s^i associated to λ^i .

Using this Observation 2.6, we can prove Proposition 1.9.

Proof of Proposition 1.9. The class $r\sigma_1$, with $r > 0$, is represented by the intersection of $G(k, n)$ in its Plücker embedding with a hypersurface of degree r . Hence, $r\sigma_1^m$ is represented by the intersection of $G(k, n)$ with a general hypersurface of degree r and $m - 1$ general hyperplanes. By Bertini's Theorem ([Ha], II.8.18), this intersection is smooth. This proves (1).

(2) is a special case of Observation 2.6. Let $a_s = \sum_{l=1}^s i_l$. Let F_\bullet and G_\bullet be two general flags defining Σ_λ and Σ_μ , respectively. If $b_i + c_{j-i} \geq n - k$, then $F_{n-k+a_i-b_i} \cap G_{n-a_i-c_{j-i}} = \{0\}$ for every i . Consequently, the intersection of the two Schubert varieties $\Sigma_\lambda \cap \Sigma_\mu$ is isomorphic to the product of Grassmannians

$\prod_l G(i_l, n_l)$, where $n_l = n - k + i_l - b_l - c_{j-l+1}$ for $1 \leq l \leq j$. Hence, this intersection is smooth.

The Schubert cycles $\sigma_{(n-k)^{k-1}, 0}$ and $\sigma_{(n-k-1)^k}$ are projective spaces \mathbb{P}^{n-k} and \mathbb{P}^k , respectively. Smooth complete intersections of type $(r, 1^{a-1})$ and $(r, 1^{s-1})$ in these projective spaces have cohomology classes $r\sigma_{(n-k)^{k-1}, a}$ and $r\sigma_{(n-k)^s, (n-k-1)^{k-s}}$, respectively. This proves (3).

The incidence correspondence $I := \{(L, X) \mid L \subset X\}$, where L is a linear space and X is a complete intersection of degrees d_1, \dots, d_j is smooth. By generic smoothness ([Ha], III.10.7), the general fiber of the second projection is empty or smooth. This proves the first part of (4). The second part of (4) is well-known.

Let V_i of dimension n_i be j general linear spaces in an $n = \sum_{i=1}^j n_i$ dimensional linear space. Since V_i are general, the span of any $j-1$ of them intersects the last one trivially. Given varieties $Y_i \subset G(k_i V_i)$, consider the following variety

$$Y = \{[W] \in G(k, n) \mid [W \cap V_i] \in Y_i\}.$$

Then Y is isomorphic to $\prod_{i=1}^j Y_i$. Given a point $(W_1, \dots, W_j) \in \prod_{i=1}^j Y_i$, then the span $\overline{W_1 \cdots W_j}$ is a k -dimensional subspace in Y . Conversely, for $W \in Y$, $(W \cap V_1, \dots, W \cap V_j) \in \prod_{i=1}^j Y_i$. It is clear that these maps are inverses of each other. The statement about the cohomology classes is straightforward. This completes the proof of Proposition 1.9. \square

3. SMOOTH SCHUBERT VARIETIES ARE RIGID

Proposition 3.1. *Let λ be a partition for $G(k, n)$ such that $\lambda_1 = \dots = \lambda_s = n - k$ and $\lambda_k = r$. Let X be a subvariety of $G(k, n)$ representing the cohomology class σ_λ . Then the linear spaces parameterized by X contain a fixed s -dimensional linear space and are contained in a fixed $(n - r)$ -dimensional linear space.*

Proof. For the proof of this proposition, it is more convenient to think of the Grassmannian $G(k, n)$ as the Grassmannian $\mathbb{G}(k-1, n-1)$ of projective $(k-1)$ -dimensional linear spaces in \mathbb{P}^V . Let λ be a partition with $\lambda_s = n - k$ and $\lambda_k = r$. Let X be a variety representing the class σ_λ . Consider the incidence correspondence

$$I_X = \{(p, [W]) \mid [W] \in X, p \in W\} \subset \mathbb{P}^{n-1} \times \mathbb{G}(k-1, n-1).$$

Let π_1 and π_2 denote the two projections to \mathbb{P}^{n-1} and $\mathbb{G}(k-1, n-1)$, respectively. Since a Schubert class is indecomposable, X is an irreducible variety. The fibers of π_2 are projective spaces of dimension $k-1$, hence I_X is irreducible. Consequently, the first projection $\pi_1(I_X)$ is an irreducible, projective variety in \mathbb{P}^{n-1} .

Let $L \cong \mathbb{P}^{r-1}$ be a general linear space. The class of the locus in $\mathbb{G}(k-1, n-1)$ that parameterizes the set of \mathbb{P}^{k-1} that intersect L is the Pieri class $\sigma_{n-k-r+1}$. Since the cup product of this class with σ_λ is zero, we conclude that a general linear space of dimension $r-1$ does not intersect $\pi_1(I_X)$. Therefore, $\dim(\pi_1(I_X)) \leq n-r-1$. On the other hand, by Pieri's formula ([?], 14.6.1), the cup product of σ_λ with σ_{n-k-r} is not zero, hence $\dim(\pi_1(I_X)) = n-r-1$. In order to conclude that the linear spaces parameterized by X are contained in a fixed codimension r linear space, it suffices to prove that the degree of $\pi_1(I_X)$ (with its reduced induced structure) is equal to one.

