LECTURE 5

1. Isotropic Grassmannians

In this section, we discuss the geometry of isotropic Grassmannians. Let V be an n-dimensional complex vector space. Let Q be a non-degenerate quadratic form. The form Q may either be symmetric or skew-symmetric. Since the rank of a skew-symmetric form is always even, in the latter case, we must assume that n is even. There are some differences in the discussion depending on whether Q is symmetric or skew-symmetric. Furthermore, when Q is symmetric, there are slight variations in detail depending on whether n is even or odd. We will now discuss these cases separately.

1.1. **Preliminaries on quadrics.** Let Q be a smooth quadric hypersurface in $\mathbb{P}^{n-1}$. Set $m = \lfloor \frac{n}{2} \rfloor$. The largest dimensional linear spaces contained in Q have projective dimension $m-1$. If n is odd, then the maximal dimensional linear spaces on Q form an irreducible family of dimension $\frac{m(m+1)}{2}$. If n is even, then the maximal dimensional linear spaces contained in Q form two isomorphic families of dimension $\frac{m(m-1)}{2}$. Two linear spaces belong to the same irreducible component if and only if their dimension of intersection is equal to $m-1$ modulo 2 (see [GH] p. 735).

More generally, we will be interested in linear spaces on quadric hypersurfaces with singularities. A quadric hypersurface in $\mathbb{P}^{n-1}$ of corank r (or, equivalently, with a singular locus of dimension $r-1$) is the cone over a smooth quadric hypersurface in $\mathbb{P}^{n-1-r}$ with vertex an $(r-1)$-dimensional projective linear space. If Q is a quadric hypersurface of corank r in $\mathbb{P}^{n-1}$, then the largest dimensional linear space on Q has dimension $\lfloor \frac{n-r-2}{2} \rfloor + r$. The space of linear spaces of maximal dimension on Q is irreducible if $n-r$ is odd and has two irreducible components if $n-r$ is even. Setting $l = \frac{n-r-3}{2}$ in the former case and $l = \frac{n-r-2}{2}$ in the latter case, the dimension of each irreducible component of the space of maximal dimensional linear spaces is $\frac{(l+1)(l+2)}{2}$ and $\frac{l(l+1)}{2}$, respectively. In the latter case, two linear spaces belong to the same irreducible component if and only if their dimension of intersection is equal to $l+r$ modulo 2. These claims follow from the previous paragraph since Q is a cone over a smooth quadric hypersurface in $\mathbb{P}^{n-1-r}$.

Notation 1.1. Denote the Fano variety of s-dimensional projective linear spaces contained in a quadric hypersurface $Q \in \mathbb{P}^{n-1}$ of corank r by $F^r_{s,n}(Q)$.

Let $Q \subset \mathbb{P}^{n-1}$ be a quadric hypersurface of corank r. Let s be a positive integer less than or equal to $\lfloor \frac{n-r-2}{2} \rfloor + r$. Consider the incidence correspondence of pairs of a point p of Q and an s-dimensional linear space containing p:

$$I = \{ \ (x, [\Lambda]) \mid x \in \Lambda \subset Q \ \} \ \subset \ Q \times F^r_{s,n}(Q).$$

The automorphism group of Q acts transitively on the smooth points of Q. The s-planes that contain a smooth point p lie in the tangent linear space H at p. $Q \cap H$ is a quadric hypersurface of corank $r+1$. The intersection with a hyperplane complementary to p is a quadric hypersurface of corank r and intersects all the s-planes containing p in an $(s-1)$-dimensional linear space. We conclude that the space of s-dimensional linear spaces containing p has the same dimension as the space of $(s-1)$-dimensional linear spaces lying on a quadric hypersurface in $\mathbb{P}^{n-3}$ of corank r. Therefore, by induction, we can calculate the general fiber dimension of the projection of I to Q and determine the dimension of I. The second projection maps I onto $F^r_{s,n}(Q)$ with fiber dimension s. We thus obtain a recursion relation for the dimension of $F^r_{s,n}(Q)$.

A priori we need to check that the s-dimensional linear spaces that intersect the vertex in dimension greater than $s-1-\lfloor\frac{n-r-2}{2}\rfloor$ do not form another irreducible component (potentially of different dimension) of $F^r_{s,n}(Q)$. It is easy to see that linear spaces that intersect the vertex in larger than the expected dimension are limits of linear spaces that intersect the vertex in the expected dimension. Observe that every linear space on a quadric is contained in a maximal dimensional linear space. Take a linear space Λ that intersects the vertex in the linear space Ω. Assume that the dimension of Ω is larger than expected. Take a linear space Δ in Λ complementary to Ω. Take a linear space Γ of dimension $\lfloor\frac{n-r-2}{2}\rfloor$ which contains Δ, but does not intersect the vertex of Q. Since the Grassmannian of s-planes in the span of Γ and Ω is irreducible, the claim follows.

In case $s<\frac{n-r-2}{2}$, the space of s-dimensional linear spaces on Q is irreducible. If $s\geq\frac{n-r-2}{2}$ the recursion stops when we obtain a quadric of rank r in $\mathbb{P}^{r+1}$ or $\mathbb{P}^r$ with multiplicity 2. The former case occurs if $n-r$ is even and the latter case occurs if $n-r$ is odd. This allows us to calculate the dimensions of the spaces of s-dimensional linear spaces on Q recursively. It also proves that when $s\geq\frac{n-r-2}{2}$, the spaces of s-dimensional linear spaces on Q is irreducible if $n-r$ is odd and has two components if $n-r$ is even. We have thus proved the following:

Lemma 1.2. *Let Q be a quadric hypersurface in $\mathbb{P}^{n-1}$ of corank r. If $s<\frac{n-r-2}{2}$, then $F^r_{s,n}(Q)$ is irreducible of dimension*

$$(s+1)\ \frac{2n-3s-4}{2}.$$

If $s\geq\frac{n-r-2}{2}$ and $n-r$ is even, then $F^r_{s,n}(Q)$ has two irreducible components each of dimension

$$(s+1)\ \frac{n-2s+r-2}{2}\ +\ \frac{(n-r-2)\ (n-r)}{8}.$$

If $s\geq\frac{n-r-2}{2}$ and $n-r$ is odd, then $F^r_{s,n}(Q)$ is irreducible of dimension

$$(s+1)\ \frac{n-2s+r-3}{2}\ +\ \frac{(n-r-1)\ (n-r+1)}{8}.$$

1.2. **Preliminaries on orthogonal Grassmannians.** Let W be an n-dimensional vector space endowed with a non-degenerate, symmetric, bilinear form Q. Set $m=\lfloor\frac{n}{2}\rfloor$. Let $0<k\leq m$ denote a positive integer. Let $OG(k,n)$ denote the k-dimensional subspaces of W isotropic with respect to the form Q, unless $n=2k$. In the latter case, the parameter space of k-dimensional isotropic subspaces of W has two isomorphic irreducible components. $OG(k,n)$ denotes one of these irreducible components.

The orthogonal Grassmannian $OG(k,n)$ is isomorphic to one irreducible component of the Fano variety $F^0_{k-1,n}(Q)$ of $(k-1)$-dimensional projective linear spaces on a smooth quadric hypersurface. The non-degenerate quadratic form Q defines the smooth quadric hypersurface in $\mathbb{P}^{n-1}$. A linear space is isotropic with respect to Q if and only if its projectivization is contained in the quadric hypersurface defined by Q. In particular, by the discussion in §1.1, the dimension of $OG(k,n)$ is

$$\frac{k(2n-3k-1)}{2}$$

The cohomology of $OG(k,n)$ is generated by the classes of Schubert varieties. There are minor differences in the cohomology of $OG(k,n)$ depending on the parity of n due to the fact that when n is even, the half-dimensional isotropic subspaces form two connected components. For even n, the notation has to distinguish between these two connected components. For simplicity, we will first discuss the case of odd n, then describe the necessary modifications for even n.

We begin by describing the Schubert varieties in $OG(k,2m+1)$. Let λ denote a sequence

$$m \geq \lambda_1 > \lambda_2 > \cdots > \lambda_s > 0$$

of strictly decreasing integers, where $s \leq k$. Given λ, there is an associated sequence

$$m-1 \geq \tilde{\lambda}_{s+1} > \cdots > \tilde{\lambda}_m \geq 0$$

of strictly decreasing integers defined by requiring that there does not exist any parts λ_i for which $\tilde{\lambda}_j + \lambda_i = m$. In other words, the associated partition is obtained by removing the integers $m-\lambda_1, \ldots, m-\lambda_s$ from the sequence $m-1, m-2, \ldots, 0$. For example, if $m=6$, then the partition associated to $(6,4)$ is $(5,4,3,1)$. The Schubert varieties in $OG(k,2m+1)$ are parameterized by pairs (λ, μ), where λ is a strictly decreasing partition of length s and μ

$$m-1 \geq \mu_{s+1} > \mu_{s+2} > \cdots > \mu_k \geq 0$$

is a subpartition of $\tilde{\lambda}$ (i.e., the parts of μ are a subset of the parts of $\tilde{\lambda}$) of length $k-s$. We will call such pairs of partitions *allowed pairs*. Observe that for maximal isotropic Grassmannians $OG(m,2m+1)$, the partition $\mu = \tilde{\lambda}$ is uniquely determined by the partition λ. Consequently, in the literature it is standard to omit the sequence μ and parametrize Schubert varieties by strict partitions λ. We will find it useful to record the dimensions of all the flag elements where a jump in dimension occurs, so we add μ to the notation. For non-maximal Grassmannians there are several notations in use. The advantage of our notation is that it minimizes the amount of calculation needed to determine the dimensions of the flag elements where a jump in dimension occurs. Since μ is a subpartition of $\tilde{\lambda}$ we can assume that it occurs as $\tilde{\lambda}_{i_{s+1}}, \cdots, \tilde{\lambda}_{i_k}$. Given a pair (λ, μ), the discrepancy $\mathrm{dis}(\lambda,\mu)$ of the pair is defined by

$$\mathrm{dis}(\lambda,\mu) = (m-k)s + \sum_{j=s+1}^{k} (m-k+j-i_j).$$

Fix an isotropic flag $F_\bullet$

$$0 \subset F_1 \subset F_2 \subset \cdots \subset F_m \subset F_{m-1}^{\perp} \subset \cdots \subset F_1^{\perp} \subset W.$$

Here $F_i^{\perp}$ denotes the orthogonal complement of F_i with respect to the bilinear form. In terms of the geometry of the quadric hypersurface $Q \subset \mathbb{P}^{n-1}$ we can describe $F_j^{\perp}$ as follows. A one-dimensional isotropic subspace corresponds to a point $p \in Q \subset \mathbb{P}^{n-1}$. The annihilator of that subspace corresponds to the tangent space to Q at the point p. We can take Q to be given by the equation $\sum_{i=1}^{n} X_i^2 = 0$. We can assume the isotropic subspace is generated by $v = (1, i, 0, \ldots, 0)$. The annihilator of v is given by vectors $(v_1, v_2, \ldots, v_n)$ such that $v_1 + iv_2 = 0$. On the other hand, the tangent space to the quadric hypersurface at p corresponding to v is given by $X_1 + iX_2 = 0$. So the annihilator of a vector consists precisely of those vectors lying in the tangent hyperplane to the quadric at the point corresponding to the vector. To find $F_j^{\perp}$ we take the intersection of all the tangent hyperplanes at the points of F_j. The intersection is the projective linear space $\mathbb{P}^{n-1-j}$ everywhere tangent to Q along the projectivization of F_j.

The Schubert variety $\Omega_\lambda^\mu(F_\bullet)$ is defined as the closure of the locus

$$\{[\Lambda] \in OG(k, 2m+1) |\dim(\Lambda \cap F_{m+1-\lambda_i}) = i \text{ for } 1 \leq i \leq s, \dim(\Lambda \cap F_{\mu_j}^{\perp}) = j \text{ for } s < j \leq k\}.$$

The codimension of a Schubert variety is given by $\sum_{i=1}^{s} \lambda_i + \operatorname{dis}(\lambda, \mu)$. We will denote the cohomology class of Ω_λ^μ by σ_λ^μ.

The description of the Schubert varieties in $OG(k, 2m)$ requires minor modifications to account for the fact that the space of m-dimensional isotropic subspaces have two irreducible components. Let λ denote a sequence

$$m - 1 \geq \lambda_1 > \lambda_2 > \cdots > \lambda_s \geq 0$$

of strictly decreasing integers where $s \leq k$. When $k = m$ and m is even (respectively, odd), we will assume that s is even (respectively, odd). Given λ, we can define an associated sequence $\tilde{\lambda}$ of strictly decreasing integers

$$m - 1 \geq \tilde{\lambda}_{s+1} > \cdots > \tilde{\lambda}_m \geq 0$$

satisfying the condition that there does not exists λ_i such that $\lambda_i + \tilde{\lambda}_j = m - 1$. In other words, to obtain $\tilde{\lambda}$ remove from the sequence $m-1, \ldots, 0$ the integers $m-1-\lambda_1, \ldots, m-1-\lambda_s$. The Schubert varieties in $OG(k, 2m)$ are parameterized by pairs (λ, μ), where λ is a strictly decreasing partition of length s and μ

$$m - 1 \geq \mu_{s+1} > \mu_{s+2} > \cdots > \mu_k \geq 0$$

is a subpartition of $\tilde{\lambda}$ of length $k - s$. We will call such pairs of partitions *allowed pairs*. As above, for maximal isotropic Grassmannians $OG(m, 2m)$, the partition $\mu = \tilde{\lambda}$ is uniquely determined by the partition λ, so it is often omitted from the notation. The pair (λ, μ) is a subpartition of a pair $(\lambda', \tilde{\lambda}')$ of total length m defined as follows. If m and s have the same parity, then $\lambda = \lambda'$. If m and s have different parities, λ' has length $s+1$ and differs from λ in that it includes the smallest number between 0 and $m-1$ not already occurring in λ and not adding to $m-1$ with any of the parts in μ. The discrepancy $\operatorname{dis}(\lambda, \mu)$ of the pair (λ, μ) is defined as follows: Since (λ, μ) is a subpartition of $(\lambda', \tilde{\lambda}')$, we can assume that the parts occur as $\lambda'_{i_1}, \ldots, \lambda'_{i_s}, \tilde{\lambda}'_{i_{s+1}}, \cdots, \tilde{\lambda}'_{i_k}$. The discrepancy is defined as

$$\operatorname{dis}(\lambda, \mu) = \sum_{j=1}^{k} (m - k + j - i_j).$$

We will make the convention that F_m denotes an m-dimensional isotropic subspace in one of the irreducible components. By abuse of notation, we will denote by $F_{m-1}^{\perp}$ an m-dimensional isotropic subspace in the other irreducible component.

Note that strictly speaking the intersection of the quadric hypersurface with $F_{m-1}^{\perp}$ consists of the union of two m-dimensional isotropic subspaces one in each irreducible component. Our slight abuse of notation will make notation more compact. We will use this convention without further mention in the rest of the paper. The Schubert variety $\Omega_\lambda^\mu(F_\bullet)$ is defined as the closure of the locus

$$\{[\Lambda] \in OG(k,2m) |\dim(\Lambda \cap F_{m-\lambda_i}) = i \text{ for } 1 \leq i \leq s, \ \dim(\Lambda \cap F_{\mu_j}^{\perp}) = j \text{ for } s < j \leq k\}.$$

The codimension of a Schubert variety is given by $\sum \lambda_i' + \operatorname{dis}(\lambda, \mu)$. We will denote the cohomology class of Ω_λ^μ by σ_λ^μ.

The cohomology classes σ_λ^μ, as (λ, μ) varies over all allowed pairs, form an additive basis of the cohomology ring of $OG(k,n)$. Given an allowed pair (λ, μ) for $OG(k, 2m+1)$, there is a dual allowed pair (λ^c, μ^c) defined by

$$\lambda_1^c = m - \mu_k, \ \ldots, \ \lambda_{k-s}^c = m - \mu_{s+1}, \ \mu_{k-s+1}^c = m - \lambda_s, \ \ldots, \ \mu_k^c = m - \lambda_1.$$

Similarly, if (λ, μ) is an allowed pair for $OG(k,2m)$, define the dual pair (λ^c, μ^c) by setting

$$\lambda_1^c = m-1-\mu_k, \ \ldots, \ \lambda_{k-s}^c = m-1-\mu_{s+1}, \ \mu_{k-s+1}^c = m-1-\lambda_s, \ \ldots, \ \mu_k^c = m-1-\lambda_1.$$

If (λ, μ) and (λ^c, μ^c) are dual allowed pairs, then $\sigma_\lambda^\mu \cdot \sigma_{\lambda^c}^{\mu^c}$ is equal to the Poincaré dual of the point class.

In this section, we recall basic facts concerning the geometry of isotropic Grassmannians.

Let $n = 2m$ be a positive, even integer. Let V be an n-dimensional vector space over $\mathbb{C}$. Let Q be a non-degenerate, skew-symmetric form on V. By Darboux's Theorem, we can choose a basis for V such that in this basis Q is expressed as $\sum_{i=1}^m x_i \wedge y_i$. A subspace W of V is called *isotropic* if $w^T Q v = 0$ for any two vectors $v, w \in W$. The dimension of an isotropic subspace of V is at most m. Given a vector space W, the orthogonal complement $W^\perp$ of W is defined as the set of $v \in V$ such that $v^T Q w = 0$ for every $w \in W$. If the dimension of W is k, then the dimension of $W^\perp$ is $n-k$ and the restriction of Q to $W^\perp$ has rank $n-2k$ (or, equivalently, corank k).