Let λ^* denote the partition dual to λ given by $\lambda_i^* = n - k - \lambda_{k-i+1}$. Since the cup product of X with σ_{λ^*} is equal to the Poincaré dual of the point class, by Kleiman's Transversality Theorem ([K11]), a general Schubert variety representing

σ_{λ^*} intersects X in a reduced point. Furthermore, since $\pi_1(I_X)$ is a variety of dimension $n - r - 1$, a general projective linear space of dimension r intersects $\pi_1(I_X)$ in finitely many reduced points.

Fix a general linear space $\mathbb{P}F_{r+1} \cong \mathbb{P}^r$ such that the intersection of $\mathbb{P}F_{r+1}$ with $\pi_1(I_X)$ consists of finitely many reduced points q_1, \dots, q_j and there exists a partial flag F_\bullet beginning with F_{r+1} such that the Schubert variety with class σ_{λ^*} defined with respect to F_\bullet intersects X in a reduced point. By the transitivity of the monodromy group ([HM], 6.45), we may assume that any geometric property that occurs at one of the intersection points q_i also occurs at the others. By construction, there is a linear space $[\Lambda] \in X$ such that $[\Lambda] = X \cap \Sigma_{\lambda^*}(F_\bullet)$. We must necessarily have $\mathbb{P}\Lambda \cap \mathbb{P}F_{r+1} = q_i$ for one of the intersection points $q_i \in \pi_1(I_X) \cap \mathbb{P}F_{r+1}$. In particular, by the transitivity of the monodromy group, we may assume that for each q_i in $\mathbb{P}F_{r+1} \cap \pi_1(I_X)$, there exists a linear space Λ_{q_i} parameterized by X and a Schubert cycle $\Sigma_{\lambda^*}(F_\bullet^{q_i})$ defined with respect to a partial flag containing F_{r+1} such that X and $\Sigma_{\lambda^*}(F_\bullet^{q_i})$ intersect transversely at $[\Lambda_{q_i}]$. Note that $\dim(\Lambda_{q_i} \cap F_{r+1}) = 1$.

The locus of partial flags

$$G_\bullet = F_{r+1} = G_{n-k+1-\lambda_1^*} \subset G_{n-k+2-\lambda_2^*} \cdots \subset G_{n-\lambda_k^*}$$

starting with the fixed general linear space F_{r+1} such that the Schubert variety $\Sigma_{\lambda^*}(G_\bullet)$ intersects X transversely is a non-empty, Zariski-open subset U of a partial flag variety. Since a non-empty Zariski open subset of an irreducible variety is connected, U is connected. There is a natural morphism from U to the points of intersection $\mathbb{P}F_{r+1} \cap \pi_1(I_X)$ sending a partial flag G_\bullet to $\mathbb{P}(\Lambda \cap F_{r+1})$, where $[\Lambda] = X \cap \Sigma_{\lambda^*}(G_\bullet)$. This map surjects onto $\mathbb{P}F_{r+1} \cap \pi_1(I_X)$. Since the image of a connected set is connected, we conclude that the intersection of $\pi_1(I_X)$ with $\mathbb{P}F_{r+1}$ is connected. Consequently, $\mathbb{P}F_{r+1} \cap \pi_1(I_X)$ is a single point. Therefore, $\pi_1(I_X)$ has degree one and is a linear space. This proves that the linear spaces parameterized by X have to be contained in a fixed codimension r linear space.

The fact that the linear spaces parameterized by X contain a fixed s -dimensional linear space follows by duality. The Grassmannian $G(k, n)$ is isomorphic to the Grassmannian $G(n-k, n)$. Under this isomorphism a Schubert class σ_λ is replaced by $\sigma_{\lambda'}$, where λ' is the partition given by the columns of the Young diagram associated to λ . In particular, λ' satisfies $\lambda'_{n-k} = s$. By the previous paragraph, all the linear spaces parameterized by the image of X under this isomorphism are contained in a fixed codimension s linear space. Under the duality, this linear space produces a fixed linear space of dimension s contained in all the linear spaces parameterized by X . This concludes the proof. \square

As a corollary, we recover a well-known rigidity result (see [Br], [Ho1] or [Ho2]).

Corollary 3.2. *Any proper subvariety of $G(k, n)$ representing the Schubert class σ_λ with $\lambda = ((n-k)^s, p^{k-s})$ is a Schubert variety. Consequently, the smooth Schubert varieties in Grassmannians are rigid.*

Proof. Let X be a variety representing the cohomology class σ_λ where $\lambda = ((n-k)^s, p^{k-s})$. By Proposition 3.1, every linear space parameterized by X contains a fixed linear space of dimension s and is contained in a fixed linear space of codimension p . The dimension of X is $(k-s)p$. Since the Schubert variety parameterizing

k -dimensional linear spaces containing a fixed s -dimensional linear space and contained in a fixed codimension p linear space is irreducible of dimension $(k-s)p$ and contains X , we conclude that X must be equal to the Schubert variety. \square

Remark 3.3. In [Br] and [Ho1], Bryant and Hong, using more involved differential geometric arguments, prove that smooth Schubert varieties other than non-maximal linear spaces in Grassmannians are Schur rigid.

Next we would like to define some operations on partitions that will allow us to state a reduction lemma.

Definition 3.4. Let $\lambda = (\lambda_1, \dots, \lambda_k)$ be a partition for $G(k, n)$.