The Grassmannian $SG(k,n)$ parameterizing k-dimensional isotropic subspaces of V is a homogeneous variety for the symplectic group $Sp(n)$. The Grassmannian $SG(m,n)$ parameterizing maximal isotropic subspaces has dimension

$$\dim(SG(m,n)) = \frac{m(m+1)}{2}.$$

This can be seen inductively. The dimension of $SG(1,2) \cong \mathbb{P}^1$ is one since every vector is isotropic with respect to Q. Consider the incidence correspondence

$$I = \{(w, W) \mid w \in \mathbb{P}(W) \text{ and } [W] \in SG(m,n)\}$$

parameterizing a pair of a maximal isotropic subspace W and a point w of $\mathbb{P}(W)$. The first projection of the incidence correspondence I maps to $\mathbb{P}(V)$ with fibers isomorphic to $SG(m-1, n-2)$. The second projection maps the incidence correspondence to $SG(m,n)$ with fibers isomorphic to $\mathbb{P}(W)$. By the Theorem on the Dimension of Fibers [S, I.6.7] and induction, we conclude that the dimension of $SG(m,n)$ is $\frac{m(m+1)}{2}$.

The dimension of the isotropic Grassmannian $SG(k,n)$ is

$$\dim SG(k,n) = \frac{m(m+1)}{2} + \frac{(m-k)(3k-m-1)}{2} = nk - \frac{3k^2-k}{2}.$$

To see this, consider the incidence correspondence

$$I = \{(W_1, W_2) \mid W_1 \in SG(k,n), W_2 \in SG(m,n), W_1 \subset W_2\}$$

parameterizing two-step flags consisting of a k-dimensional isotropic space contained in a maximal isotropic space. Since every k-dimensional isotropic space can be completed to a maximal isotropic space, the first projection is onto $SG(k,n)$. The fibers of the first projection are isomorphic to the isotropic Grassmannian $SG(m-k, n-2k)$. The second projection is onto $SG(m,n)$ with fibers isomorphic to $G(k,m)$. The Theorem on the Dimension of Fibers [S, I.6.7] and the previous paragraph imply the claim.

More generally, we will need to study spaces parameterizing k-dimensional linear spaces isotropic with respect to a degenerate skew form Q_n^r of corank r on an n-dimensional vector space. Naturally, $n-r$ needs to be even. Since the restriction of Q_n^r to a linear space complementary to its kernel is non-degenerate, we conclude that the largest dimensional isotropic subspace has dimension $r+\frac{n-r}{2}$. Set $h=\frac{n-r}{2}$. Then the space of $(r+h)$-dimensional isotropic linear spaces with respect to Q_n^r is isomorphic to $SG(h,2h)$ and has dimension $\frac{h(h+1)}{2}$. Considering the incidence correspondence

$$I = \{(W_1, W_2) \mid W_1 \subset W_2 \text{ isotropic with respect to } Q_n^r,$$

$$\dim(W_1) = k, \text{ and } \dim(W_2) = h + r\},$$

we see that the space of k-dimensional isotropic subspaces of Q_n^r has dimension $\frac{h(h+1)}{2} + k(h+r-k)$ if $k \geq h$ and $\frac{h(h+1)}{2} + k(h+r-k) - \frac{(h-k)(h-k+1)}{2}$ if $k < h$.

The cohomology of $SG(k,n)$ is generated by the classes of Schubert varieties. Let $0 \leq s \leq k$ be a non-negative integer. Let $\lambda_\bullet : 0 < \lambda_1 < \lambda_2 < \cdots < \lambda_s \leq m$ be a sequence of increasing positive integers. Let $\mu_\bullet : m > \mu_{s+1} > \mu_{s+2} > \cdots > \mu_k \geq 0$ be a sequence of decreasing non-negative integers such that $\lambda_i \neq \mu_j + 1$ for any $1 \leq i \leq s$ and $s < j \leq k$. Then the Schubert varieties in $SG(k,n)$ may be indexed by pairs of admissible sequences $(\lambda_\bullet; \mu_\bullet)$. Fix an isotropic flag

$$F_\bullet = F_1 \subset F_2 \subset \cdots F_m \subset F_{m-1}^\perp \subset \cdots F_1^\perp \subset V.$$

The Schubert variety $\Sigma_{\lambda_\bullet;\mu_\bullet}(F_\bullet)$ is defined as the Zariski closure of the set of linear spaces

$$\{W \in SG(k,n) \mid \dim(W \cap F_{\lambda_i}) = i \text{ for } 1 \leq i \leq s, \dim(W \cap F_{\mu_j}^\perp) = j \text{ for } s < j \leq k\}.$$

In the literature, it is customary to denote Schubert classes in the cohomology of $SG(m,n)$ by strictly decreasing partitions $m \geq a_1 > a_2 > \cdots > a_s > 0$ of length $s \leq m$. In our notation, the sequence $a_\bullet$ translates to the sequence $\lambda_\bullet$ by setting $a_i = m+1-\lambda_i$. Note that when $n = 2m$, the sequence $\lambda_\bullet$ determines the sequence $\mu_\bullet$ by the requirement that $\lambda_i \neq \mu_j + 1$ for any $1 \leq i \leq s$ and $s < j \leq m$. Therefore, it is common to omit the sequence $\mu_\bullet$ from the notation. We will not follow this convention. In Schubert calculus, many authors prefer to record Schubert classes so that the codimension will be easily accessible. Our notation has the advantage that it is preserved under natural maps between Grassmannians arising from linear embeddings between ambient vector spaces.

We will index Schubert classes in the cohomology of the Grassmannian $G(k,n)$ by increasing sequences of non-negative integers $a_\bullet : 0 < a_1 < a_2 < \cdots < a_k \leq n$. The Schubert variety $\Sigma_{a_\bullet}(F_\bullet)$ with respect to a flag $F_\bullet$ parameterizes k-dimensional subspaces W of V that satisfy $\dim(W \cap F_{a_i}) \geq i$ for $1 \leq i \leq k$.

2. The restriction problem

The orthogonal or the symplectic Grassmannian naturally includes in the ordinary Grassmannian. The restriction problem asks for computing the induced map on cohomology in terms of the Schubert classes. This is the geometric analogue of the restriction problem in representation theory. Our solution to this problem will be via specialization.

In the case of orthogonal Grassmannians, the quadratic form Q defines a smooth degree two hypersurface Q in $\mathbb{P}W$. We will interpret $OG(k,n)$ as the Fano variety of $(k-1)$-dimensional projective linear subspaces on Q. We will also need to study singular quadric hypersurfaces. Over the complex numbers, the projective equivalence class of a quadric hypersurface is determined by its dimension and corank. Let $Q_{d_i}^{r_i}$ denote a quadratic form of corank r_i obtained by restricting Q to a vector space of dimension d_i. Let L_{n_j} denote an isotropic linear space of (vector space) dimension n_j. A restriction variety in $OG(k,n)$ is defined in terms of a sequence

$$L_{n_1} \subset \cdots \subset L_{n_s} \subset Q_{d_{k-s}}^{r_{k-s}} \subset \cdots \subset Q_{d_1}^{r_1}$$

of isotropic linear spaces and quadrics. (In Definitions 4.2 and 4.9, we will specify the conditions that these linear spaces and quadrics need to satisfy. For the purposes of the introduction we ignore these subtleties.) The restriction variety parameterizes the isotropic linear spaces that intersect L_{n_j} in a subspace of dimension j and $Q_{d_i}^{r_i}$ in a subspace dimension $k-i+1$ for every $1 \leq j \leq s$ and $1 \leq i \leq k-s$. Schubert varieties are examples of restriction varieties with the property that the quadrics in the sequence are as singular as possible (i.e., $d_i + r_i = n$). The strategy to calculate the class of a restriction variety is to specialize the quadrics in the sequence one at a time to become more singular until they are maximally singular. When we specialize the quadrics, the restriction variety breaks into a union of simpler restriction varieties. The process is governed by the following basic facts about quadrics.

• **The corank bound.** Let $Q_{d_2}^{r_2} \subset Q_{d_1}^{r_1}$ be two linear sections of Q such that the singular locus of $Q_{d_1}^{r_1}$ is contained in the singular locus of $Q_{d_2}^{r_2}$. Then $r_2 - r_1 \leq d_1 - d_2$. In particular, the corank of a sub-quadric in Q is bounded by its codimension.

• **The linear space bound.** The largest dimensional isotropic linear space with respect to a quadratic form Q_d^r has dimension $\lfloor \frac{d+r}{2} \rfloor$. A linear space of dimension j intersects the singular locus of Q_d^r in a subspace of dimension at least $\max(0, j - \lfloor \frac{d-r}{2} \rfloor)$.

• **Irreducibility.** A sub-quadric Q_d^{d-2} of Q is reducible and equal to the union of two linear spaces of (vector space) dimension $d-1$ meeting along a linear space of dimension $d-2$. If $n = 2k$, then the linear spaces constituting Q_{k+1}^{k-1} belong to two distinct connected components.

• **The variation of tangent spaces.** Let a quadric Q^r_d be singular along a codimension j linear subspace M of a linear space L. Then the image of the Gauss map of Q^r_d restricted to the smooth points of L has dimension at most $j-1$. In other words, the tangent spaces to Q^r_d along the smooth points of L vary at most in a $(j-1)$-dimensional family.

The corank bound determines the order of the specialization. We increase the corank of the smallest dimensional quadric $Q^{r_i}_{d_i}$ that satisfies $d_i + r_i < d_{i-1} + r_{i-1}$ by one, i.e., we replace $Q^{r_i}_{d_i}$ in the sequence with $Q^{r_i+1}_{d_i}$. The algorithm is obtained by describing the flat limit of this specialization. Suppose that a general linear space parametrized by the restriction variety intersects the singular locus of $Q^{r_i}_{d_i}$ in a subspace of dimension x_i. The linear spaces parametrized by the flat limit intersect the singular locus of $Q^{r_i+1}_{d_i}$ in a subspace of dimension x_i or x_i+1. The limit has more than one component when both cases are possible. 'The linear space bound' and 'the variance of tangent spaces' dictate which of the possibilities occur. In addition, if $r_i = d_i - 3$, then by the 'irreducibility' property, the new quadric $Q^{r_i+1}_{d_i}$ is reducible forcing the limit to possibly have more components. Surprisingly, each of these components occur with multiplicity one in the limit. The algorithm is obtained by inductively applying this specialization to each irreducible component. We refer the reader to §5 for the precise statement of the algorithm and detailed examples.

The case of $SG(k,n)$ is similar. The computation depends on four very simple geometric principles. We now explain these principles. Let Q^r_d denote a d-dimensional vector space such that the restriction of Q has corank r. Let $\mathrm{Ker}(Q^r_d)$ denote the kernel of the restriction of Q to Q^r_d. Let L_j denote an isotropic subspace of dimension j with respect to Q. Let $L_j^\perp$ denote the set of $w \in V$ such that $w^T Q v = 0$ for all $v \in L_j$.

Evenness of rank. The rank of a non-degenerate skew-symmetric form is even. Hence, $d-r$ is even for Q^r_d. Furthermore, if $d=r$, then Q^r_d is isotropic.

The corank bound. Let $Q^{r_1}_{d_1} \subset Q^{r_2}_{d_2}$ and let $r_2' = \dim(\mathrm{Ker}(Q^{r_2}_{d_2}) \cap Q^{r_1}_{d_1})$. Then $r_1 - r_2' \leq d_2 - d_1$. In particular, $d + r \leq n$ for Q^r_d.

The linear space bound. The dimension of an isotropic subspace of Q^r_d is bounded above by $\lfloor \frac{d+r}{2} \rfloor$. Furthermore, an m-dimensional linear space L satisfies $\dim(L \cap \mathrm{Ker}(Q^r_d)) \geq m - \lfloor \frac{d-r}{2} \rfloor$.

The kernel bound. Let L be an $(s+1)$-dimensional isotropic space such that $\dim(L \cap \mathrm{Ker}(Q^r_d)) = s$. If an isotropic linear subspace M of Q^r_d intersects $L - \mathrm{Ker}(Q^r_d)$, then M is contained in $L^\perp$.

These four principles dictate the order of the specialization and determine the limits that occur. Given a flag, we will specialize the smallest dimensional non-isotropic subspace Q^r_d, whose corank can be increased subject to the corank bound, keeping all other flag elements unchanged. We will replace Q^r_d with $\tilde{Q}^{r+2}_d$. The branching rule simply says that under this specialization, the limit L' of a linear space L satisfying rank conditions with respect to the original flag satisfies the same rank conditions with the unchanged flag elements and either $\dim(L' \cap \mathrm{Ker}(\tilde{Q}^{r+2}_d)) = \dim(L \cap \mathrm{Ker}(Q^r_d))$ or $\dim(L' \cap \mathrm{Ker}(\tilde{Q}^{r+2}_d)) = \dim(L \cap \mathrm{Ker}(Q^r_d)) + 1$. Furthermore, both of these cases occur with multiplicity one unless the latter leads to a smaller dimensional variety or the former violates the linear space bound. See Sections 3 and 8 for an explicit statement of the rule and for examples.

3. Combinatorics

In this section, we present the rule for computing restriction coefficients of $OG(k, n)$ and $SG(k, n)$ combinatorially.

3.1. **The orthogonal case.**

Consider a sequence of n integers written from left to right. We say that a bracket or brace is in position i if i of the integers are to the left of the bracket or brace.

Definition 3.1. Let $0 \leq s \leq k < n$ be integers. A *orthogonal sequence of brackets and braces* of type (k, n) is a sequence of n natural numbers, s right brackets] and $k - s$ right braces } such that:

- Every bracket or brace occupies a positive position and each position is occupied by at most one bracket or brace.
- Every number i in the sequence satisfies $0 \leq i \leq k-s$. The positive integers in the sequence are non-decreasing from left to right and are to the left of every zero in the sequence.
- Every bracket is to the left of every brace.
- If $2k = n$, a bracket in the k-th position may either be a bracket] or a bracket decorated with a prime $]'$.

For example, 1]1]122]33]0000}00}00}000 is an orthogonal sequence of brackets and braces of type $(7, 18)$ with $s = 4$. To be concrete, the first rule forbids 0]]0, 0}}0 (two brackets or two braces in the same position), 00]}00 (a bracket and a brace in the same position),]100 (a bracket that is not in a position). The second rule forbids numbers that look like 1132 (3 is not allowed to be to the left of 2) or 11200300 (3 should be to the left of any zero). The third rule forbids 000}00]0 (a brace cannot be to the left of a bracket).

Notation 3.2. We order the brackets in the sequence from left to right and the braces in the sequence from right to left. In our example, $1]^1 1]^2 122]^3 33]^4 0000\}^3 00\}^2 00\}^1 000$ the small numbers above the brackets and braces indicate their order. Let $\rho(i, j)$ denote the number of integers to the right of the i-th brace and to the left of the j-th brace. Let $\rho(i, 0)$ denote the number of integers to the right of the i-th brace. In our example, $\rho(3, 2) = 2, \rho(2, 1) = 2, \rho(1, 0) = 3$. Let $p(]^i)$ and $p(\}^i)$ denote the number of integers to the left of the i-th bracket and i-th brace, respectively. These record the positions of the brackets and braces. In our running example, $p(]^1) = 1, p(]^2) = 2, p(]^3) = 5, p(]^4) = 7$ and $p(\}^3) = 11, p(\}^2) = 13, p(\}^1) = 15$. Let $l(i)$ denote the number of integers in the sequence that are equal to i. Let $l(\leq i)$ denote the number of *positive* integers in the sequence that are less than or equal to i. In our running example, $l(1) = 3, l(2) = 2, l(3) = 2, l(\leq 2) = 5, l(\leq 3) = 7$. When we are discussing more than one sequence, we will write ρ_D, p_D and l_D for the invariants of the sequence D.

We are now ready to define quadric diagrams, which are the main combinatorial objects of this paper. The first three conditions in the definition do not play a role in the algorithm. They are included for precision and the reader may ignore them in a first reading. The last three conditions are crucial and the reader should remember them.

Definition 3.3. A *quadric diagram* for $OG(k,n)$ is an orthogonal sequence of brackets and braces of type (k,n) with s brackets such that the following conditions hold.

(D1) $l(i) \leq \rho(i, i-1)$ for $1 \leq i \leq k-s$.
(D2) $2p(]^s) \leq p(\}^{k-s}) + l(\leq k-s)$.
(D3) Suppose that the integer $0 < i < k-s$ occurs in the sequence. If $i+1$ does not occur in the sequence, either $i = 1$ and every position after a 1 is occupied by a bracket, or $l(j) = \rho(j, j-1)$ for every $j > i+1$ and $\rho(i+1, i) = 1$.
(D4) There are at least three zeros to the left of $\}^{k-s}$.
(D5) Let x_i be the number of brackets such that $p(]^j) \leq l(\leq i)$. Then

$$x_i \geq k - i + 1 - \frac{p(\}^i) - l(\leq i)}{2}.$$

(D6) The two integers immediately to the left of a bracket are equal. If there is only one integer to the left of a bracket and $s < k$, then the integer is 1.

Remark 3.4. Quadric diagrams index restriction varieties, which will be introduced in the next section and are the main geometric objects of study in this paper.

Example 3.5. Let us give a few examples to clarify the meaning of these conditions. The first condition says that the number of times i appears in the sequence is less than or equal to the number of integers between the i-th and $(i-1)$-st braces. In particular, the following are forbidden 2220000}00}0 ($l(2) = 3$, but $\rho(2,1) = 2$), 11000}0 ($l(1) = 2$, but $\rho(1,0) = 1$). Let the right most bracket be at position $p(]^s)$ and the left most brace be at position $p(\}^{k-s})$. The second condition says that twice $p(]^s)$ is less than or equal to the sum of $p(\}^{k-s})$ and the number of positive integers in the sequence. For example, 00]00}0, 100]00}0 are allowed, but 000]00}0 is not ($2p(]^1) = 6 > p(\}^1) = 5$). The third condition is a consequence of the order in the algorithm. The reader does not have to pay attention to it except in a few places in the proof of the algorithm, where it simplifies the dimension counts. The rule says that if a positive integer occurs in the sequence, then all the larger integers (less than or equal to $k-s$) also occur in the sequence except in two very special cases. For example, 1]1]330000}00}0}00 (all the 1s are followed by brackets) and 1]1330000}00}0}00 (2 is missing, but $l(3) = \rho(3,2) = 2$ and $l(2) = \rho(2,1) - 1 = 0$) are allowed, but 1]130000}00}0}00 is not (2 is missing, but $l(3) = 1 \neq \rho(3,2) = 2$). These conditions are preserved during the algorithm. The reader may ignore them in a first reading.

The last three conditions are the important conditions that the reader has to remember. The fourth condition is self-evident. It allows 11]00]00}00 or 33000}00}00}0, but does not allow 1100}00. The sixth condition is also self-evident. It allows for 1]22]33]0000}00}0}0 or 22]22]2000}00000}0, but disallows 2]22]000}000}0 (there is only one integer to the left of $]^1$, but it is not 1) or 1234]0000}0}0}0}0 (the two numbers preceding the bracket are not equal). The fifth condition is the one that is hardest to visually verify without resorting to some counting. In words, it says that the number of integers that are to the right of the right-most i and to the left of the i-th brace has to be at least twice the total number of brackets and braces that are at positions greater than $l(\leq i)$ and less than or equal to $p(\}^i)$. For example, it disallows 10]00}0 (There are three zeros to the right of the 1 that are to the left of

$\}$. There is one bracket and one brace in positions greater than 1 and less than or equal to 4. However, $3 \ngeq 4$).

We are now ready to state the algorithm. We begin by defining a new set of sequences of brackets and braces associated to D. The new sequences D^a and D^b defined below may fail to be quadric diagrams, but we address such instances below.

Definition 3.6. If there exists an index i in D such that $l(i) < \rho(i, i-1)$, let $\kappa = \max(i \mid l(i) < \rho(i, i-1))$. Let D^a be the sequence of brackets and braces obtained by changing the $(l(\leq \kappa)+1)$-st integer in the sequence D to κ.

If $p_{D^a}(]^s) > l_{D^a}(\leq \kappa)$, let $\eta = \min(i \mid p_{D^a}(]^i) > l_{D^a}(\leq \kappa))$. Let D^b be the sequence of brackets and braces obtained from D^a by moving the bracket $]^\eta$ to the position $l_{D^a}(\leq \kappa)$.

To clarify, let us give some examples. Let $D = 233]0000\}00\}0\}0$. Then $\kappa = 1$. We change the integer in the position $l(\leq 1)+1$ (in this case the left most 2) to 1 to obtain $D^a = 133]0000\}00\}0\}0$. We slide the first bracket in D^a to the right of the 1 we added to the immediate right of it to obtain $D^b = 1]330000\}00\}0\}0$. Note that in this case both D^a and D^b are quadric diagrams.

Next let $D = 00]0]0000\}0$. Here $\kappa = 1$, so we turn the left most 0 into 1 to obtain $D^a = 10]0]0000\}0$. We slide the first bracket to the right of the 1 to its immediate right to obtain $D^b = 1]00]0000\}0$. Here note that D^b is a quadric diagram, but D^a fails condition (D6). We have to turn D^a into a quadric diagram. Here is the algorithm that turns D^a into a quadric diagram.

Algorithm 3.7. • If D^a fails condition (D5), discard it. D^a does not lead to any quadric diagrams.
• If D^a satisfies condition (D5) but not condition (D6), change the $(l(\leq \kappa)+1)$-st integer in the sequence to κ and move $\}^\kappa$ one position to the left. Repeat until you reach a sequence of brackets and braces that satisfies condition (D6). Label the resulting sequence D^c. If D^c is a quadric diagram, we refer to it as a quadric diagram derived from D^a. Otherwise, proceed to the next step.
• If D^a or D^c satisfy conditions (D5) and (D6), but fail condition (D4), replace D^a or D^c with two identical diagrams D^{a_1} and D^{a_2} obtained by replacing $\}^{k-s}$ (in D^a or D^c) with $]^{s+1}$ in position $p(\}^{k-s})-1$ and turning the digits equal to $k-s$ to 0. If $2p(]^{s+1}) = n$, then we use $(]')^{s+1}$ instead of $]^{s+1}$ in D^{a_2}. We refer to D^{a_1} and D^{a_2} as quadric diagrams derived from D^a. Furthermore, if $2k = n$ and $2p(]^{s+1}) = n$, then discard the diagram with $]^{s+1}$ (respectively, $(]')^{s+1}$) if $s+1 \neq k$ mod (2) (respectively, if $s+1 = k$ mod (2)).

In our example, we first turn $D^a = 10]0]0000\}0$ to $11]0]000\}00$. This diagram still fails condition (6), so we repeat to obtain $11]1]00\}000$. Now condition (D6) is satisfied, but condition (D4) fails. Since $n = 8 = 2 \cdot 4$, we obtain the two diagrams $00]0]0]0000$ and $00]0]0]'0000$. These are the two diagrams derived from D^a.

Let $D = 000\}000\}000\}$, then $\kappa = 3$. We turn the left most 0 into 3 to obtain $D^a = 300\}000\}000\}$. In this case, there are no brackets to the left of the 3, so there is no D^b. The sequence D^a fails condition (D4). Since n is odd, we replace D^a with two identical quadric diagrams $D^{a_1} = 00]0000\}000\}$ and $D^{a_2} = 00]0000\}000\}$.

Let $D = 00]0000\}00\}0$. Then $D^a = 20]0000\}00\}0$ and $D^b = 2]00000\}00\}0$. Neither of these diagrams satisfy condition (D6). We already know that we should replace D^a with $22]000\}000\}0$. Here is how to modify D^b.

Algorithm 3.8. • If D^b does not satisfy condition (D6), let $]^j$ be the bracket for which it fails. Let i be the integer immediately to the left of $]^j$. Replace i with $i-1$ and move $\}^{i-1}$ one position to the left. As long as the resulting sequence does not satisfy condition (D6), repeat this process either until the resulting sequence is a quadric diagram (in which case this is the quadric diagram derived from D^b) or two braces occupy the same position. In the latter case, no quadric diagrams are derived from D^b.

In our example, we replace $D^b = 2]00000\}00\}0$ with $1]00000\}0\}00$, which is a quadric diagram. If our example had been $D = 00]0000\}0\}0$, then $D^b = 2]00000\}0\}0$. Replacing 2 with 1 and moving $\}^1$ to the left would produce $1]00000\}\}00$. Hence, in this case no quadric diagrams are derived from D^b.

We need one final definition. Given a sequence of brackets and braces such that $p(]^s) > l(\kappa)$, if $l(\leq i) < p(]^{x_\kappa+1})$ for all i, set $y_{x_\kappa+1} = k-s+1$. Otherwise, let $y_{x_\kappa+1} = \max(i \mid l(\leq i) \leq p(]^{x_\kappa+1}))$. If there is a 0 to the left of $]^{x_\kappa+1}$, then $y_{x_\kappa+1}$ is $k-s+1$. Otherwise, $y_{x_\kappa+1}$ is the largest integer that occurs to the right of $]^{x_\kappa+1}$, which is the first bracket occurring in a position greater than $l(\leq \kappa)$. The condition $p(]^{x_\kappa+1})-l(\leq \kappa)-1 = y_{x_\kappa+1}-\kappa$ will play an important role. In words, this condition says that the number of integers larger than κ to the left of $]^{x_\kappa+1}$ is one more than the cardinality of the set of integers greater than κ (or zero) occurring to the left of $]^{x_\kappa+1}$. In view of condition (D3), a sequence satisfying this equality looks like $\cdots \kappa+1\ \kappa+2\ \cdots \kappa+l-1\ \kappa+l\ \kappa+l]\cdots$ or $\cdots \kappa+1\ \kappa+2\ \cdots \kappa+l-1\ 00]\cdots$, where we have drawn the part of the sequence starting with the left most $\kappa+1$ and ending with $]^{x_\kappa+1}$. We are now ready to state the algorithm.

Algorithm 3.9. Let D be a quadric diagram. If $l(i) = \rho(i, i-1)$ for every $1 \leq i \leq k-s$, then return D and stop. Otherwise, let D^a and D^b be as above.

(1) If $p(]^{x_\kappa+1}) - l(\leq \kappa) - 1 > y_{x_\kappa+1} - \kappa$ or $p(]^s) \leq l(\kappa)$ in D, then return the quadric diagrams that are derived from D^a.
(2) If D^a violates condition (D5), then return the quadric diagrams that are derived from D^b.
(3) Otherwise, return the quadric diagrams that are derived from both D^a and D^b.

Remark 3.10. In the proof of Theorem 5.12, we will check in detail that Algorithm 3.9 always returns at least one quadric diagram. Briefly, D^a does not lead to a quadric diagram only if it violates condition (D5). In that case, by the definition of κ, there has to be equality in condition (D5) for all indices $\kappa \leq i \leq k-s$ in the diagram D. Then, condition (D4) implies that there has to be a bracket to the right of κ in D^a; and condition (D6) implies that $p(]^{x_\kappa+1}) - l(\leq \kappa) - 1 = y_{x_\kappa+1} - \kappa$ in D. Finally, while running Algorithm 3.36, if two braces occupy the same position, then condition (D5) is violated for the index $\kappa - 1$ in the diagram D. These considerations imply that there is a quadric diagram derived from D^b by Algorithm 3.9 (see paragraph 6 of the proof of Theorem 5.12 for more details).

3.2. **The symplectic case.**

Notation 3.11. Let $0 \leq s \leq k$ be an integer. A *sequence of n natural numbers of type s* for $SG(k, n)$ is a sequence of n natural numbers such that every number is less than or equal to $k-s$. We write the sequence from left to right with a small gap to the right of each number in the sequence. We refer to the gap after the

i-th number in the sequence as the *i-th position*. For example, 1 1 2 0 0 0 0 0 and 3 0 0 2 0 1 0 0 are two sequences of 8 natural numbers of types 1 and 0, respectively, for $SG(3,8)$.

Definition 3.12. Let $0 \leq s \leq k$ be an integer. A *sequence of brackets and braces of type s* for $SG(k,n)$ consists of a sequence of n natural numbers of type s, s brackets] ordered from left to right and $k-s$ braces } ordered from right to left such that:

(1) Every bracket or brace occupies a position and each position is occupied by at most one bracket or brace.
(2) Every bracket is to the left of every brace.
(3) Every positive integer greater than or equal to i is to the left of the i-th brace.
(4) The total number of integers equal to zero or greater than i to the left of the i-th brace is even.

Example 3.13. 11]200}0}00 and 300}20}10}0 are typical examples of sequences of brackets and braces for $SG(3,8)$ that have the two examples from Notation 3.11 as their sequences of natural numbers. When writing a sequence of brackets and braces, we often omit the gaps not occupied by a bracket or a brace.

Example 3.14. Let us give several non-examples to clarify Definition 3.12. The first condition disallows diagrams such as]0000} (the first bracket is not in a position), 0]]000, 000}}0, 00]}00 (two brackets, two braces, or a bracket and a brace occupy the same position, respectively). The second condition disallows diagrams such as 00}0]000 (a brace cannot be to the left of a bracket). The third condition disallows diagrams such as 100}30}20}0 (3 is to the right of the third brace and 2 is to the right of the second brace). The fourth condition disallows diagrams such as 1]2000}0}00 (the number of zeros to the left of the second brace, and the number of zeros and twos to the left of the first brace are odd).

Notation 3.15. By convention, the brackets are indexed from left to right and the braces are indexed from right to left. We write $]^i$ and $\}^i$ to denote the i-th bracket and i-th brace, respectively. Their positions are denoted by $p(]^i)$ and $p(\}^i)$. The position of a bracket or a brace is equal to the number of integers to its left. For notational convenience, we declare that, in a sequence of brackets and braces of type s for $SG(k,n)$, the brace $\}^{k-s+1}$ denotes $]^s$ and an integer in the sequence equal to $k-s+1$ should be read as 0. Let $l(i)$ denote the number of integers in the sequence that are equal to i. Let r_i be the total number of positive integers less than or equal to i that are to the left of $\}^i$. For $0 < j < i$, let $\rho(i,j) = p(\}^j) - p(\}^i)$ and let $\rho(i,0) = n - p(\}^i)$. Equivalently, $\rho(i,0)$ (respectively, $\rho(i,j)$) denotes the number of integers to the right of the i-th brace (respectively, to the right of the i-th brace and to the left of the j-th brace).

Example 3.16. For the sequence of brackets and braces 300}20}10}0 for $SG(3,8)$, the positions are $p(\}^3) = 3, p(\}^2) = 5, p(\}^1) = 7$. We have $r_i = l(i) = 1$, for $1 \leq i \leq 3$, $\rho(i,i-1) = 2$, for $2 \leq i \leq 3$, and $\rho(1,0) = 1$.

Example 3.17. For the sequence of brackets and braces 1]22]00}00}0 for $SG(4,8)$, the positions are $p(]^1) = 1, p(]^2) = 3, p(\}^2) = 5, p(\}^1) = 7$. We have $r_1 = l(1) = 1$, $l(2) = 2$, and $r_2 = 3$. Moreover, $\rho(2,1) = 2$ and $\rho(1,0) = 1$.

Definition 3.18. Two sequences of brackets and braces are *equivalent* if the lengths of their sequence of numbers are equal, the brackets and braces occur at the same positions, and the collection of digits that occur between any consecutive brackets and/or braces are the same up to reordering.

Example 3.19. The sequences 1221]00200}000}00, 1122]20000}000}00 and the sequences 003}02}01}0, 300}20}10}0 are equivalent pairs of sequences. We can depict an equivalence class of sequences by the representative where the digits are listed so that between any two consecutive brackets and/or braces the positive integers precede the zeros and are listed in non-decreasing order. We will always use this *canonical representative* and often blur the distinction between the equivalence class and this representative.

Definition 3.20. A sequence of brackets and braces is *in order* if the sequence of numbers consists of a sequence of non-decreasing positive integers followed by zeros except possibly for one i immediately to the right of $\}^{i+1}$ for $1 \leq i < k-s$. Otherwise, we say that the sequence is *not in order*. A sequence is *in perfect order* if the sequence of numbers consists of non-decreasing positive integers followed by zeros.

Example 3.21. The sequences 300}20}10}000, 11]22]00}00}00, 1]33]0000}200}0}0 are in order. Furthermore, 11]22]00}00}00 is in perfect order. The sequences 11]00]100}000, 1]20000}1}0}00, 122]100}00}00 are not in order.

Definition 3.22. A sequence of brackets and braces is *saturated* if $l(i) = \rho(i, i-1)$ for $1 \leq i \leq k-s$.

Example 3.23. The sequences 11]22]00}00}00 and 1]22]100}00}00 are saturated, whereas, 22]00}00}00 and 1]0000}00}000 are not.

The next definition is a technical definition that plays a role in the proof and is a consequence of the order in which the game is played. The reader can define a symplectic diagram as a sequence of brackets and braces that occurs in the game and refer to the conditions only when necessary.

Definition 3.24. A *symplectic diagram* for $SG(k, n)$ is a sequence of brackets and braces of type s for $SG(k, n)$ for some $0 \leq s \leq k$ such that:

(S1) $l(i) \leq \rho(i, i-1)$ for $1 \leq i \leq k-s$.

(S2) Let τ_i be the sum of $p(]^s)$ and the number of positive integers between $]^s$ and $\}^i$. Then
$$2\tau_i \leq p(\}^i) + r_i.$$

(S3) Either the sequence is in order or there exists at most one integer $1 \leq \eta \leq k-s$ such that the sequence of integers is non-decreasing followed by a sequence of zeros except for at most one occurrence of η between $]^s$ and $\}^{\eta+1}$ and at most one occurrence of $i < \eta$ after $\}^{i+1}$.

(S4) Let ξ_j denote the number of positive integers between $\}^j$ and $\}^{j-1}$. If an integer i occurs to the left of all the zeros, then either $i = 1$ and there is a bracket in the position following it, or there exists at most one index j_0 such that $\rho(j, j-1) = l(j)$ for $j_0 \neq j > \min(i, \eta)$ and $\rho(j_0, j_0-1) \leq l(j_0)+2-\xi_{j_0}$. Moreover, any integer η violating order occurs to the right of $\}^{j_0}$.

Remark 3.25. Conditions (S1) and (S2) are necessary to guarantee that symplectic diagrams represent geometrically meaningful objects. Conditions (S3) and (S4) are consequences of the order the game is played and describe the most complicated possible diagrams that can occur. The reader can ignore these conditions. They are necessary to carry out the dimension counts and to prove that the algorithm is defined at each step. They are not needed in order to run the algorithm.