- (1) If $\lambda_1 < n - k$, then define the partition $\lambda + \underline{1}$ to be the partition $(\lambda_1 + 1, \dots, \lambda_k + 1)$ for $G(k, n)$ obtained by adding one to each part.
- (2) If $\lambda_k = 0$, define the partition λ^+ to be the partition $(n - k, \lambda_1, \dots, \lambda_{k-1})$ for $G(k, n)$ obtained by omitting the part $\lambda_k = 0$ and adding the part $n - k$ to the resulting partition.
- (3) If $\lambda_1 = n - k$, define the partition λ^- to be the partition for $G(k - 1, n - 1)$ obtained by omitting the part λ_1 from λ .

Lemma 3.5 (Reduction Lemma). *Let σ_λ be a Schubert class in $G(k, n)$.*

- (1) *If $\lambda_1 < n - k$ and $\sigma_{\lambda + \underline{1}}$ cannot be represented by a smooth, proper subvariety of $G(k, n)$, then σ_λ cannot be represented by a smooth, proper subvariety of $G(k, n)$.*
- (2) *If $\lambda_k = 0$ and σ_{λ^+} cannot be represented by a smooth, proper subvariety of $G(k, n)$, then σ_λ cannot be represented by a smooth, proper subvariety of $G(k, n)$.*
- (3) *If $\lambda_1 = n - k$ and σ_{λ^-} cannot be represented by a smooth, proper subvariety of $G(k - 1, n - 1)$, then σ_λ cannot be represented by a smooth, proper subvariety of $G(k, n)$.*

Proof. Suppose that $\lambda_1 < n - k$ and that X is a smooth, proper subvariety representing σ_λ . By Kleiman's Transversality Theorem, the intersection of X with a general smooth Schubert variety $\Sigma_{(1^k)}$ is a smooth, proper subvariety of $G(k, n)$. By the Littlewood-Richardson rule ([?], 14.6.2), $X \cap \Sigma_{(1^k)}$ represents the cohomology class $\sigma_{\lambda + \underline{1}}$. This proves Part (1) of the lemma.

Suppose that $\lambda_k = 0$ and that X is a smooth, proper subvariety representing σ_λ . By Kleiman's Transversality Theorem, the intersection of X with a general smooth Schubert variety $\Sigma_{(n-k, 0^{k-1})}$ is a smooth, proper subvariety of $G(k, n)$. This variety represents the class σ_{λ^+} . This proves Part (2) of the lemma.

Finally, if $\lambda_1 = n - k$, by Proposition 3.1, any variety X representing the class σ_λ is contained in a linearly embedded sub-Grassmannian $G(k - 1, n - 1)$. The class of X in $G(k - 1, n - 1)$ is σ_{λ^-} . Part (3) of the lemma follows. \square

4. RIGIDITY OF SCHUBERT VARIETIES

Lemma 4.1. *Let X be a proper variety representing a Schubert class σ_λ . Then there exists a flat, one-parameter family $\mathcal{Y} \rightarrow B$ over a smooth curve such that the general fiber is isomorphic to X and the special fiber \mathcal{Y}_0 is isomorphic to a Schubert variety Σ_λ .*

Proof. The Grassmannian $G(k, n)$ has a cell-decomposition by Schubert cells, where each cell is isomorphic to an affine space. Suppose that X is contained in the closure of a cell. If X is equal to the closure of the cell, then X is a Schubert variety. Otherwise, by projecting from a point not on X , X can be degenerated to a variety contained in a cell of lower dimension. Consequently, using the action of a one-parameter subgroup of $GL(n)$, every subvariety of $G(k, n)$ can be degenerated to a scheme supported on a union of Schubert varieties (usually with multiplicities). By the properness of the Hilbert scheme a flat limit always exists. Since the class σ_λ is indecomposable in cohomology, a variety X representing σ_λ is irreducible. Furthermore, the flat limit of the family is supported on a Schubert variety Σ_λ and is generically reduced. Since the Schubert variety Σ_λ is normal, Hironaka's Lemma ([Ha], Theorem III.9.11, [Ko] Theorem 2 or [Hi]) guarantees that there are no embedded components in the limit. \square

Proposition 4.2. *Let $\lambda = ((n-k)^r, (n-k-1)^{k-r-1}, n-k-2)$ with $k-r > 2$ or $\lambda = ((n-k)^{k-2}, n-k-1, r)$ with $r < n-k-2$ be a partition for $G(k, n)$. Then the only proper varieties of $G(k, n)$ representing the Schubert class σ_λ are Schubert varieties. In particular, any variety representing σ_λ in $G(k, n)$ is singular.*

Proof. Let X be a proper subvariety of $G(k, n)$ representing $\sigma_{(n-k)^{k-2}, n-k-1, r}$. We begin by showing that X is a variety of minimal degree. The dimension of X is $n-k-r+1$. By Pieri's formula, the degree of X in the Plücker embedding is $n-k-r$. By Lemma 4.1, X can be specialized to a Schubert variety. By semi-continuity ([Ha], III.12.8), the dimension of the linear span of X is at least the dimension of the linear span of a Schubert variety representing the same class. We will next observe that a Schubert variety with class $\sigma_{(n-k)^{k-2}, n-k-1, r}$ is a minimal variety, i.e., it has the maximal possible dimensional linear span given its degree and dimension. (Recall that an irreducible variety of dimension n and degree d spans a linear space of dimension at most $n+d-1$ ([EH]).) Since X has the same degree and dimension as the Schubert variety and the dimension of its linear span is at least that of the Schubert variety, we conclude that X must be a minimal variety.