Example 3.26. Let us give some examples to clarify Definition 3.24. Condition (S1) allows for diagrams such as 11]22]2]00}000}00 but disallows 22]3300}2}00}000 (there are two 3's and three 2's in the sequence but $\rho(2,3) = 1$ and $\rho(1,2) = 2$). Condition (S2) disallows diagrams such as 000]10}0 ($r_1 = 1$, $\tau_1 = 4$, but $2 \cdot 4 > 5 + 1$). Condition (S3) allows for 2344]300}00}00}10}0 (a non-decreasing sequence of positive integers 2344 followed by a sequence consisting of one 3, one 1 and zeros), but disallows 22]110000}2200}0000}00 (there are two 1s and two 2s following the non-decreasing sequence 22) or 22]133]00}00}00}0 (there are two 3s following the non-decreasing sequence 22). Condition (S4) allows for diagrams such as 11]3300}00}1}000, 1]1]33]00}00}00}00, however, it disallows diagrams such as 144]00}00}00}00}0 (1 occurs in the initial non-decreasing part of the sequence, but 2 and 3 do not occur. 1 is not followed by a bracket and $l(3) = 0 \neq \rho(3,2) = 2$, $l(2) = 0 \neq \rho(2,1) = 2$).

The next definition is crucial for the game and the reader should remember these conditions.

Definition 3.27. A symplectic diagram is called *admissible* if it satisfies the following additional conditions.

(A1) The two integers to the left of a bracket are equal. If there is only one integer to the left of a bracket and $s < k$, then the integer is one.

(A2) Let x_i be the number of brackets $]^h$ such that every integer to the left of $]^h$ is positive and less than or equal to i. Then

$$x_i \geq k - i + 1 - \frac{p(\}^i) - r_i}{2}.$$

Example 3.28. Condition (A1) disallows diagrams such as 11]23]00}00}00}00 (the digits preceding the second bracket are not equal), 2]200}00}00 (there is a bracket in position 1, but the first digit is not 1). Condition (A2) is hard to visualize without resorting to counting. Let p be the position of the rightmost bracket such that every digit to the left of p is positive and less than or equal to i. In words, condition (A2) says that the total number of zeros and integers greater than i in the sequence is at least twice the number of brackets and braces in positions $p+1$ through $p(\}^i)$. The following diagrams violate condition (A2): 22}00}00 ($x_2 = 0$, $p(\}^2) = r_2 = 2$, but $0 < 1$), 200}2}00} (the number of braces up to $p(\}^2) = 4$ is 2; the number of zeros is 2, but $2 < 2 \cdot 2$), 11]33]00}00}1}000 (the total number of brackets and braces between positions 3 and $9 = p(\}^1)$ is 4. The number of zeros and integers greater than 1 is 6, but $2 \cdot 4 > 6$).

Remark 3.29. The admissible symplectic diagrams are the main combinatorial objects in this paper. They represent symplectic restriction varieties, which are the main geometric objects of the paper and will be defined in the next section. The symplectic diagram records a non-necessarily isotropic flag. The corresponding symplectic restriction variety parameterizes isotropic spaces that satisfy certain rank conditions with respect to this flag. The definition of an admissible symplectic diagram reflects the basic facts about isotropic subspaces discussed in the introduction, as we will see in the next section.

Definition 3.30. The *symplectic diagram $D(\sigma_{\lambda;\mu})$ associated to the Schubert class $\sigma_{\lambda;\mu}$ in $SG(k,n)$ is the saturated symplectic diagram in perfect order, where the brackets occur at positions $\lambda_1, \dots, \lambda_s$ and the braces occur at positions $n-\mu_{s+1}, \cdots, n-\mu_k$.*

Example 3.31. The symplectic diagram associated to $\sigma_{2,4;4,2}$ in $SG(4,10)$ is 11]22]00}00}00.

Lemma 3.32. *The diagram $D(\sigma_{\lambda;\mu})$ is an admissible symplectic diagram.*

Proof. Let $n = 2m$. Since $0 < \lambda_1 < \cdots < \lambda_s \leq m < n - \mu_{s+1} < \cdots < n - \mu_k$, the brackets and braces occur in different positions and the brackets are to the left of the braces. Since the sequence is saturated and in perfect order, the number of integers in the sequence equal to i is $\mu_{k-i+1} - \mu_{k-i+2} \leq \mu_{s+1} < m$ (with the convention that $\mu_{k+1} = 0$), for $1 \leq i \leq k-s$ and occur to the left of $\}^{k-s}$. Finally, the number of integers equal to zero or greater than or equal to i to the left of $\}^i$ is $n - 2\mu_{k-i+1} = 2(m - \mu_{k-i+1})$. Therefore, $D(\sigma_{\lambda;\mu})$ satisfies all 4 conditions in Definition 3.12.

By definition, $D(\sigma_{\lambda;\mu})$ is saturated, so $l(i) = \rho(i, i-1)$ and conditions (S1) and (S4) hold. Since the diagram is in perfect order, (S3) holds and

$$\tau_i = \max(\lambda_s, \mu_{s+1}) \leq m.$$

On the other hand, $p(\}^i) + r_i = n - \mu_{k-i+1} + \mu_{k-i+1} = n = 2m \geq 2\tau_i$. Therefore, $D(\sigma_{\lambda;\mu})$ satisfies all the conditions in Definition 3.24.

Finally, since $\lambda_j \neq \mu_i + 1$ for any i, j, the two integers preceding a bracket must be equal. Furthermore, if $\lambda_1 = 1$, $\mu_1 \geq 1$. Hence, condition (A1) holds. For $1 \leq i \leq k-s$, $k-i+1-(p(\}^i)-r_i)/2 = k-i+1+\mu_{k-i+1}-m$. From the sequence $0, 1, \dots, m-1$, remove the integers $\lambda_1 - 1, \lambda_2 - 1, \dots, \lambda_s - 1$ to obtain a sequence $\alpha_m < \alpha_{m-1} < \cdots < \alpha_{s+1}$. By assumption $\mu_{k-i+1} = \alpha_j$ for some $j \geq k-i+1$. Hence, $k-i+1+\mu_{k-i+1}-m \leq \alpha_j - (m-j) = x_i$. To see the last equality, observe that x_i is the number of integers λ_h that are less than or equal to $\mu_{k-i+1} = \alpha_j$. This number is equal to the number of integers $(\alpha_j - (m-j))$ between 0 and α_j that do not occur in the sequence $\alpha_m, \dots, \alpha_j$. Hence, condition (A2) holds. We conclude that $D(\sigma_{\lambda;\mu})$ is an admissible symplectic diagram. □

The game is defined on admissible symplectic diagrams. We will see in the next section that saturated admissible diagrams in perfect order represent Schubert varieties in $SG(k,n)$. The goal of the algorithm is to transform every admissible symplectic diagram to a collection of saturated admissible diagrams in perfect order. Given an admissible symplectic diagram D, we will associate to it one or two sequences D^a and/or D^b of brackets and braces. Initially, neither D^a nor D^b has to be admissible. We will shortly describe an algorithm that modifies D^a and D^b so that they become admissible. The game records a degeneration of the flag elements represented by D.

Definition 3.33. Let D be an admissible symplectic diagram of type s for $SG(k,n)$. For the purposes of this definition, read any mention of $k-s+1$ as 0 and any mention of $\}^{k-s+1}$ as $]^s$.

(1) If D is not in order, let η be the integer in condition (S3) violating the order.

  (i) If every integer $\eta < i \leq k-s$ occurs to the left of η, let ν be the leftmost integer equal to $\eta + 1$ in the sequence of D. Let D^a be the canonical representative of the diagram obtained by interchanging η and ν.

(ii) If an integer $\eta < i \leq k-s$ does not occur to the left of η, let ν be the leftmost integer equal to $i+1$. Let D^a be the canonical representative of the diagram obtained by swapping η with the leftmost 0 to the right of $\}^{i+1}$ not equal to ν and changing ν to i.

(2) If D is in order but is not a saturated admissible diagram in perfect order, let κ be the largest index for which $l(i) < \rho(i, i-1)$.

(i) If $l(\kappa) < \rho(\kappa, \kappa-1) - 1$, let ν be the leftmost digit equal to $\kappa+1$. Let D^a be the canonical representative of the diagram obtained by changing ν and the leftmost 0 to the right of $\}^{k+1}$ not equal to ν to κ.

(ii) If $l(\kappa) = \rho(\kappa, \kappa-1) - 1$, let η be the integer equal to $\kappa - 1$ immediately to the right of $\}^{\kappa}$.

(a) If κ occurs to the left of η, let ν be the leftmost integer equal to κ in the sequence of D. Let D^a be the canonical representative of the diagram obtained by changing ν to $\kappa - 1$ and η to zero.

(b) If κ does not occur to the left of η, let ν be the leftmost integer equal to $\kappa+1$. Let D^a be the canonical representative of the diagram obtained by swapping η with the leftmost 0 to the right of $\}^{\kappa+1}$ not equal to ν and changing ν to κ.

Let p be the position in D immediately to the right of ν. If there exists a bracket at a position $p' > p$ in D^a, let $q > p$ be the minimal position occupied by a bracket in D^a. Let D^b be the diagram obtained from D^a by moving the bracket at position q to position p. Otherwise, D^b is not defined.

Example 3.34. Let $D = 2300\}10\}0\}0$, then $\eta = 1$ violates the order and $\nu = 2$ and 3 occur to the left of it. Hence, we are in case (1)(i) and $D^a = 1300\}20\}0\}0$ is obtained by swapping 1 and 2. Similarly, let $D = 200]200\}00\}$, then the second 2 violates the order and $D^a = 220]000\}00\}$, $D^b = 22]0000\}00\}$.

Let $D = 124400\}00\}1\}0\}00$, the 1 in the ninth place violates the order and 3 does not occur to its left, so we are in case (1)(ii) and $D^a = 123400\}10\}0\}0\}00$.

Let $D = 22]00\}00\}00$, then D is in order and $\kappa = 1$. Since $l(1) = 0 < \rho(1,0) - 1$, we are in case (2)(i) and $D^a = 12]00\}10\}00$ and $D^b = 1]200\}10\}00$.

Let $D = 3300\}200\}0\}$, then D is in order and $\kappa = 3$. Since $l(3) = 2 = \rho(3,2) - 1$, we are in case (2)(ii)(a) and $D^a = 2300\}000\}0\}$.

Finally, let $D = 330000\}00\}1\}0$, then D is in order and $\kappa = 2$. Since $l(2) = 0 = \rho(2,1) - 1$ and 2 does not occur in the sequence, we are in case (2)(ii)(b) and $D^a = 230000\}10\}0\}0$.

We will soon check that both D^a and D^b are symplectic diagrams; however, they do not have to be admissible. We now describe algorithms for turning them into admissible diagrams.

Algorithm 3.35. If D^a is not an admissible symplectic diagram, perform the following steps to turn it into an admissible diagram.

Step 1. If D^a does not satisfy condition (A2), let i be the maximal index for which condition (A2) fails. Define a new diagram D^c as follows. Let the two rightmost integers equal to i in D^a be in the places $\pi_1 < \pi_2$. Delete $\}^i$ and move the i in place π_2 to place $\pi_1 + 1$. Slide the integers in places $\pi_1 < \pi < \pi_2$ and brackets and braces in positions $\pi_1 < p < \pi_2$ one to the right. Add a bracket at position $\pi_1 + 1$. Subtract one from the integers $i < h \leq k-s$; and if $i = k-s$, change the integers equal to $k-s$ to 0. Let D^c be the resulting diagram and replace D^a with D^c. If D^a satisfies condition (A2), proceed to the next step.

Step 2. If D^a fails condition (A1), let $]^j$ be the smallest index bracket for which it fails and let i be the integer preceding $]^j$. Change this i to $i-1$ ($k-s$ if $i=0$) and move $\}^{i-1}$ ($\}^{k-s}$ if $i=0$) one position to the left. Repeat this procedure until the sequence of brackets and braces satisfies condition (A1). Let the resulting sequence be D^c. In both steps, we refer to D^c as a *quadric diagram derived from* D^a.

Algorithm 3.36. If D^b does not satisfy condition (A1), run Step 2 of Algorithm 3.35 on D^b. Explicitly, let $]^j$ be the minimal index bracket for which (A1) fails. Let i be the integer immediately to the left of $]^j$. Replace i with $i-1$ and move $\}^{i-1}$ one position to the left. As long as the resulting sequence does not satisfy condition (A1), repeat this process either until the resulting sequence is an admissible symplectic diagram (in which case, this is *the symplectic diagram derived from* D^b) or two braces occupy the same position. In the latter case, no admissible symplectic diagrams are derived from D^b.

Example 3.37 (Examples of Algorithm 3.35)**.** Let $D = 22]33]00\}00\}00\}00$. Then the diagram $D^a = 12]33]00\}00\}10\}00$ fails condition (A2) since $x_1 = 0 < 1 = 5-(10-2)/2$. Hence, according to Step 1 of Algorithm 3.35, we replace D^a with $11]1]22]00\}00\}000$ (delete $\}^1$, move the 1 in position 9 to position 2 and slide everything in positions 2-8 one position to the right, add a bracket in position 2, and subtract 1 from the integers greater than 1). The latter is an admissible diagram.

Let $D = 00\}00\}00$. Then $D^a = 22\}00\}00$ fails condition (A2) since $x_2 = 0 < 1-(2-2)/2$. Hence, Step 1 of Algorithm 3.35 replaces D^a with $00]00\}00$ (delete $\}^2$ and add a bracket in position 2), which is admissible.

Similarly, if $D = 11]33]00\}00\}00\}00$, then the diagram $D^a = 11]23]00\}20\}00\}00$ fails condition (A2) since $x_2 = 1 < 2$. Hence, according to Step 1 of Algorithm 3.35, we replace D^a with $11]22]2]00\}000\}00$, which is admissible.

If $D = 22]2]200\}0000\}00$, then the diagram $D^a = 12]2]200\}1000\}00$ is not admissible since it fails condition (A1) for $]^1$. Step 2 of Algorithm 3.35 replaces D^a first with $11]2]200\}100\}000$ (change the 2 preceding $]^1$ to 1 and move $\}^1$ one position to the right). Note that this diagram fails condition (A1) for $]^2$. Hence, Step 2 replaces it with $11]1]200\}10\}0000$ (change the 2 preceding $]^2$ to 1 and move $\}^1$ one position to the left). This diagram is admissible, hence it is the diagram derived from D^a.

Example 3.38 (Examples of Algorithm 3.36)**.** Let $D = 11]33]00\}00\}00\}00$, then $D^b = 11]2]300\}20\}00\}00$ fails condition (A1). Algorithm 3.36 replaces it with $11]1]300\}20\}0\}000$, which is admissible.

Let $D = 00]0000\}00\}00\}$, then $D^b = 3]30000\}00\}00\}$ does not satisfy condition (A1) since the digit to the left of $]^1$ has to be 1. Algorithm 3.36 replaces D^b first with $2]30000\}0\}000\}$, which still fails condition (A1). Hence, Algorithm 3.36 replaces this diagram with $1]30000\}0\}00\}0$, which is admissible.

If $D = 00]0000\}2\}0\}$, then $D^a = 30]2000\}0\}0\}$ and $D^b = 3]20000\}0\}0\}$. They both fail condition (A1). When we run Algorithm 3.36 on D^b, we turn the 3 into 2 and slide $\}^2$ one position to the left. In that case, we obtain $1]30000\}\}00\}$. Since two braces occupy the same position, no diagrams are derived from D^b in this case. When we run Algorithm 3.35 on D^a, we obtain the admissible diagram $33]200\}00\}0\}$.

Let D be an admissible symplectic diagram and let ν be as in Definition 3.33. Let $\pi(\nu)$ denote the place of ν in the sequence of integers. If $p(]^s) > \pi(\nu)$, then $]^{x_{\nu-1}+1}$ is the first bracket to the right of ν. If the integer to the immediate left of $]^{x_{\nu-1}+1}$ is positive, let $y_{x_{\nu-1}+1}$ be this integer. Otherwise, let $y_{x_{\nu-1}+1} = k - s + 1$. The condition $p(]^{x_{\nu-1}+1}) - \pi(\nu) - 1 = y_{x_{\nu-1}+1} - \nu$ plays an important role. In words, this condition says that the number of values larger than ν or equal to zero that the integers to the left of $]^{x_{\nu-1}+1}$ attain is one more than the cardinality of the set of integers consisting of zero and integers larger than ν occurring to the left of $]^{x_{\nu-1}+1}$. In view of conditions (S3), (S4) and (A1), a sequence satisfying this equality looks like

$$\cdots \nu\ \nu+1 \cdots \nu+l-1\ \nu+l\ \nu+l] \cdots \quad \text{or} \quad \cdots \nu\ \nu+1\ \cdots \nu+l\ 00] \cdots ,$$

where we have drawn the part of the sequence starting with the left most ν and ending with $]^{x_{\nu-1}+1}$. We are now ready to state the algorithm.

Algorithm 3.39. Let D be an admissible, symplectic diagram of type s for $SG(k,n)$. If D is saturated and in perfect order, return D and stop. Otherwise, let D^a and D^b be defined as in Definition 3.33.

(1) If $p(]^s) \leq \pi(\nu)$ or $p(]^{x_{\nu-1}+1}) - \pi(\nu) - 1 > y_{x_{\nu-1}+1} - \nu$ in D, then return the admissible symplectic diagrams that are derived from D^a.
(2) Otherwise, return the admissible symplectic diagrams that are derived from both D^a and D^b.

We run the algorithm on two symplectic diagrams.

Example 3.40.

$$\begin{array}{ccccc} 00\}00\}00 & \to & 00]00\}00 & \to & 00]0]000 \\ & & \downarrow & & \\ & & 1]100\}00 & & \end{array}$$

In this example, first $D^a = 22\}00\}00$ is not admissible since the diagram fails condition (A2). Therefore, we replace it by $00]00\}00$. Next, $D^a = 10]10\}00$ and $D^b = 1]100\}00$. D^a is not admissible since it does not satisfy condition (A2). Hence, we replace it by the admissible diagram $00]0]000$. D^b is admissible. Note that the last two diagrams are saturated and in perfect order, so the algorithm terminates. We will soon see that this calculation shows $i^*\sigma_{2,4} = \sigma_{2,3;} + \sigma_{1;2}$ in $SG(2,6)$.

Finally, we give a larger example in $SG(3,10)$ that illustrates the inductive structure of the game.

Example 3.41.

$$\begin{array}{cccccc} \multicolumn{5}{r}{300\}20\}10\}000 \to 200\}00\}10\}000 \to 200]00\}10\}000 \to} & 1]0000\}00\}000 \\ & & & & \downarrow & \downarrow \\ & & & & 100]00\}00\}000 & 1]2200\}00\}000 \\ & & & \swarrow & \searrow & \downarrow \\ & & 100]0]000\}000 & & 11]200\}0\}0000 & 1]1200\}10\}000 \\ & \swarrow & \searrow & & \downarrow & \downarrow \\ \underline{000]0]0]00000} & & 11]00]100\}000 & & \underline{11]11]00\}0000} & 1]1100\}00\}000 \\ & \swarrow & \searrow & & & \downarrow \\ \underline{11]1]0000\}000} & & \underline{11]11]00\}0000} & & & \underline{1]1100]00\}000} \end{array}$$

We will see that this calculation shows $i^*\sigma_{3,5,7} = \sigma_{3,4,5;} + \sigma_{2,3;3} + 2\sigma_{2,4;4} + \sigma_{1,5;3}$ in $H^*(SG(3,10),\mathbb{Z})$.

Definition 3.42. A *degeneration path* is a sequence of admissible symplectic diagrams

$$D_1 \to D_2 \to \cdots \to D_r$$

such that D_{i+1} is one of the outcomes of running Algorithm 3.39 on D_i for $1 \le i < r$.

The main theorem of this paper is the following.

Theorem 3.43. *Let D be an admissible symplectic diagram for $SG(k,n)$. Let $V(D)$ be the symplectic restriction variety associated to D. Then, in terms of the Schubert basis of $SG(k,n)$, the cohomology class $[V(D)]$ can be expressed as*

$$[V(D)] = \sum c_{\lambda;\mu}\sigma_{\lambda;\mu},$$

where $c_{\lambda;\mu}$ is the number of degeneration paths starting with D and ending with the symplectic diagram $D(\sigma_{\lambda;\mu})$.

Theorem **??** stated in the introduction is a corollary of Theorem 3.43.

Definition 3.44. Let $\sigma_{a_\bullet}$ be a Schubert class in $G(k,n)$. If $a_j < 2j-1$ for some $1 \leq j \leq k$, then $i^*\sigma_{a_\bullet} = 0$ and we do not associate a symplectic diagram to $\sigma_{a_\bullet}$. Suppose that $a_j \geq 2j-1$ for $1 \leq j \leq k$. Let u be the number of i such that $a_i = 2i-1$. For j such that $a_j \neq 2j-1$, let u_j be the number of integers $i < j$ such that $a_i = 2i-1$. Let v_j be the number of integers $i > j$ such that $a_i = 2i-1$. Then the diagram $D(a_\bullet)$ associated to $i^*\sigma_{a_\bullet}$ is a diagram consisting of u brackets at positions $1, 2 \cdots, u$ and a brace for each $a_j > 2j-1$ at position $a_j - u_j + v_j$. The sequence of integers consists of u integers equal to 1 followed by zeros except for one integer equal to $k-j-v_j+1$ immediately following the first bracket or brace to the right of $\}^{k-j-v_j+1}$ (or in the first position if $j + v_j = 1$) for each odd $a_j > 2j-1$.

Example 3.45. The diagram $D(\sigma_{3,5,7})$ in $SG(3,8)$ is 300}20}10}0. The diagram $D(\sigma_{1,3,6,7,10})$ in $SG(5,10)$ is 1]1]1]00}00}000. The diagram $D(\sigma_{1,3,7,8,9,12})$ in $SG(6,14)$ is 1]1]1]300}0}00}00000

Remark 3.46. The reader will notice that $D(\sigma_{a_\bullet})$ is the diagram obtained by running Algorithm 3.35 on the diagram that has a brace at positions a_j and whose sequence consists of zeros except for one $k-j+1$ immediately to the right of $\}^{k-j+2}$ when a_j is odd.

Lemma 3.47. *If $a_j \geq 2j-1$ for $1 \leq j \leq k$, then $D(a_\bullet)$ is an admissible symplectic diagram.*

Proof. The brackets occur at positions $1, \dots, u$. Let a_j and a_{j+l} be two consecutive integers in the sequence $a_\bullet$ satisfying $a_i > 2i-1$. Then the positions of the corresponding braces are $a_j - u_j + v_j$ and $a_{j+l} - u_{j+l} + v_{j+l}$. Since $u_{j+l} = u_j + l - 1$ and $v_{j+l} = v_j - l + 1$, the positions of the two braces differ by the quantity $\beta = a_{j+l} - a_j - 2l + 2$. If $l = 1$, $\beta > 0$. If $l > 1$, then $a_j < a_{j+1} = 2j+1$. Since $a_{j+l} \geq 2j + 2l$, β is also positive. The first brace corresponds to the smallest index j_0 such that $a_{j_0} > 2j_0 - 1$ and occurs at position $a_{j_0} - (j_0 - 1) + (u - j_0 + 1) = u + a_{j_0} - 2j_0 + 2 \geq u + 2$. The number of positive integers less than or equal to $k - j - v_j + 1$ to the left of $\}^{k-j-v_j+1}$ is u (respectively, $u+1$) if a_j is even (respectively, odd). Hence, the number $a_j - u_j + v_j - u(-1) = a_j - 2u_j(-1)$ (where -1 occurs if a_j is odd) of integers equal to zero or greater $k - j - v_j + 1$ to the left of $\}^{k-j-v_j+1}$ is even. Therefore, conditions (1)-(4) of Definition 3.12 hold.

By construction, $l(i) \leq 1$ for $i > 1$ and $l(1) = u(+1)$ depending on whether the largest $a_j > 2j-1$ is even (or odd). In either case, one easily sees that $l(1) \leq \rho(1,0)$. The number of positive integers to the left of $\}^{k-j-v_j+1}$ is equal to u plus the number o_j of odd $a_l < a_j$ such that $a_l > 2l-1$. Since $2(u_j + o_j) \leq 2j \leq a_j$, we have that $2(u + o_j) \leq a_j - u_j + v_j + u = a_j + 2v_j$ and condition (S2) holds. The sequence is in order and the only integers other than $k-u$ occurring in the initial part of the sequence are ones, which are followed by brackets. We conclude that all the conditions in Definition 3.24 hold.

Since any bracket is preceded by 1, condition (A1) holds. Finally, for $\}^{k-j-v_j+1}$, the quantity $j + v_j - \frac{a_j - u_j + v_j - u(-1)}{2} = j + u - \frac{a_j(-1)}{2} \leq u$ (where -1 occurs if a_j is odd) since $a_j > 2j-1$. We conclude that $D(a_\bullet)$ is an admissible symplectic diagram. □

Corollary 3.48. *Let $\sigma_{a_\bullet}$ be a Schubert class in $G(k,n)$. If $a_j < 2j-1$ for some $1 \leq j \leq k$, then set $i^*\sigma_{a_\bullet} = 0$. Otherwise, let $D(\sigma_{a_\bullet})$ be the diagram associated to*

$\sigma_{a_\bullet}$. *Express*

$$i^*\sigma_{a_\bullet} = \sum c_{\lambda;\mu}\sigma_{\lambda;\mu}$$

in terms of the Schubert basis of $SG(k,n)$. Then $c_{\lambda;\mu}$ is the number of degeneration paths starting with $D(\sigma_{a_\bullet})$ and ending with the symplectic diagram $D(\sigma_{\lambda;\mu})$.

Proof. In Lemma 7.20, we will prove that the intersection of $SG(k,n)$ with a general Schubert variety in $G(k,n)$ with class $\sigma_{a_\bullet}$ is a restriction variety of the form $V(D(\sigma_{a_\bullet}))$. The corollary is immediate from this lemma and Theorem 3.43. $\square$

We conclude this section by proving that Algorithm 3.39 is well-defined and terminates. The proof of Theorem 3.43 is geometric and will be taken up in the next two sections.

Proposition 3.49. *Algorithm 3.39 replaces an admissible symplectic diagram with one or two admissible symplectic diagrams.*

Proof. If D is a saturated symplectic diagram in perfect order, then the algorithm returns D and there is nothing further to check. We will first check that D^a and D^b are (not necessarily admissible) symplectic diagrams. The diagram D^b is obtained from D^a by moving a bracket to the left. Conditions (2), (3), (4) of Definition 3.12 and conditions (S1), (S2), (S3) and (S4) of Definition 3.24 are preserved under moving a bracket to the left. Since $\nu \neq 1$ is the leftmost integer in D equal to a given integer, by condition (A1) for D, there cannot be a bracket at position p in D or D^a. Hence, condition (1) is satisfied for D^b. We conclude that if D^a is a symplectic diagram, then D^b is also a symplectic diagram. We will now check that D^a is a symplectic diagram in each case.

In case (1)(i), by condition (S3) for D, let η be the unique integer that violates the order. Since η is violating the order, η is to the left of $\}^{\eta+1}$. D^a is obtained by swapping η and ν, the leftmost integer equal to $\eta+1$. This operation does not change the positions of the brackets and braces and keeps $l(i)$ fixed for every i. After the swap, every integer i is still to the left of $\}^i$ for every i since η was to the left of $\}^{\eta+1}$. Furthermore, the operation also preserves or decreases τ_i for every i. We thus conclude that conditions (1) through (4) of Definition 3.12 and condition (S1), (S2) and (S4) of Definition 3.24 hold for the diagram D^a. After the swap, η is part of the non-decreasing initial sequence in D^a. Hence, the diagram D^a is either in order or $\eta+1$ is the only integer violating the order. Condition (S3) holds for D^a. We conclude that D^a is a symplectic diagram.

In case (1)(ii), let η be the unique integer that violates the order. Assume that $\eta < i \leq k-s$ does not occur to the left of η. Then i does not occur anywhere in the sequence and, in condition (S4) for D, $i = j_0$. We claim that the i-th and $(i-1)$-st braces in D must look like $\cdots\}^i\eta\}^{i-1}\cdots$. By conditions (S3) and (S4) for D, η is to the right of $\}^i$ and to the left of $\}^{\eta+1}$. If η is between $\}^{i+h}$ and $\}^{i+h-1}$ for $h \neq -1$, then since $\rho(i+h, i+h-1) = l(i+h)$ by condition (S4), the parity in condition (4) is violated for $\}^{i+h-1}$. We conclude that η is between $\}^i$ and $\}^{i-1}$. Furthermore, $1 \leq \rho(i,i-1) \leq l(i)+2-\xi_i = 1$ by condition (S4). The formation of D^a does not affect conditions (1) through (3) in Definition 3.12. Condition (4) holds for D^a since the formation of D^a changes the number of integers that are equal to zero or greater than j to the right of $\}^j$ only when $j=i$ and for $\}^i$ it changes the number by two. Since the formation of D^a only increases $l(i)$ by one and decreases or preserves $l(j)$ for $j \neq i$, D^a satisfies (S1). Similarly, τ_i increases by one and all other τ_j remain fixed or decrease. On the other hand, r_i increases by two, hence D^a satisfies condition (S2). There is one exception. If $i = k - s$ and every integer to the left of $]^s$ is positive, τ_{k-s} increases by two. Then, $\tau_{k-s} = r_{k-s}$, hence $2\tau_{k-s} \le p(\}^{k-s}) + r_{k-s}$ and D^a satisfies (S2). The diagram D^a is either in order or η is still the only integer violating the order, hence D^a satisfies (S3). Finally, the formation of D^a changes $l(i) = 1$ and decreases $l(i+1)$ by one. Hence, $l(i) = \rho(i, i-1)$ for D^a. By condition (S4) for D, we have that $\rho(j, j-1) = l(j)$ in D^a for any j for which the equality held for D except for $j = i+1$. Furthermore, $\xi_{i+1} = 1$ in D^a, so $\rho(i+1, i) = l(i+1) + 1 = l(i+1) + 2 - \xi_{i+1}$ in D^a. Hence (S4) holds for D^a. We conclude that D^a is a symplectic diagram.

From now on assume that D is in order. Then there cannot be $i \ge \kappa$ such that i is immediately to the right of $\}^{i+1}$. Suppose there exists such an i. The number $\chi(i)$ and $\chi(i+1)$ of zeros and integers greater than i, respectively $i+1$, to the left of $\}^i$, respectively $\}^{i+1}$, has to be even. However, $\chi(i) = \chi(i+1) + l(i+1) + \rho(i+1, i) - 1$. Since by assumption $\rho(i+1, i) = l(i+1)$, we conclude that either $\chi(i)$ or $\chi(i+1)$ cannot be even leading to a contradiction.

In case (2)(i), changing ν to κ and the first zero to the right of $\}^{\kappa+1}$ does not change the positions of brackets and braces, it decreases $l(\kappa+1)$ by one and increases $l(\kappa)$ by two. Furthermore, the sequence D^a is still in order, unless $\kappa = k - s$ and there are zeros to the left of $]^s$. In the latter case, the κ to the right of $]^s$ is the unique integer violating order. Since by assumption $l(\kappa) < \rho(\kappa, \kappa-1) - 1$ in D, $l(\kappa) \le \rho(\kappa, \kappa - 1)$ in D^a. The parity of the integers equal to zero or greater than i also remains constant for all $1 \le i \le k - s$. We conclude that conditions (1) through (4) in Definition 3.12 and conditions (S1) and (S3) in Definition 3.24 hold for D^a. The quantity τ_i remains constant for $i > \kappa$ and increases by one for $i \le \kappa$ unless $\kappa = k-s$, $l(k-s) \ge p(]^s)$ and τ_{k-s} increases by two. In the latter case, τ_{k-s} is less than or equal to both r_{k-s} and $p(\}^{k-s})$ and (S2) holds. In the former case, r_κ increases by two, hence (S2) holds for the index κ. Since $\rho(\kappa, \kappa-1) < l(\kappa) - 1$, (S2) also holds for indices $i < \kappa$. If there exists an index $i < \kappa$ in D such that i is not a 1 followed by a bracket, then in condition (S4) for D, we have that $j_0 = \kappa$. Furthermore, $\rho(\kappa, \kappa-1) = l(\kappa) + 2$. Hence, the formation of D^a preserves the equalities in condition (S4) except for $j = \kappa$ or $\kappa + 1$. In D^a, we have that $\rho(\kappa, \kappa-1) = l(\kappa)$ and $\rho(\kappa+1, \kappa) = l(\kappa+1) + 1 = l(\kappa+1) + 2 - \xi_{\kappa+1}$. We conclude that condition (S4) holds for D^a. Therefore, D^a is a symplectic diagram.

Finally, the argument showing that D^a is a symplectic diagram in case (2)(ii)(a) is identical to the argument in case (1)(i) and the argument in case (2)(ii)(b) is identical to the case (1)(ii), so we leave them for the reader. We conclude that both D^a and D^b are symplectic diagrams. However, they need not be admissible. We now check that Algorithms 3.35 and 3.36 preserve the fact that the resulting sequences are symplectic diagrams and output admissible symplectic diagrams.

D^a may fail to be admissible either because it fails condition (A1) or (A2) in Definition 3.27. The formation of D^a from D does not change the quantities x_h, $p(\}^h)$. In cases (1)(i) and (2)(ii)(a) the quantity r_h either remains the same or decreases. Hence, in these cases D^a satisfies condition (A2). In case (1)(ii), r_h remains the same or decreases except for r_i, which increases by two. Hence, the inequality in condition (A2) can only be violated for the index i by one. If it is violated, we conclude that in D, we have $x_i = k - i + 1 - \frac{p(\}^i) - r_i}{2}$. Recall that in this case D looks like $\cdots \}^i \eta \}^{i-1} \cdots$. Since i does not appear in D, $x_i = x_{i-1}$. Writing the inequality in (A2) for D and the index $i-1$ and noting that $r_{i-1} = r_i + 1$ and $p(\}^{i-1}) = p(\}^i) + 1$, we see that $x_i = x_{i-1} \geq k - i + 2 - \frac{p(\}^i) - r_i}{2} = x_i + 1$. Since D satisfies (A2), this is a contradiction. We conclude that D^a satisfies (A2) also in the case (1)(ii). By similar reasoning, in cases (2)(i) and (2)(ii)(b), D^a can violate the inequality in (A2) only for the index κ by one. After Step 1 of Algorithm 3.35, all the inequalities in condition (A2) remain unchanged or improve and $\}^\kappa$ is eliminated. We conclude that after Step 1, the resulting diagram satisfies (A2). When the inequality in (A2) is violated for D^a, it is violated for the index κ by at most 1. When we form D^b in cases (2)(i) and (2)(ii)(b), x_κ also increases by one. Hence, D^b, when it exists, always satisfies (A2).