To see that the Schubert variety with class $\sigma_{(n-k)^{k-2}, n-k-1, r}$ is a minimal variety, it suffices to show that the linear span is a projective linear space of dimension at least $2n-2k-2r$. Fix an ordered basis e_1, \dots, e_n for the vector space V . Let F_\bullet be flag where F_i is the span of the first i basis elements e_1, \dots, e_i . The Schubert variety $\Sigma_{(n-k)^{k-2}, n-k-1, r}(F_\bullet)$ contains the points

$$p_{1, \dots, k-2, i, j} = e_1 \wedge e_2 \wedge \dots \wedge e_{k-2} \wedge e_i \wedge e_j,$$

where $i \in \{k-1, k\}$, $i < j$ and $j \in \{k, \dots, n-r\}$. Hence, the span of the Schubert variety in the Plücker embedding contains the span of these Plücker coordinate points. Since there are $2n-2k-2r+1$ such Plücker coordinate points, the projective linear space spanned by the Schubert variety has dimension at least $2n-2k-2r$. Since this is the maximal possible, the dimension of the linear span must equal $2n-2k-2r$. We thus conclude that X is a minimal variety in projective space.

By our assumptions, the degree of X is greater than two. Moreover, the dimension of X is larger than its degree. By the classification of varieties of minimal degree, X must be a cone. In particular, X is singular. Suppose that the vertex of the cone is p . (Note that X has a unique singular point. Observe that the Schubert

variety with the same class has a unique singular point. By Proposition 4.1 and the semi-continuity of the dimension of the singular locus, the dimension of the singular locus of X must be zero. Hence, X is a cone with a unique vertex point.)

By Proposition 3.1, X is contained in a linearly embedded sub-Grassmannian $G(2, n - k - r + 2)$ and has class $\sigma_{n-k-1-r,0}$ in this sub-Grassmannian. Since X is a cone, the Zariski tangent space of X at p is equal to the linear span of X , which is a projective linear space of dimension $2n - 2k - 2r$. Since the Zariski tangent space to $G(2, n - k - r + 2)$ at p also has dimension $2n - 2k - 2r$ and contains the Zariski tangent space to X at p , we conclude that $X \subset Y = T_p G(2, n - k - r + 2) \cap G(2, n - k - r + 2)$. Comparing the Plücker equations of a Schubert variety with class $\sigma_{n-k-1-r,0}$ with the equations of the Zariski tangent space to $G(2, n - k - r + 2)$ at p , it is immediate that Y is a Schubert variety with class $\sigma_{n-k-1-r,0}$. Since both X and Y are irreducible, projective varieties of the same dimension, we conclude that they must be equal. Therefore, X is a Schubert variety. The claim about $\sigma_{(n-k)r, (n-k-1)^{k-r-1}, n-k-2}$ follows by the duality between $G(k, n)$ and $G(n-k, n)$. This concludes the proof of the proposition. \square

Proposition 4.3. *Let $\lambda = (a^s, 0^{k-s}) \neq (1, 0^{k-1})$. Then the only proper subvarieties of $G(k, n)$ representing the cohomology class σ_λ are Schubert varieties. In particular, σ_λ cannot be represented by a smooth, proper subvariety of $G(k, n)$ provided $k > s > 0$, $a \neq n - k$ and $(a^s, 0^{k-s}) \neq (1, 0^{k-1})$.*

Proof. We first prove that if $0 < s < k$, $a \neq n - k$ and $(a^s, 0^{k-s}) \neq (1, 0^{k-1})$, then σ_λ is singular. It suffices to prove this for the classes $\sigma_{(n-k-1)^{k-1}, 0}$ in $G(k, n)$. If X is a smooth variety representing $\sigma_{a^s, 0^{k-s}}$, then the intersection of X with a general smooth Schubert variety representing the class $\sigma_{(n-k)^{k-s-1}, (n-k-1-a)^{s+1}}$ yields a smooth variety representing the class $\sigma_{(n-k)^{k-s-1}, (n-k-1)^s, n-k-1-a}$. By Proposition 3.1, this class is supported on a linearly embedded sub-Grassmannian $G(s+1, k+a+1)$ and has the class $\sigma_{(k+a-s-1)^s, 0}$ in this sub-Grassmannian. Note that a Schubert variety $\sigma_{(n-k-1)^{k-1}, 0}$ in $G(k, n)$ embedded in its Plücker embedding is a cone over the Segre embedding of $\mathbb{P}^{k-1} \times \mathbb{P}^{n-k-1}$. By Lemma 4.1, any variety representing this class is a deformation of the cone over the Segre variety. To conclude the proof, we appeal to the following theorem of Thom, Grauert-Kerner, Schelssinger, Kleiman-Landolfi:

Rigidity of cones over Segre varieties. [KL, Theorem 2.2.8] *Let K be a field and let $X = \mathbb{P}_K^n \times \mathbb{P}_K^m$ be embedded in projective space by the Segre morphism. If $n \geq 1$ and $m \geq 2$, then the cone over X is rigid over K .*

By our assumptions, if $k = 2$, then $a > 1$ and $n - k - 1 \geq 2$. Similarly, if $n - k = 2$, then $k > 2$. We conclude that the assumptions of Theorem 2.2.8 of [KL] are satisfied. Since the cone over the Segre variety admits no infinitesimal deformations, X is a cone over a Segre variety and hence is not smooth. Furthermore, the singular point p_0 of X corresponds to a k -plane Λ . Since X is a cone, for every point $p \in X$, the line joining p and p_0 is contained in X . Since a line in the Grassmannian $G(k, n)$ consists of k -planes that contain a fixed $(k-1)$ -plane and are contained in a fixed $(k+1)$ -plane, we conclude that every k -plane corresponding to a point of X must intersect Λ in a $(k-1)$ -dimensional linear space. Hence, X is contained in the Schubert variety $\Sigma_{(n-k-1)^{k-1}, 0}$. Since they are both irreducible varieties of the same dimension, X equals the Schubert variety $\Sigma_{(n-k-1)^{k-1}, 0}$.