Observe that the operation in Step 1 of Algorithm 3.35 preserves the fact that D^a is a symplectic diagram. By construction, conditions (1)-(4) and (S1) and (S2) hold. The diagram resulting after Step 1 is in order, hence (S3) holds. The operation renames $l(i)$ as $l(i-1)$ for $i > \kappa + 1$ and $\rho(i+1, i)$ as $\rho(i, i-1)$ for $i > \kappa + 1$. The operation does not change the quantities $l(i)$ and $\rho(i, i-1)$ when $i < \kappa$ and replaces $l(\kappa)$ and $l(\kappa+1)$ with their sum under the name $l(\kappa)$. The quantities $\rho(\kappa, \kappa-1)$ and $\rho(\kappa+1, \kappa)$ are replaced by $\rho(\kappa, \kappa-1) + \rho(\kappa+1, \kappa) - 1$ and renamed $\rho(\kappa, \kappa-1)$. Hence, the equalities in condition (S4) are preserved. Since (A1) also holds for the resulting diagram D^c, we conclude that if D^a fails condition (A2), then Step 1 of Algorithm 3.35 produces an admissible symplectic diagram.

Observe that changing a digit to the left of a bracket and moving a brace one unit to the left, increases x_i and r_i by one and decreases $p(\}^i)$ by one. Hence, it preserves the inequality in condition (A2). It also preserves the conditions (1) through (4) and (S1) through (S4), with the possible exception of (1) in case $p(\}^{i+1}) = p(\}^i) - 1$. Condition (A1) is violated for D^a when there is a bracket in position $p(\nu) + 1$ and it is violated only for that bracket. After l applications of Step 2 of Algorithm 3.35, Condition (A1) is still violated if there exists brackets at positions $p(\nu)+1, p(\nu)+2, \cdots, p(\nu)+l$. Since there are a finite number of brackets, this process stops and the resulting diagram satisfies condition (A1). In this case, the only brace that moves is $\}^{\nu-1}$. Since $l(\nu) \leq \rho(\nu, \nu-1)$ in D, the intermediate sequences and the resulting sequence all satisfy condition (1). If D^b does not satisfy condition (A1), then the only bracket that can violate it is the one in position $p(\nu)$. In this case, Algorithm 3.36 successively decreases the integer to the right of the bracket in $p(\nu)$ by one until it either becomes equal to the integer to its right or to one in case there isn't an integer to its right. Hence, this algorithm terminates in finitely many steps. A diagram might violate condition (1) in the process, but in that case the diagram is discarded. Hence, after finitely many steps either the diagram is discarded or results in an admissible symplectic diagram. We conclude that Algorithm 3.39, replaces D with one or two admissible symplectic diagrams. $\square$

Proposition 3.50. *After finitely many applications of Algorithm 3.39, every admissible symplectic diagram is transformed to a collection of admissible symplectic diagrams in perfect order.*

Proof. If the diagram D is not in order, after one application of the algorithm either the diagram is in order or the integer violating the order increases or the position of the integer violating the order in the sequence decreases. Since these steps cannot go on indefinitely, after finitely many steps, the diagram is in order. Furthermore, during the process either the number of braces decreases or the number of positive integers less than or equal to i, for $1 \leq i \leq k-s$ in the initial part of the sequence remains constant or increases. If the diagram is in order, then at each application of the algorithm either the number of braces decreases or the number of positive integers less than or equal to i, for $1 \leq i \leq k-s$, in the initial part of the sequence increases. Since these cannot go on indefinitely, we conclude that repeated applications of the algorithm transform an admissible symplectic diagram into a collection of admissible symplectic diagrams in perfect order. Hence, the algorithm terminates in finitely many steps. $\square$

4. Restriction varieties in the orthogonal Grassmannians

In this section, we introduce restriction varieties in orthogonal Grassmannians and discuss their basic properties. Restriction varieties are subvarieties of $OG(k,n)$ that parameterize isotropic k-planes that intersect elements of a given flag in specified dimensions. We do not require the flag to be isotropic; however, we need to impose some basic numerical restrictions in order to obtain geometrically meaningful subvarieties.

Notation 4.1. Let W be a vector space of dimension n. Let Q be a non-degenerate, symmetric bilinear form on W. We denote an isotropic linear space of (vector space) dimension n_j by L_{n_j}. In case $2n_j = n$, L_{n_j} and L'_{n_j} denote isotropic linear spaces in different connected components. Let $Q^{r_i}_{d_i}$ denote a sub-quadric of corank r_i cut out by a d_i-dimensional linear section of Q. We denote the singular locus of $Q^{r_i}_{d_i}$ by $Q^{r_i,sing}_{d_i}$. For convenience, we let $r_0 = 0$ and $d_0 = n$.

Definition 4.2. A sequence of linear spaces and quadrics $(L_\bullet, Q_\bullet)$ associated to $OG(k,n)$ is a totally ordered set

$$L_{n_1} \subsetneq L_{n_2} \subsetneq \cdots \subsetneq L_{n_s} \subsetneq Q^{r_{k-s}}_{d_{k-s}} \subsetneq \cdots \subsetneq Q^{r_1}_{d_1}$$

of isotropic linear spaces L_{n_j} (or possibly L'_{n_s} in case $2n_s = n$) and sub-quadrics $Q^{r_i}_{d_i}$ of Q such that

(1) $2n_s \leq d_{k-s} + r_{k-s}$.
(2) $2(k-i+1) \leq r_i + d_i$ for every $1 \leq i \leq k-s$.
(3) $r_{i+1} + d_{i+1} \leq r_i + d_i \leq n$ for every $1 \leq i < k-s$.
(4) $Q^{r_{i-1},sing}_{d_{i-1}} \subseteq Q^{r_i,sing}_{d_i}$ for every $1 < i \leq k-s$.
(5) $\dim(L_{n_j} \cap Q^{r_i,sing}_{d_i}) = \min(n_j, r_i)$.
(6) Let x_1 denote the number of isotropic subspaces in the sequence contained in the singular locus of $Q^{r_1}_{d_1}$. For every $1 \leq i \leq k-s$, either $r_i = r_1 = x_1$, or $r_l - r_i \geq l - i - 1$ for every $l > i$. Furthermore, if $r_l = r_{l-1} > x_1$ for some l, then $d_i - d_{i+1} = r_{i+1} - r_i$ for all $i \geq l$ and $d_{l-1} - d_l = 1$.

Remark 4.3. Conditions (1), (2) and (3) express basic facts about quadrics. Conditions (1) and (2) express the "Linear space bound" that the dimension of an isotropic linear space with respect to a quadratic form of corank r in d variables is at most half of $d+r$. Since $L_{n_s} \subset Q^{r_{k-s}}_{d_{k-s}}$, Condition (1) needs to be satisfied. Below, in defining restriction varieties, we will require the isotropic k-planes to intersect $Q^{r_i}_{d_i}$ in a subspace of dimension $k-i+1$. Hence, Condition (2) needs to be satisfied.

Condition (3) expresses the "Corank bound" that a hyperplane section of a quadric of corank r can have corank at most $r+1$. Conditions (4) and (5) express that the singular loci of the quadrics $Q_{d_i}^{r_i}$ are in the most special position. The singular locus of the quadric $Q_{d_i}^{r_i}$ contains the singular locus of all the larger dimensional quadrics in the sequence. Furthermore, isotropic linear spaces in the sequence of dimension greater (resp., less) than r_i contain (resp., are contained in) the singular locus of $Q_{d_i}^{r_i}$. Finally, Condition (6) is a technical condition: If a quadric $Q_{d_i}^{r_i}$ is more singular than the linear spaces in the sequence force it to be, then each quadric contained in $Q_{d_i}^{r_i}$ is more singular than the one larger quadric containing it except in a very special case detailed in Condition (6). These conditions will automatically hold for all the varieties in our algorithm, hence the reader does not need to remember these conditions to implement the algorithm.

We will use sequences of brackets and braces introduced in the previous section for representing the geometric sequences.

Definition 4.4. Let $(L_\bullet, Q_\bullet)$ be a sequence for $OG(k,n)$. The sequence of brackets and braces associated to $(L_\bullet, Q_\bullet)$ is a sequence of non-negative integers of length n, s right brackets and $k-s$ right braces such that

(1) The sequence consists of $r_i - r_{i-1}$ integers equal to i for $1 \leq i \leq k-s$ placed in increasing order followed by a sequence of $n - r_{k-s}$ zeros.
(2) The right square brackets are placed after the n_j-th integer in the sequence for $1 \leq j \leq s$ and the right braces are placed after the d_i-th integer in the sequence for $1 \leq i \leq k-s$.

In case $2n_s = n$, we distinguish between L_{n_s} and L'_{n_s} by writing $]$ and $]'$, respectively, for the right bracket after the n_s-th digit.

Example 4.5. The sequence of brackets and braces $1]22]000\}00\}0$ represents the sequence $L_1 \subset L_3 \subset Q_6^3 \subset Q_8^1$. To determine the (vector space) dimension d_i of the span of the quadric $Q_{d_i}^{r_i}$, we count the number of digits to the left of the i-th brace. For example, there are 8 digits to the left of the right most brace, so $d_1 = 8$. There are six digits to the right of the second brace, so $d_2 = 6$. To determine r_i, we count the number of positive digits less than or equal to i. In this example, there are 3 positive digits less than or equal to 2, so $r_2 = 3$. There is a unique one, so $r_1 = 1$. Finally, to determine n_j, we count the number of digits to the left of the j-th square bracket. In this example, $n_1 = 1$, $n_2 = 3$. The reader will notice that the Zariski closure of the subvariety of $OG(4,9)$ parameterizing isotropic subspaces Λ that satisfy

$$\dim(\Lambda \cap L_1) = 1, \dim(\Lambda \cap L_3) = 2, \dim(\Lambda \cap Q_6^3) = 3, \dim(\Lambda \cap Q_8^1) = 4$$

is the Schubert variety $\Omega_{4,2}^{3,1}$. Note that the sequence μ in our notation for Schubert varieties denotes the codimensions (equivalently, coranks) of the quadrics defining the variety, so it is very easy to read from the diagram.

The sequence of brackets and braces associated to $(L_\bullet, Q_\bullet)$ is a sequence of brackets and braces in the sense of the previous section. Since $n_1 < \cdots < n_s < d_{k-s} < \cdots < d_1$, the brackets and braces occupy different positions. Since the quadrics contain the linear spaces, the brackets are to the left of all the braces. The positive integers are increasing and less than or equal to the number of braces and they are all to the left of the zeros by construction. The position of a bracket $p(]^j)$ is equal the dimension n_j of the linear space L_{n_j}. The position of a brace $p(\}^i)$ is equal to the dimension of the span d_i of the quadric $Q_{d_i}^{r_i}$. The dimension r_i of the singular locus of $Q_{d_i}^{r_i}$ is the number of positive integers $l(\leq i)$ less than or equal to i. Finally, $l(i)$ is $r_i - r_{i-1}$ and $\rho(i, i-1) = d_{i-1} - d_i$. Hence, these sequences satisfy conditions (D1) (which is equivalent to Condition (3)), (D2) (which is equivalent to condition (2)) and (D3) (which is equivalent to Condition (6)).

Definition 4.6. Given a sequence $(L_\bullet, Q_\bullet)$, let x_i denote the number of isotropic linear spaces L_{n_j} of the sequence contained in $Q_{d_i}^{r_i, sing}$. Similarly, let y_j be the integer such that $r_{y_j - 1} < n_j \leq r_{y_j}$. If $r_i < n_j$ for every $1 \leq i \leq k-s$, set $y_j = k - s + 1$.

Remark 4.7. We will require the $(k-i+1)$-dimensional subspace contained in $Q_{d_i}^{r_i}$ to intersect $Q_{d_i}^{r_i, sing}$ in a subspace of dimension x_i. The index y_j is the smallest index i such that L_{n_j} is contained in the singular locus of $Q_{d_i}^{r_i}$. By conditions (4) and (5), every quadric of index at least y_j will be everywhere singular along L_{n_j}.

We need some further assumptions on the sequence $(L_\bullet, Q_\bullet)$ before it reflects the properties of the corresponding variety.

Example 4.8. Consider the sequence $L_3 \subset Q_5^1 \subset Q_6^1$ depicted by

$$100]00\}0\}0.$$

By 'the linear space bound', any isotropic 3-plane in $OG(3,7)$ which is contained in Q_6^1 necessarily must contain the singular point of Q_6^1. (Geometrically, any plane in a five dimensional quadric cone contains the vertex.) Hence the sequence $L_1 \subset Q_5^1 \subset Q_6^1$

$$1]0000\}0\}0$$

better reflects the geometric properties of isotropic 3-planes contained in Q_6^1. Similarly, consider the sequence $Q_4^2 \subset Q_6^0$ depicted by

$$2200\}00\}0.$$

The quadric Q_4^2 is reducible. (Geometrically, a quadric surface which is singular along a line is the union of two planes.) Hence, two sequences of the form $L_3 \subset Q_6^0$

$$000]000\}0$$

better reflect the geometry of the corresponding variety.

These examples motivate the following definition.

Definition 4.9. A sequence $(L_\bullet, Q_\bullet)$ associated to $OG(k,n)$ is *admissible* if the linear spaces and quadrics satisfy the following additional conditions:

(7) $r_{k-s} \leq d_{k-s} - 3$.

(8) For every $1 \leq i \leq k-s$,

$$x_i \geq k - i + 1 - \frac{d_i - r_i}{2}.$$

(9) For any $1 \leq j \leq s$, there does not exist $1 \leq i \leq k-s$ such that $n_j - r_i = 1$.

Remark 4.10. If Condition (7) is violated, then $Q^{r_{k-s}}_{d_{k-s}}$ would either be reducible or non-reduced. Condition (8) expresses the fact that in a quadric $Q^{r_i}_{d_i}$, a linear space of dimension $k-i+1$ has to intersect the singular locus in dimension at least $k-i+1-\frac{d_i-r_i}{2}$ (see Remark 4.7). Condition (9) expresses the fact that if $n_j - r_i = 1$ for some pair, then the tangent spaces to $Q^{r_i}_{d_i}$ would be constant along L_{n_j}. Hence the $(k-i+1)$-dimensional subspace contained in $Q^{r_i}_{d_i}$ would actually be contained in $Q^{r_i+1}_{d_i-1}$ with singular locus L_{n_j}. The reader should remember these three conditions in order to implement the algorithm.

Lemma 4.11. *The sequence of brackets and braces associated to an admissible sequence is a quadric diagram. Conversely, every quadric diagram corresponds to an admissible sequence* $(L_\bullet, Q_\bullet)$.

Proof. We already saw that the sequence associated to $(L_\bullet, Q_\bullet)$ is a sequence of brackets and braces that satisfies the conditions (D1), (D2) and (D3). Conditions (7), (8) and (9) translate to the conditions (D4), (D5) and (D6). If $r_{k-s} \leq d_{k-s}-3$, then there are at least three zeros to the left of $\}^{k-s}$ since the total number of positive integers in the sequence (r_{k-s}) is three less than the position of $\}^{k-s}$. Using the facts that $d_i = p(\}^i)$ and $r_i = l(\leq i)$, Conditions (8) and (D5) are direct translations of each other. Finally, if the two digits preceding a bracket $]^j$ are $a < b$, then $n_j - r_a = 1$ contradicting Condition (9). If a bracket is at the first position, then $n_1 = 1$. If $r_1 = 0$, then $n_1 - r_1 = 1$ contradicting Condition (9). Hence, the digit preceding $]^1$ must be 1. We conclude that conditions (D6) and (9) are equivalent. Finally, observe that Condition (8) implies Condition (2). We have included Condition (2) to simplify certain statements in the proof of the algorithm. We conclude that the data defining quadric diagrams and admissible sequences are equivalent. $\square$

Definition 4.12. Let $(L_\bullet, Q_\bullet)$ be an admissible sequence for $OG(k,n)$. A *restriction variety* $V(L_\bullet, Q_\bullet)$ is the subvariety of $OG(k,n)$ defined as the Zariski closure of the following quasi-projective variety

$$V(L_\bullet, Q_\bullet)^0 := \{ [W] \in OG(k,n) \mid \dim(W \cap L_{n_j}) = j, \dim(W \cap Q^{r_i}_{d_i}) = k-i+1, \dim(W \cap Q^{r_i, sing}_{d_i}) = x_i \}.$$

Example 4.13. Schubert varieties in $OG(k,n)$ are restriction varieties defined with respect to sequences satisfying $d_i + r_i = n$ for all $1 \leq i \leq k-s$ (see Lemma 4.18). The intersection of a general Schubert variety in $G(k,n)$ with $OG(k,n)$ (when non-empty) is a restriction variety associated to a sequence where $s = 0$ and $r_i = 0$ for $1 \leq i \leq k$ (see Proposition 6.2 for the precise statement). Hence, restriction varieties are a class of varieties that interpolate between the restriction of Schubert varieties in $G(k,n)$ and Schubert varieties in $OG(k,n)$.

Remark 4.14. A restriction variety does not have to be irreducible. For example,

$$000\}0\}0$$

in $OG(2,5)$ consists of two irreducible components. (Geometrically, the corresponding restriction variety parametrizes lines on a smooth quadric surface in $\mathbb{P}^3$.) When the inequality in Condition (8) is an equality for an index i, then the $(d_i + r_i)/2$-dimensional linear spaces in $Q_{d_i}^{r_i}$ form two irreducible components. The $(k-i+1)$-dimensional subspaces contained in $Q_{d_i}^{r_i}$ may be distinguished by their parity of the dimension of their intersection with linear spaces in each of these components.

Definition 4.15. Let $(L_\bullet, Q_\bullet)$ be an admissible sequence. An index $1 \leq i \leq k-s$ such that

$$x_i = k - i + 1 - \frac{d_i - r_i}{2}$$

is called a special index. For each special index, a marking $m_\bullet$ of $(L_\bullet, Q_\bullet)$ designates one of the irreducible components of $\frac{d_i+r_i}{2}$-dimensional linear spaces of $Q_{d_i}^{r_i}$ as even and the other one as odd, such that

- If $d_{i_1} + r_{i_1} = d_{i_2} + r_{i_2}$, for two special indices $i_1 < i_2$, and the component containing a linear space V is designated even for i_2, then the component containing V is designated even for i_1 as well; and
- If $2n_s = d_i + r_i$ for a special index i, then the component to which L_{n_s} belongs is assigned the parity of s; and
- If $n = 2k$, $m_\bullet$ assigns the component containing L_k the parity that characterizes the component $OG(k, 2k)$.

A marked restriction variety $V(L_\bullet, Q_\bullet, m_\bullet)$ is the Zariski closure of the subvariety of $V(L_\bullet, Q_\bullet)^0$ parameterizing k-dimensional isotropic subspaces W, where, for each special index i, W intersects subspaces of dimension $\frac{d_i+r_i}{2}$ of $Q_{d_i}^{r_i}$ designated even (respectively, odd) by $m_\bullet$ in a subspace of even (respectively, odd) dimension.

Proposition 4.16. *The marked restriction variety $V(L_\bullet, Q_\bullet, m_\bullet)$ associated to a marked admissible sequence is an irreducible variety of dimension*

$$\dim(V(L_\bullet, Q_\bullet, m_\bullet)) = \sum_{j=1}^{s}(n_j - j) + \sum_{i=1}^{k-s}(d_i + x_i - 2s - 2i) \tag{1}$$

Proof. We prove this proposition by induction on k. Suppose $k = 1$. If $s = 1$, then clearly the variety is isomorphic to projective space of dimension $n_1 - 1$ and the proposition holds. If $s = 0$, then the variety is isomorphic to a quadric hypersurface in $\mathbb{P}^{d_1-1}$ singular along a linear space of codimension at least three (by Condition (7) in Definition 4.9). Since such a quadric is irreducible of dimension $d_1 - 2$, the base case of the induction follows.

Now suppose that the proposition holds up to $k-1$. If $k - s = 0$, then the proposition is immediate. In that case, the isotropic subspaces are contained in the Grassmannian $G(k, n_k)$ and the restriction variety is an ordinary Schubert variety $(\Sigma_{n_k - n_1 - k+1, \dots, n_k - n_{k-1} - 1})$ in $G(k, n_k)$. The irreducibility and the dimension follow from these considerations. We may assume that $k - s > 0$. Let $(L_\bullet, Q'_\bullet)$ be the sequence for $OG(k-1, n)$ obtained from $(L_\bullet, Q_\bullet)$ by omitting $Q_{d_1}^{r_1}$ from the sequence (and subtracting one from the indices of the quadrics). Observe that $(L_\bullet, Q'_\bullet)$ is also an admissible sequence: Conditions (1)-(9) remain valid when we omit the largest quadric. Let $m'_\bullet$ be the restriction of the marking $m_\bullet$ to this new sequence, where m' designates the same components of linear spaces as even if $r_i + d_i < r_1 + d_1$ and swaps the designation for linear spaces with $r_i + d_i = r_1 + d_1$. Let $V(L_\bullet, Q_\bullet, m_\bullet)^0$ denote the intersection of $V(L_\bullet, Q_\bullet, m_\bullet)$ with $V(L_\bullet, Q_\bullet)^0$, the Zariski open set used to define $V(L_\bullet, Q_\bullet)$. We then have a morphism $f : V(L_\bullet, Q_\bullet, m_\bullet)^0 \to V(L_\bullet, Q'_\bullet, m'_\bullet)^0$ by taking the intersection of the linear spaces of dimension k in $V(L_\bullet, Q_\bullet, m_\bullet)^0$ with $Q^{r_2}_{d_2}$. By induction, we can assume that the the image is an irreducible variety of dimension predicted by the proposition. We now study the fibers of this morphism. Fix a point $[W]$ in the image. By assumption, the dimension of intersection of W with the singular locus of $Q^{r_1}_{d_1}$ is x_1. Then any k-dimensional linear space containing W has to be contained in the quadric Q' cut out on Q_1 by the linear space everywhere tangent to W. This is a quadric of corank $r_1 + k - 1 - x_1$ in a linear space of dimension $d_1 - (k - 1 - x_1)$. We have to choose a k-plane containing W. We can choose a linear section Q'' of Q' complementary to W. Choosing a k-plane is equivalent is to choosing a point on Q''. Hence, the dimension of the fiber is $d_1 - k + 1 + x_1 - 2 - k + 1$. Furthermore, by Condition (8)

$$x_1 \geq k - \frac{d_1 - r_1}{2}.$$

If the inequality is strict, it follows that

$$r_1 + k - 1 - x_1 < (d_1 - k + 1 + x_1) - 2,$$

hence Q'' and consequently the fiber is irreducible. If equality holds, then Q'' is a union of two linear spaces. The marking $m_\bullet$ selects one of these components by specifying the parity of the dimension of intersection with the k-dimensional linear space. Hence, the fiber is irreducible. This concludes the proof. $\square$

Remark 4.17. Since Equation 1 does not depend on the marking $m_\bullet$, every irreducible component of the restriction variety $V(L_\bullet, Q_\bullet)$ has dimension

$$\sum_{j=1}^{s} (n_j - j) + \sum_{i=1}^{k-s} (d_i + x_i - 2s - 2i).$$

Observe that $V(L_\bullet, Q_\bullet)$ has an irreducible component for every marking $m_\bullet$. The markings $m_\bullet$ parameterize the irreducible components of $V(L_\bullet, Q_\bullet)$. Correspondingly, given a sequence D of brackets and braces, we define $\dim(D)$ by the expression

$$\sum_{j=1}^{s} (p(]^j) - j) + \sum_{i=1}^{k-s} (p(\}^i) + x_i - 2s - 2i).$$

Lemma 4.18. *Schubert varieties in $OG(k, n)$ are the restriction varieties where the admissible sequence defining the restriction variety satisfies $r_i + d_i = n$ for every $1 \leq i \leq k - s$. When $n = 2k$, we also require that the k-dimensional linear spaces to intersect the k-dimensional linear space L_k in the sequence in a subspace of the correct parity.*

Proof. Set $\alpha = \lfloor (n-1)/2 \rfloor$. Let the sequence λ be defined by setting $\lambda_j = \alpha + 1 - n_j$. Let the sequence μ be given by setting $\mu_{k-i+1} = r_i$. We claim that the restriction variety $V(L_\bullet, Q_\bullet)$ is the Schubert variety Ω^{μ}_{λ}. Since the sequence satisfies Conditions (4) and (5), it suffices to show that there does not exist n_j and r_i such that $n_j - r_i = 1$ for any i and j. This is guaranteed by Condition (9) defining admissible sequences. When $2k = n$, we require that the length of λ have the same parity as k (alternatively, we could interpret a restriction variety with the wrong parity as a Schubert variety for the other connected component of $OG(k,2k)$). Note that under the assumptions of the lemma, the restriction variety is automatically irreducible. Suppose equality holds in Condition (8) for some i_0. Then n is even. Condition (9) and the assumption on the sequence imply that equality holds for every $i \geq i_0$. In particular, equality must hold for the index $k-s$. Combining Conditions (9) and equality in Condition (8), we have

$$n_s \geq r_{k-s} + 1 + s - \left(s + 1 - \frac{d_{k-s} - r_{k-s}}{2} \right) = \frac{d_{k-s} + r_{k-s}}{2} = \frac{n}{2}.$$

Using Condition (1), we deduce that $n_s = n/2$. Hence, the marking is uniquely determined by the sequence. □

5. The Algorithm for computing the classes of restriction varieties in Grassmannians

5.1. **The strategy and examples.** The strategy to calculate the class of a restriction variety $V(L_\bullet, Q_\bullet)$ is to specialize it into a union of Schubert varieties by successively making the quadrics in the sequence more singular. By the corank bound (Condition (3)), if $r_i + d_i = r_{i-1} + d_{i-1}$, then $Q_{d_i}^{r_i}$ is as singular as it can be given that it is contained in $Q_{d_{i-1}}^{r_{i-1}}$, so its corank cannot be increased. If $r_i + d_i = n$ for all $1 \leq i \leq k-s$, then $V(L_\bullet, Q_\bullet)$ is a Schubert variety and there is nothing further to do. Otherwise, there is a smallest dimensional quadric whose corank can be increased. We increase the corank of this quadric (fixing all the other linear spaces and quadrics) by one by specializing the quadric in a pencil. As a result of this specialization, the restriction variety breaks into a union of restriction varieties each with multiplicity one. In the rest of this section, we will describe the components and show that they occur with multiplicity one. We first discuss several fundamental examples that illustrate the possibilities.

Example 5.1. We first compute the class of $V(Q_4^0)$ depicted by

$$0000\}0$$

in $OG(1,5)$. Projectively, $V(Q_4^0)$ parametrizes points on a smooth quadric hypersurface Q in $\mathbb{P}^4$ that are contained in a smooth hyperplane section Q_4^0. We specialize the hyperplane section until it becomes tangent to Q. This specialization replaces Q_4^0 with Q_4^1 (singular at the point of tangency). In the process, the restriction variety is transformed to

$$1000\}0.$$

This is the quadric diagram D^a described in §3. Observe that if the linear spaces had to intersect the singular locus, then they would just be the singular point of Q_4^1. The singular point has smaller dimension than the quadric Q_4^1. That's why in these cases the quadric diagrams derived from D^b do not occur. The cohomology class of a smooth hyperplane section is the same as that of a singular hyperplane section, hence $V(Q_4^0)$ and $V(Q_4^1)$ have the same cohomology class. Since $V(Q_4^1)$ is a Schubert variety with class σ^1 in $OG(1,5)$, this concludes the calculation.

During this process, a quadric may become reducible. As a slight variation, we compute the class of $V(Q_3^0)$ depicted by

$$000\}0$$

in $OG(1,4)$. Projectively, $V(Q_3^0)$ parametrizes points contained in a smooth conic in a smooth quadric surface Q in $\mathbb{P}^3$. We specialize the plane of the conic until it becomes tangent to Q, replacing Q_3^0 with Q_3^1. Note that Q_3^1 violates Condition (7) (its corank is two less than its ambient dimension). Geometrically, a singular conic is a union of two lines which belong to two different rulings on the quadric surface Q. The sequence of brackets and braces $100\}0$ fails condition (D4). We replace it by the two quadric diagrams $00]00$ and $00]'00$ according to §3. Hence, the restriction variety corresponding to the diagram $000\}0$ is replaced by the two restriction varieties corresdponding to

$$00]00 \quad \text{and} \quad 00]'00.$$

Geometrically, the class of a conic is the sum of the classes of two lines on the quadric one in each ruling. This concludes the calculation since the latter two varieties are Schubert varieties with classes σ_0 and σ^2, respectively. Hence, the class of $V(Q_3^0)$ in $OG(1,4)$ is $\sigma_0 + \sigma^2$. This example shows that in the algorithm, we have to replace a quadric by two linear spaces if the specialization forces the quadric to become reducible (or, equivalently, violate Condition (7)).

Example 5.2. Next we calculate the class of the restriction variety $V(L_2 \subset Q_4^0)$ in $OG(2,5)$ depicted by

$$00]00\}0.$$

Geometrically, this example corresponds to calculating the class of the inclusion of $OG(2,4)$ in $OG(2,5)$. More concretely, we calculate the class of the space of lines contained in the smooth quadric surface Q_4^0 and intersect the line L_2 (in a point). Note that Q_4^0 is a smooth hyperplane section of the ambient quadric Q in $\mathbb{P}^4$. We specialize it until it becomes tangent to Q at a point on L_2. This replaces Q_4^0 with Q_4^1, the quadric cone singular at the point of tangency. This is depicted by $D^a = 10]00\}0$, which violates condition (D5). By the linear space bound (Condition (8)), lines contained in a quadric cone in $\mathbb{P}^3$ all pass through the vertex of the cone. Hence, the degeneration replaces $V(L_2 \subset Q_4^0)$ with the restriction variety $V(L_1 \subset Q_4^1)$

$$1]000\}0.$$

This is the quadric diagram D^b defined in §3. This restriction variety is the Schubert variety with class σ_2^1. Note that in this case, this is the diagram derived from D^b and D^a does not lead to any diagrams since it violates condition (D5). Geometrically, this corresponds to the fact that the lines are required to pass through the singular point.

Example 5.3. Finally, consider the variety $V(L_2 \subset Q_6^0)$ in $OG(2,7)$.

$$\begin{array}{l} 00]0000\}0 \rightarrow 1]00000\}0 \\ \downarrow \\ 11]000\}00 \end{array}$$

Geometrically, this variety parametrizes lines on a smooth quadric Q in $\mathbb{P}^6$ that intersect a line L_2 and are contained in a smooth hyperplane section Q_6^0 of Q. As before, we specialize the linear space defining Q_6^0 to be tangent to Q along a point of the line L_2, replacing Q_6^0 with Q_6^1. In the limit, there are two possibilities. In the first case, the lines may all pass through the singular point of Q_6^1. This case

$(V(L_1 \subset Q_6^1))$ is depicted by the quadric diagram $D^b = 1]00000\}0$. In the second, case the lines intersect L_2 in a point other than the vertex. This is denoted by the sequence of brackets and braces $D^a = 10]0000\}0$. Note that this sequence fails condition (D6). By "the variation of tangent spaces", the tangent spaces to Q_6^1 are constant along the line L_2. Therefore, the lines in Q_6^1 that intersect L_2 in a point other than the singular point have to be contained in the quadric Q_5^2 given by the intersection of Q with the linear space everywhere tangent to Q along L_2. This possibility $(V(L_2 \subset Q_5^2))$ is depicted by $11]000\}00$, which is the quadric diagram derived from D^a as in §3. Both of these varieties are Schubert varieties and occur with multiplicity one in the limit. Hence, the class of $V(L_2 \subset Q_6^0)$ is $\sigma_3^1 + \sigma_2^2$.

Example 5.3 shows the basic branching. When we increase the corank of the quadric, the linear spaces intersect the new singular locus either in a larger dimensional vector space (unless this possibility leads to a smaller dimensional variety as in Example 5.1) or in the same dimensional vector space (unless this possibility is excluded by the linear space bound (Condition (8)) as in Example 5.2). Additional branching occurs when one of the quadrics become reducible (as in Example 5.1). The general rule is obtained by repeating these three fundamental examples. In fact, these examples capture all the geometric complexity of restriction varieties in orthogonal Grassmannians. Next we give a complicated example that illustrates the inductive structure of the Algorithm.

Example 5.4. Consider the restriction variety $V = V(Q_4^0 \subset Q_6^0 \subset Q_8^0)$ in $OG(3,9)$. Concretely, V is the intersection of $OG(3,9)$ with a general Schubert variety $\Sigma_{3,2,1}(F_\bullet)$ in $G(3,9)$. We calculate the class of V in terms of Schubert classes in $OG(3,9)$ as follows.

$$
\begin{array}{l}
0000\}00\}00\}0 \to 3000\}00\}00\}0 \xrightarrow{\times 2} 000]000\}00\}0 \to 200]000\}00\}0 \xrightarrow{\times 2} 000]0]0000\}0 \to 100]0]0000\}0 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 22]0000\}00\}0 \to 1]20000\}00\}0 \to 1]22000\}00\}0 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 11]0000\}0\}00 \to 11]2000\}0\}00
\end{array}
$$

We explain the salient features of this example. In the first two steps, we increase the corank of the smallest dimensional quadric Q_4^0 by one. After the second step, we obtain Q_4^2, which is a reducible quadric equal to the union of two linear spaces of dimension three. (in terms of the combinatorics of quadric diagrams $3300\}00\}00\}0$ violates condition (D4), so has to be replaced by two copies of $000]000\}00\}0$.) Correspondingly, the restriction varieties breaks into two irreducible components both isomorphic to the restriction variety $V(L_3 \subset Q_6^0 \subset Q_8^0)$. The symbol $\times 2$ above the right arrow indicates that there are two components of the limit with the same class (though they are distinct varieties, each occurring with multiplicity one). In the next two steps, we increase the corank of the quadric Q_6^0 by one. After the second step, either the linear spaces intersect the singular locus of Q_6^2 and we get the restriction variety indicated by the down arrow or the linear spaces do not intersect the singular locus of Q_6^2. In the latter case, the tangent spaces to Q_6^2 are constant along L_3. Hence, these linear spaces must intersect the quadric Q_5^3 everywhere tangent along the three dimensional linear space in a two-dimensional subspace.