By Corollary 3.2, if $s = 0$, $s = k$ or $a = n - k$, a variety X representing σ_λ is a Schubert variety. We can now prove that if $(a^s, 0^{k-s}) \neq (1, 0^{k-1})$, then X is a Schubert variety by induction on n and $k - s$. We may assume that $0 < a < n - k$ and $0 < s < k$. By induction on k and n , the singular locus of X has class $((a + 1)^{s+1}, 0^{k-s-1})$. By induction on $k - s$, the singular locus of X is a Schubert variety of k -planes intersecting a fixed linear space $\Lambda_{n-k+s+1-a}$ of dimension $n - k + s + 1 - a$. Intersect X with a general Schubert variety Σ_{1^k} defined with respect to a hyperplane H . By induction on n , the intersection $X \cap \Sigma_{1^k}$ is a Schubert variety in $G(k, n - 1)$ with class $(a^s, 0^{k-s})$. Since the singular locus of $X \cap \Sigma_{1^k}$ is the Schubert variety of k -planes intersecting $\Lambda \cap H$ in a subspace of dimension $n - k + s + 1 - a$, we conclude that $X \cap \Sigma_{1^k}$ is the Schubert variety of k -planes intersecting $\Lambda \cap H$ in a subspace of dimension $n - k + s - a$. Since this is true for a general hyperplane H , we conclude that the k -planes parameterized by X intersect Λ in a subspace of dimension $n - k + s - a$. Therefore, X is a Schubert variety. This concludes the proof. \square

Remark 4.4. Example 13 and Remark 33 of [Br] or Proposition 3.5 of [Ho2] show that when $\lambda = (a^s, 0^{k-s})$ with $1 < s < k$ and $a > 2$, the Schubert classes are Schur rigid.

Proof of Theorem 1.6. We are now ready to prove Theorem 1.6. We need to show that if λ is a non-smoothable partition for $G(k, n)$, then σ_λ cannot be represented by a smooth subvariety of $G(k, n)$. Express $\lambda = (\mu_1^{i_1}, \dots, \mu_j^{i_j})$, where $\sum i_l = k$ and $\mu_1 > \mu_2 > \dots > \mu_j$.

We may assume that $\mu_1 \neq n - k$. Otherwise, by Proposition 3.1, any subvariety of $G(k, n)$ representing σ_λ is contained in a linearly embedded sub-Grassmannian $G(k - i_1, n - i_1)$ and represents the class $(\mu_2^{i_2}, \mu_3^{i_3}, \dots, \mu_j^{i_j})$ in this sub-Grassmannian. If the latter class cannot be represented by a smooth subvariety, then σ_λ cannot be represented by a smooth subvariety of $G(k, n)$. Similarly, we may assume that $\mu_j = 0$. Otherwise, by Proposition 3.1, any subvariety representing σ_λ is contained in a linearly embedded sub-Grassmannian $G(k, n - \mu_j)$ and represents the class $((\mu_1 - \mu_j)^{i_1}, (\mu_2 - \mu_j)^{i_2}, \dots, (\mu_{j-1} - \mu_j)^{i_{j-1}}, 0^{i_j})$ in this sub-Grassmannian. If the latter class cannot be represented by a smooth subvariety, then σ_λ cannot be represented by a smooth subvariety of $G(k, n)$. Now we proceed by induction on j .

- If $j = 1$, then λ is not non-smoothable and the theorem holds. In fact, the Schubert variety Σ_λ is a linearly embedded sub-Grassmannian and smooth. We may, therefore, assume that $j > 1$.
- If $j = 2$, since we are assuming that $\mu_1 < n - k$, λ is non-smoothable if $i_1 > 1$ or if $\mu_1 > \mu_2 + 1$. In either of these cases, by Proposition 4.3, any variety representing the class σ_λ is a singular Schubert variety. Note that if $\mu_1 = n - k$, then σ_λ is the class of a linearly embedded sub-Grassmannian and can be represented by a smooth subvariety of $G(k, n)$. If $i_1 = 1$ and $\mu_1 = \mu_2 + 1$, then σ_λ is the class of a hyperplane section of a linearly embedded sub-Grassmannian and can be represented by a smooth subvariety of $G(k, n)$.
- If $j = 3$, since we are assuming that $\mu_1 < n - k$, then a non-smoothable partition λ satisfies one of the following possibilities.

- If $\mu_2 > \mu_3 + 1$ or $i_2 > 1$, then applying Case (1) of the Lemma 3.5 $(n - k - \mu_1)$ -times we conclude that if σ_λ can be represented by a smooth subvariety of $G(k, n)$, then $\sigma_{\lambda'}$, with $\lambda' = ((n - k)^{i_1}, (n - k + \mu_2 - \mu_1)^{i_2}, (n - k + \mu_3 - \mu_1)^{i_3})$, can be represented by a smooth subvariety of $G(k, n)$. Next, applying Case (3) of the Lemma 3.5 i_1 -times, we conclude that $\sigma_{\lambda'}$ can be represented by a smooth subvariety of $G(k, n)$, then $\sigma_{\lambda''}$ with $\lambda'' = ((n - k + \mu_2 - \mu_1)^{i_2}, (n - k + \mu_3 - \mu_1)^{i_3})$ can be represented by a smooth subvariety of $G(k - i_1, n - i_1)$. By the case $j = 2$, $\sigma_{\lambda''}$ cannot be represented by a smooth subvariety of $G(k - i_1, n - i_1)$. Therefore, σ_λ cannot be represented by a smooth subvariety of $G(k, n)$. From now on we may assume that $\mu_2 = \mu_3 + 1$ and $i_2 = 1$.
- If $\mu_1 > \mu_2 + 1$ or $i_1 > 1$, since we are assuming that $\mu_3 = 0$, by applying Cases (2) and (3) of Lemma 3.5 (i_3) -times, we conclude that if σ_λ can be represented by a smooth subvariety of $G(k, n)$, then $\sigma_{\lambda'}$ with $\lambda' = (\mu_1^{i_1}, \mu_2^{i_2})$ can be represented by a smooth subvariety of $G(k - i_3, n - i_3)$. Since by induction $\sigma_{\lambda'}$ cannot be represented by a smooth subvariety of $G(k - i_3, n - i_3)$, we conclude that σ_λ cannot be represented by a smooth subvariety of $G(k, n)$.

Note that if $\mu_1 = n - k$, $i_2 = 1$ and $\mu_2 = \mu_3 + 1$, then σ_λ is the class of a hyperplane section of a linearly embedded sub-Grassmannian and can be represented by a smooth subvariety of $G(k, n)$. Similarly, if $i_1 = i_2 = i_3 = 1$ and $\mu_i = \mu_{i+1} + 1$, then σ_λ can be represented by a smooth subvariety of $G(3, n)$. If $\mu_3 = r$, then σ_λ is the class of the variety parameterizing Schubert cycles $\sigma_{r-3, r-3}$ in the Plücker embedding of $G(2, r)$ that are contained in a general hyperplane section.

- Finally, if $j > 3$, we can easily reduce to the case $j = 3$ by induction. Let λ be a non-smoothable partition. Suppose that for $s < j - 1$, $i_s > 1$ or $\mu_s > \mu_{s+1} + 1$. Then applying Cases (2) and (3) of Lemma 3.5 (i_j) -times, we conclude that if σ_λ can be represented by a smooth subvariety of $G(k, n)$, then $\sigma_{\lambda'}$ with $\lambda' = (\mu_1^{i_1}, \mu_2^{i_2}, \dots, \mu_{j-1}^{i_{j-1}})$ can be represented by a smooth subvariety of $G(k - i_j, n - i_j)$. Since by induction on j , $\sigma_{\lambda'}$ cannot be represented by a smooth subvariety of $G(k - i_j, n - i_j)$, we conclude that σ_λ cannot be represented by a smooth subvariety of $G(k, n)$. Similarly, if $i_{j-1} > 1$ or $\mu_{j-1} > \mu_j + 1$, then applying Case (1) of Lemma 3.5 $(n - k - \mu_1)$ -times followed by Case (3) of Lemma 3.5 i_1 -times, we conclude that if σ_λ can be represented by a smooth subvariety of $G(k, n)$, then $\sigma_{\lambda'}$ with $\lambda' = ((n - k + \mu_2 - \mu_1)^{i_2}, \dots, (n - k + \mu_{j-1} - \mu_1)^{i_{j-1}}, (n - k + \mu_j - \mu_1)^{i_j})$ can be represented by a smooth subvariety of $G(k - i_1, n - i_1)$. Since, by induction on j , $\sigma_{\lambda'}$ cannot be represented by a smooth subvariety, σ_λ cannot be represented by a smooth subvariety of $G(k, n)$. This concludes the proof of the theorem. \square

If σ_λ can be represented by a smooth subvariety X in $G(k, n)$, then, by Theorem 1.6, λ has the form $((n - k)^{i_1}, p, p - 1, p - 2, \dots, p - s + 1, (p - s)^{i_{s+2}})$. Furthermore, Proposition 3.1 implies that the Schubert class $\sigma_{\lambda'}$ with $\lambda' = (s, s - 1, s - 2, \dots, 1, 0^{i_{s+2}})$ is also representable by a smooth subvariety of $G(k - i_1, n - p + s)$. Hence, we can restrict our discussion to classes σ_λ with $\lambda = (s, s - 1, \dots, 1, 0^l)$ in $G(s + l, n)$. The cup product of the cohomology class $[X]$ of X and the Schubert cycle σ_ν , with $\nu = ((n - s - l)^{l+1}, 0^{s-1})$, is zero. We conclude that the incidence

correspondence

$$I = \{(W_1, W_2) | W_1 \subset W_2, [W_2] \in X\} \subset G(l+1, n) \times X$$

of pairs of an $(l+1)$ -dimensional subspace W_1 and an $(s+l)$ -dimensional subspace W_2 parameterized by X and containing W_1 cannot dominate $G(l+1, n)$ under the first projection π_1 . On the other hand, considering the cup product of $[X]$ and $\sigma_{\nu'}$, with $\nu' = ((n-s-l)^l, (n-s-l-1), 0^{s-1})$, it is easy to see that the image of π_1 is a divisor in $G(l+1, n)$ with class σ_1 . Consequently, X parameterizes Schubert varieties with class $\sigma_{(n-s-l)^{l+1}}$ contained in a hyperplane section of $G(l+1, n)$ in its Plücker embedding. Hence, a Schubert cycle cannot be represented by a smooth subvariety of $G(k, n)$ unless it is the class of a linearly embedded sub-Grassmannian or the class of a variety parameterizing linearly-embedded sub-Grassmannians in a hyperplane section of a Grassmannian. Moreover, as discussed in the Introduction, the classes with $l = 1$ can always be represented by smooth subvarieties. Consequently, for Grassmannians $G(k, n)$ with $k = 2$ or 3 , Theorem 1.6 can be stated as follows.

Corollary 4.5. *Let $\sigma_{\lambda_1, \lambda_2}$ be a Schubert class in $G(2, n)$. Then the following are equivalent.*

- (1) $\sigma_{\lambda_1, \lambda_2}$ can be represented by a smooth subvariety of $G(2, n)$.
- (2) $\lambda_1 = n - 2$ or $\lambda_2 \geq \lambda_1 - 1$.
- (3) $\sigma_{\lambda_1, \lambda_2}$ is the class of a linearly embedded sub-Grassmannian or a hyperplane section of a linearly embedded sub-Grassmannian.

Corollary 4.6. *Let σ_λ be a Schubert class in $G(3, n)$. Then the following are equivalent.*

- (1) σ_λ can be represented by a smooth subvariety of $G(3, n)$.
- (2) λ is not a non-smoothable partition.
- (3) σ_λ is the class of a linearly embedded sub-Grassmannian, the hyperplane section of a linearly embedded sub-Grassmannian or the class of planes with class $\sigma_{k-3, k-3}$ contained in a general hyperplane section of the Plücker embedding of $G(2, k)$ for $k \leq n$.

Finally, we prove Theorem 1.3.

Proof of Theorem 1.3. First, observe that the condition for a partition $\lambda = (\mu_1^{i_1}, \dots, \mu_j^{i_j})$ to have an index $1 \leq s < j$ with $i_s = 1$ and $n - k > \mu_s = \mu_{s+1} + 1$ is invariant under the duality that exchanges Schubert classes in $G(k, n)$ and $G(n - k, n)$.

Let $\lambda = (\mu_1^{i_1}, \dots, \mu_j^{i_j})$ be a partition such that there exists an index $1 \leq s < j$ with $i_s = 1$ and $n - k > \mu_s = \mu_{s+1} + 1$. We construct a variety representing the cohomology class σ_λ , which is not isomorphic to a Schubert variety. The construction is identical to the construction of the variety X in Example 1.4.

First suppose that $s = 1$. Let $a_l = \sum_{m=1}^l i_m$. Fix a $(j-1)$ -step partial flag F_\bullet with vector spaces of dimensions $n - k + a_i - \mu_i$ for $2 \leq i \leq j$. Fix a hyperplane section H in the Plücker embedding of $G(a_2, F_{n-k+a_2-\mu_2})$. Consider the following variety

$$Y := \{[W] \in G(k, n) | W' \subset W \cap F_{n-k+a_2-\mu_2} \text{ for } [W'] \in H, \text{ and } \dim(W \cap F_{n-k+a_i-\mu_i}) \geq a_i, 2 \leq i \leq j\}.$$

Then Y represents the Schubert class σ_λ , but is not a Schubert variety unless the hyperplane section H is a Schubert variety. For example, if H is a smooth

hyperplane section, then the singular locus of Y has a different cohomology class than the singular locus of the Schubert variety. Y is not singular along a Schubert variety with class $\sigma_{(\mu_1+1)^2, \mu_2^{i_2-1}, \mu_3^{i_3}, \dots, \mu_j^{i_j}}$.

More generally, suppose $s > 1$. Let F be a vector space of dimension $n - k + a_{s+1} - \mu_{s+1}$. Consider the class $((\mu_1 - \mu_{s+1})^{i_1}, \dots, (\mu_s - \mu_{s+1}), 0^{i_{s+1}})$ in $G(a_{s+1}, F)$. If we consider this class under the isomorphism between $G(\dim(F) - a_{s+1}, F)$ and $G(a_{s+1}, F)$, we get a class where $s = 1$. Hence, by the previous construction, this class can be represented by a variety Z which is not a Schubert variety. Pick a partial flag

$$F \subset F_{n-k+a_{s+2}-\mu_{s+2}} \subset \dots \subset F_{n-k+a_j-\mu_j}.$$

The variety Y defined as follows has the class $(\mu_1^{i_1}, \dots, \mu_j^{i_j})$, but is not a Schubert variety.

$$Y := \{W \in G(k, n) \mid U \subset W \cap F \text{ for some } U \in Z \text{ and } \dim(W \cap F_{n-k+a_l-\mu_l}) \geq a_l \text{ for } s+1 \leq l \leq j\}.$$

This proves that if a Schubert class is rigid, then $\lambda = (\mu_1^{i_1}, \dots, \mu_j^{i_j})$ cannot have an index s with $i_s = 1$ and $n - k > \mu_s = \mu_{s+1} + 1$.

Suppose that $\lambda = (\mu_1^{i_1}, \dots, \mu_j^{i_j})$ is a partition that does not have an index $1 \leq s < j$ such that $i_s = 1$ and $n - k > \mu_s = \mu_{s+1} + 1$. Let X be a variety that represents σ_λ . We would like to prove that X is a Schubert variety. We will prove this by induction on j , n and the sequence i_1, \dots, i_j . On the sequences (i_1, \dots, i_j) with $\sum i_s = k$, we use the ordering $(i'_1, \dots, i'_l) < (i_1, \dots, i_j)$ if $l < j$ or if $l = j$, $i'_s = i_s$ for $s < m$ and $i'_m > i_m$ for some $m > 0$. If $j = 1$, then by Corollary 3.2, X is a Schubert variety.

If $j = 2$, then by Proposition 4.3, X is a Schubert variety. Suppose that the proposition is true for all j up to j_0 . If $\mu_1 = n - k$ or $\mu_j > 0$, by Proposition 3.1, we may assume that X is contained in a linearly embedded sub-Grassmannian. Hence, by induction on j and n , we can reduce to the case $\mu_1 < n - k$ and $\mu_j = 0$. By intersecting X with a general Schubert variety Σ_{n-k} and a general Schubert variety Σ_{1^k} and induction on i_j and n , we conclude that the singular locus of X has cohomology class $\sum \sigma_{\lambda_s}$, where λ_s ranges over all singular partitions associated to λ . In Proposition 4.1, the deformation is given by the group action. Hence, except for the central fiber, the members of the family are isomorphic. In particular, each irreducible component of the singular locus forms a flat family away from the central fiber. The flat limit for each irreducible component has to be contained in the singular locus of the Schubert variety and must be connected in codimension one. Since the singular locus of X and the Schubert variety have the same cohomology class and the components of the singular locus of the Schubert variety in loci of codimension greater than one, we conclude that the singular locus of X must be a union of irreducible components one for each singular partition associated to λ .

In particular, the singular locus of X contains a variety Y with class $\sigma_{\lambda'}$, where

$$\lambda' = (\mu_1^{i_1}, \dots, (\mu_{t-1} + 1)^{i_{t-1}+1}, \mu_t^{i_t-1}, \dots, \mu_j^{i_j}).$$

Since $\mu_1 < n - k$, we may set $t = 2$. Note that λ' is also a rigid partition unless $i_2 = 2$ and $\mu_2 = \mu_3 + 1$. Assume first that either $i_2 > 2$ or $\mu_2 > \mu_3 + 1$. Then, by induction on the sequence (i_1, \dots, i_j) , Y is a Schubert variety in $G(k, n)$ defined with respect to a flag

$$F_\bullet = F_{n-k+a_1-\mu_1} \subset F_{n-k+a_2-\mu_2} \subset \dots \subset F_{n-k+a_j-\mu_j}.$$

We have to make one exception: if $i_2 = 1$, then $F_{n-k+a_2-\mu_2}$ should be omitted from the sequence. Let us assume that $i_2 > 1$. Then requiring k -planes to intersect $F_{n-k+a_i-\mu_i}$ in a subspace of dimension a_i defines a Schubert variety Σ_λ with the same class as X . Intersect the partial flag F_\bullet with a general codimension one linear space L to obtain a partial flag in L . Correspondingly, intersect the variety X with the Schubert variety Σ_{1^k} defined with respect to L . This gives rise to a variety Z_L with class σ_λ in $G(k, n-1)$. By induction on n , Z_L is a Schubert variety and it is defined with respect to the partial flag $F_\bullet \cap L$. As we vary L , the varieties Z_L cover X and we conclude that any linear space parameterized by X intersects the linear space $F_{n-k+a_i-\mu_i}$ in a subspace of dimension at least a_i . Since both X and Σ_λ are irreducible varieties of the same dimension and X is contained in Σ_λ , we conclude that $X = \Sigma_\lambda$.

If $i_2 = 1$, the same argument shows that the linear spaces parameterized by X intersect $F_{n-k+a_i-\mu_i}$ in a subspace of dimension a_i with the exception of $i = 2$ (which is omitted from the partial flag F_\bullet). If $j > 3$, let R be the Zariski closure of the variety R^0 in $G(k - i_j, F_{n-k+a_{j-1}-\mu_{j-1}})$ defined by

$$R^0 = \{W' \in G(k - i_j, F_{n-k+a_{j-1}-\mu_{j-1}}) \mid \\ W' = W \cap F_{n-k+a_{j-1}-\mu_{j-1}} \text{ for } W \in X \text{ with } \dim(W \cap F_{n-k+a_{j-1}-\mu_{j-1}}) = k - i_j\}.$$

Then R has cohomology class σ_ν with $\nu = (\mu_1^{i_1}, \dots, \mu_{j-1}^{i_{j-1}})$ and is a Schubert variety by induction on j . It follows that X is also a Schubert variety. If $j = 3$, consider the singular locus Y of X with class $\sigma_{\lambda'}$, where $\lambda' = (\mu_1^{i_1}, (\mu_2 + 1)^{i_2+1}, (\mu_3 - 1)^{i_3-1})$. By induction on i_3 , Y is a Schubert variety (even when $i_1 = 1$ and $\mu_1 = \mu_2 + 2$ since all the k -planes are required to intersect $F_{n-k+a_1-\mu_1}$ in a subspace of dimension a_1). Y determines the missing partial flag element $F_{n-k+a_2-\mu_2}$. The argument in the previous paragraph allows us to conclude that X is a Schubert variety.

Finally, suppose $i_2 = 2$ and $\mu_2 = \mu_3 + 1$. Then let $u < j$ be the smallest index such that $i_u > 2$ or $\mu_u > \mu_{u+1} + 1$. If there does not exist such an index, then $i_s = 2$ and $\mu_s = \mu_{s+1} + 1$ for all $1 < s < j$. Using the duality between $G(k, n)$ and $G(n - k, n)$, we are reduced to the case in the previous paragraph. We may, therefore, assume that there exists such an index u . Repeating the argument for the index $t = u$ instead of $t = 2$, it is easy to conclude that X is a Schubert variety. This concludes the proof of the theorem. \square

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