Note that the latter quadric Q_5^3 is reducible, the union of two linear spaces. Hence, in this case there are two components which are indicated after the right arrow. (In terms of the combinatorics of quadric diagrams we have $D^a = 220]000\}00\}0$ and $D^b = 22]0000\}00\}0$. D^b is a quadric diagram, but D^a fails condition (D6), so we replace it with $D^c = 222]00\}000\}0$, which fails condition (D4). We have to replace D^c by two copies of $000]0]0000\}0$). The rest of the calculation is similar to the previous examples. We conclude that the class of the variety is equal to

$$4\sigma_{2,1}^1 + 2\sigma_4^{3,1} + 2\sigma_3^{3,2}.$$

5.2. **The algorithm.** We now give the algorithm for computing the class of the variety $V(L_\bullet, Q_\bullet)$ in terms of Schubert classes in $OG(k,n)$. First, we begin with a slogan that can help guide the reader through the combinatorics.

The Rule in Slogan Form: Increase the dimension of the singular locus of the smallest dimensional quadric allowed by the corank bound (Condition (3)) by one. The linear spaces intersect the new singular locus either in a subspace of the same dimension as before or in one larger dimension, unless one of these possibilities leads to a smaller dimensional variety or is precluded by the linear space bound (Condition (8)).

This section and §3 make this slogan precise.

Definition 5.5. Let $(L_\bullet, Q_\bullet)$ be an admissible sequence. We say that the quadric $Q_{d_i}^{r_i}$ is saturated if $r_i = n - d_i$. $V(L_\bullet, Q_\bullet)$ is saturated if every quadric $Q_{d_i}^{r_i}$, $1 \leq i \leq k-s$, in its definition is saturated. If the admissible sequence contains a quadric which is not saturated, define the active index κ to be the largest index i for which $r_i - r_{i-1} < d_{i-1} - d_i$ (where, by convention, we set $r_0 = 0$ and $d_0 = n$).

Remark 5.6. By Lemma 4.18, a saturated restriction variety is a Schubert variety. If a quadric $Q_{d_i}^{r_i}$ in the definition of a restriction variety is not saturated, then $Q_{d_j}^{r_j}$ is not saturated for any $j \geq i$. In particular, the smallest dimensional quadric $Q_{d_{k-s}}^{r_{k-s}}$ is not saturated. The quadric $Q_{d_\kappa}^{r_\kappa}$ is the smallest dimensional quadric in the sequence $(L_\bullet, Q_\bullet)$ which is not maximally singular given the larger quadrics containing it.

We will compute the class of $V(L_\bullet, Q_\bullet)$ by successively increasing r_κ by one, where κ is the active index. This corresponds to a specialization of the flag defining the restriction variety. In the process, $V(L_\bullet, Q_\bullet)$ will specialize into a union of restriction varieties. Applying the degeneration to each of the resulting varieties, we will be able to decompose any restriction variety into a union of Schubert varieties.

Degeneration 5.7. Let $Sing(Q)$ denote the singular locus of a quadric Q. To avoid multiple indices set $L = L_{n_{x_\kappa+1}}$. Let $p \in L \cap Sing(Q_{d_{\kappa+1}}^{r_{\kappa+1}})$. Suppose that $p \notin Sing(Q_{d_\kappa}^{r_\kappa})$. Recall that L is the smallest dimensional isotropic linear space in $(L_\bullet, Q_\bullet)$ that is not entirely contained in $Sing(Q_{d_\kappa}^{r_\kappa})$. It is understood that if $\kappa = k-s$, the condition that $p \in Sing(Q_{d_{\kappa+1}}^{r_{\kappa+1}})$ is vacuous. Similarly, if $x_\kappa = s$, then $p \in Q_{d_\kappa}^{r_\kappa} \cap Sing(Q_{d_{\kappa+1}}^{r_{\kappa+1}})$, but $p \notin Sing(Q_{d_\kappa}^{r_\kappa})$.

Let $S = Span(Q_{d_\kappa}^{r_\kappa})$ and let $U = Sing(Q_{d_\kappa}^{r_\kappa})$. Let $T = T_p Q_{d_{\kappa-1}}^{r_{\kappa-1}}$ be the tangent space to $Q_{d_{\kappa-1}}^{r_{\kappa-1}}$ at p. By Condition (5), $Q_{d_{\kappa-1}}^{r_{\kappa-1}}$ is smooth at p so the tangent hyperplane exists. Moreover, since p is not a singular point of $Q^{r_\kappa}_{d_\kappa}$, T cannot contain $Q^{r_\kappa}_{d_\kappa}$. We conclude that $T \cap S := M$ is a codimension one linear space. On the other hand, since $Q^{r_{\kappa+1}}_{d_{\kappa+1}}$ is singular at p, M automatically contains $Q^{r_{\kappa+1}}_{d_{\kappa+1}}$. Let $N = Span(M, U)$. Note that N has dimension d_κ. Consider the pencil of linear spaces determined by N and S. Since N and S have M in common, they span a linear space of dimension $d_\kappa + 1$. In this linear space and in appropriate coordinates, this pencil can be expressed as $tx + (1-t)y$, where $y = 0$ defines N and $x = 0$ defines S. This pencil of linear spaces cut out a pencil $Q^{r_\kappa(t)}_{d_\kappa}(t)$ of sub-quadrics on Q. When $t = 1$, this is the original quadric $Q^{r_\kappa}_{d_\kappa}$. When $t = 0$, it is a quadric of corank $r_\kappa + 1$. Note that all of these quadrics contain $Q^{r_{\kappa+1}}_{d_{\kappa+1}}$ and are contained in $Q^{r_{\kappa-1}}_{d_{\kappa-1}}$. Consequently, there exists a one-parameter family of sequences $(L_\bullet(t), Q_\bullet(t))$, where only the quadric $Q^{r_\kappa(t)}_{d_\kappa}(t)$ varies in the pencil just constructed. At a general t, the sequence is projectively equivalent to the original sequence. At the special point $t = 0$, the sequence $(L_\bullet(0), Q_\bullet(0))$ is equivalent to a sequence where r_κ has been replaced by $r_\kappa + 1$. Correspondingly, there is a one-parameter family of restriction varieties $V(t)$ defined with respect to the flags $(L_\bullet(t), Q_\bullet(t))$. As long as $t \neq 0$, these varieties are isomorphic. Hence, they form a flat family. By the properness of the Hilbert scheme, there exists a flat limit $V(0)$. Our algorithm is obtained by describing $V(0)$.

Notation 5.8. For the rest of the paper, we will always use Degeneration 5.7. Given an admissible sequence $(L_\bullet, Q_\bullet)$, $(L_\bullet(t), Q_\bullet(t))$ will denote the position of the flag at time t under this degeneration. To simplify notation, we will use $(L^a_\bullet, Q^a_\bullet)$ to denote the special position of the flag at $t = 0$. The dimension of the linear spaces and the dimension and corank of the quadrics in $(L^a_\bullet, Q^a_\bullet)$ will be denoted by n'_j, d'_i and r'_i, respectively. Note that except for r'_κ, these invariants equal to those of $(L_\bullet, Q_\bullet)$ and $r'_\kappa = r_\kappa + 1$.

Observe that the sequence of brackets and braces associated to $(L^a_\bullet, Q^a_\bullet)$ is D^a defined in §3. The degeneration increases r_κ by one. This is represented by changing the integer in the $(r_\kappa+1)$-st place in the quadric diagram corresponding to $(L_\bullet, Q_\bullet)$ to κ.

The reader should note that the sequence $(L^a_\bullet, Q^a_\bullet)$ does not have to be admissible. The algorithm will consist of decomposing $(L^a_\bullet, Q^a_\bullet)$ into a collection of admissible sequences $(L^j_\bullet, Q^j_\bullet)$. The flat limit will be supported along the union of the restriction varieties corresponding to these sequences. We replace $V(L_\bullet, Q_\bullet)$ by a collection of restriction varieties $V(L^j_\bullet, Q^j_\bullet)$ each occurring with multiplicity one. Hence, the cohomology class of $V(L_\bullet, Q_\bullet)$ is the sum of the cohomology classes of $V(L^j_\bullet, Q^j_\bullet)$. The varieties $V(L^j_\bullet, Q^j_\bullet)$ will be "closer" to Schubert varieties. By "closer" we mean that the admissible sequence $(L^j_\bullet, Q^j_\bullet)$ will have either $s^j = s + 1$ (one more linear space and one fewer quadric); or $r^j_i \geq r_i$ with strict inequality for at least one i (one of the quadrics will have a higher dimensional singularity). If we keep applying the algorithm to each of the varieties that are output, the varieties will eventually become saturated. Hence, we will express the class of $V(L_\bullet, Q_\bullet)$ as a sum of Schubert cycles.

A reminder about our notation: Recall that κ denotes the active index of $(L_\bullet, Q_\bullet)$. x_i denotes the number of isotropic subspaces of the sequence contained in the singular locus of $Q^{r_i}_{d_i}$. In particular, if $x_i < s$, then $L_{n_{x_i+1}}$ denotes the smallest dimensional isotropic space in the sequence strictly containing $Q^{r_i,sing}_{d_i}$ (in the quadric diagram notation, $L_{n_{x_i+1}}$ is depicted by the left most bracket such that one of the digits to its left is zero or greater than i). y_j denotes the index of the largest dimensional quadric containing L_{n_j} in its singular locus or $y_j = k-s+1$ if there are none (in terms of quadric diagrams, y_j is the positive digit to the immediate left of the j-th bracket or $y_j = k-s+1$ if this digit is zero.) The condition $n_{x_\kappa+1} - r_\kappa - 1 = y_{x_\kappa+1} - \kappa$ means that the codimension of $Q^{r_\kappa,sing}_{d_\kappa}$ in $L_{n_{x_\kappa+1}}$ is one more than the number of quadrics in the sequence that contain $Q^{r_\kappa}_{d_\kappa}$ but do not contain $L_{n_{x_\kappa+1}}$ in their singular locus.

Algorithm 5.9. We now give the algorithm that describes the maximal dimensional components of the flat limit of Degeneration 5.7.

Step 1. If $V(L_\bullet, Q_\bullet)$ is saturated (i.e., a Schubert variety), output $V(L_\bullet, Q_\bullet)$ and stop. The algorithm terminates. Otherwise,

- Let $(L^a_\bullet, Q^a_\bullet)$ be the sequence obtained by replacing $Q^{r_\kappa}_{d_\kappa}$ in $(L_\bullet, Q_\bullet)$ with $Q^{r_\kappa+1}_{d_\kappa} = Q^{r'_\kappa}_{d'_\kappa}$.
- If $x_\kappa < s$, then let $(L^b_\bullet, Q^b_\bullet)$ be the sequence obtained by replacing $L_{n_{x'_\kappa+1}}$ in $(L^a_\bullet, Q^a_\bullet)$ with $L_{r'_\kappa}$ (the singular locus of $Q^{r'_\kappa}_{d'_\kappa}$).

and proceed to Step 2.

Step 2. Depending on the case, replace $V(L_\bullet, Q_\bullet)$ by the following union of restriction varieties and stop.

• If $x_\kappa = s$ or if $n_{x_\kappa+1} - r_\kappa - 1 > y_{x_\kappa+1} - \kappa$ in the sequence $(L_\bullet, Q_\bullet)$, replace $V(L_\bullet, Q_\bullet)$ with the restriction varieties obtained by running Algorithm 5.10 on $(L^a_\bullet, Q^a_\bullet)$.

• If $(L^a_\bullet, Q^a_\bullet)$ violates Condition (8) (i.e., $x'_\kappa < k - \kappa + 1 - \frac{d'_\kappa - r'_\kappa}{2}$), then replace $V(L_\bullet, Q_\bullet)$ with the restriction varieties obtained by running Algorithm 5.10 on $(L^b_\bullet, Q^b_\bullet)$.

• If $x_\kappa < s$, $n_{x_\kappa+1} - r_\kappa - 1 = y_{x_\kappa+1} - \kappa$ in the sequence $(L_\bullet, Q_\bullet)$ and Condition (8) is satisfied for $(L^a_\bullet, Q^a_\bullet)$ (i.e., $x'_\kappa \geq k - \kappa + 1 - \frac{d'_\kappa - r'_\kappa}{2}$), then replace $V(L_\bullet, Q_\bullet)$ with the restriction varieties obtained by running Algorithm 5.10 on both sequences $(L^a_\bullet, Q^a_\bullet)$ and $(L^b_\bullet, Q^b_\bullet)$.

Algorithm 5.10 (Normalizing the sequence)**.** Given a sequence $(L^\alpha_\bullet, Q^\alpha_\bullet)$ equal to $(L^a_\bullet, Q^a_\bullet)$ or $(L^b_\bullet, Q^b_\bullet)$ defined in Algorithm 5.9, run the following loop on the sequence. We will call the process of replacing the sequence $(L_\bullet, Q_\bullet)$ by the sequences produced by this algorithm *normalizing the sequence.*

i. If the sequence $(L^\alpha_\bullet, Q^\alpha_\bullet)$ is admissible, output the sequence $(L^\alpha_\bullet, Q^\alpha_\bullet)$ and stop. Otherwise, proceed to [ii].

ii. If $r_{k-s} + 2 \geq d_{k-s}$ (i.e., Condition (7) is violated) in $(L^\alpha_\bullet, Q^\alpha_\bullet)$, replace $(L^\alpha_\bullet, Q^\alpha_\bullet)$ by two sequences $(L^i_\bullet, Q^i_\bullet)$ for $i = 1, 2$, where $(L^i_\bullet, Q^i_\bullet)$ is the sequence obtained from $(L^\alpha_\bullet, Q^\alpha_\bullet)$ by replacing $Q^{r_{k-s}}_{d_{k-s}}$ with $L_{d_{k-s}-1}$ unless $2(d_{k-s} - 1) = n$. If $2(d_{k-s} - 1) = n$, then in one of the sequences replace $Q^{r_{k-s}}_{d_{k-s}}$ with $L_{d_{k-s}-1}$ and in the other with $L'_{d_{k-s}-1}$. If in addition $2k = n$, discard the sequence that parameterizes linear spaces that has the wrong parity for the dimension of intersection with L_k. For each of the sequences $(L^i_\bullet, Q^i_\bullet)$, return to Step [i] and run the loop again setting $(L^\alpha_\bullet, Q^\alpha_\bullet) = (L^i_\bullet, Q^i_\bullet)$. If $r_{k-s} + 2 < d_{k-s}$ (i.e., Condition (7) holds), proceed to [iii].

iii. If Condition (9) is violated for $(L^\alpha_\bullet, Q^\alpha_\bullet)$, while Condition (9) is violated, let μ be the largest index for which it is violated. Form a new sequence $(L^\beta_\bullet, Q^\beta_\bullet)$ by replacing $Q^{r_\mu}_{d_\mu}$ in $(L^\alpha_\bullet, Q^\alpha_\bullet)$ with $Q^{r_\mu+1}_{d_\mu-1}$. Discard the sequence $(L^\beta_\bullet, Q^\beta_\bullet)$ if $d_{\mu+1} = d_\mu - 1$ in $(L^\alpha_\bullet, Q^\alpha_\bullet)$. If there are no remaining sequences, the algorithm terminates and does not our put any sequences. If $(L^\beta_\bullet, Q^\beta_\bullet)$ satisfies Condition (9), proceed to Step [i] and run the loop again setting $(L^\alpha_\bullet, Q^\alpha_\bullet) = (L^\beta_\bullet, Q^\beta_\bullet)$.

We already observed that the sequence $(L^a_\bullet, Q^a_\bullet)$ is represented by the sequence of brackets and braces D^a defined in §3. Next observe that $(L^b_\bullet, Q^b_\bullet)$ is represented by D^b defined in §3. $(L^b_\bullet, Q^b_\bullet)$ is obtained from $(L^a_\bullet, Q^a_\bullet)$ by replacing the smallest dimensional linear space containing the singular locus of $Q^{r'_\kappa}_{d_\kappa}$ with the singular locus of $Q^{r'_\kappa}_{d_\kappa}$. This corresponds to shifting the left most bracket whose position is greater than $l_{D^a}(\leq \kappa)$ to the position $l_{D^a}(\leq \kappa)$.

The problem, as observed in §3, is that D^a and D^b need not be quadric diagrams. Equivalently, $(L^a_\bullet, Q^a_\bullet)$ and $(L^b_\bullet, Q^b_\bullet)$ may fail to be admissible. Algorithm 5.10 replaces them by admissible sequences. The sequence $(L^a_\bullet, Q^a_\bullet)$ may fail to satisfy Conditions (7), (8), or (9). If it fails to satisfy Condition (8), this sequence does not lead to a variety supported on the flat limit. If it fails to satisfy Condition (7), then Algorithm 5.10 in Step (ii) replaces the sequence by two sequences. The geometric meaning of this step is that the quadric $Q^{r'_\kappa}_{d_\kappa}$ is reducible consisting of a union of two linear spaces. When n is even and the linear spaces have dimension $n/2$, they belong to two different connected components. These are distinguished in the algorithm.

When $(L^a_\bullet, Q^a_\bullet)$ fails to satisfy Condition (9) such as in the sequence represented by 10]0]0]00000}0, the loop in Step iii of Algorithm 5.10 increases the dimension of the singular locus of the quadric failing Condition (9) by one and decreases its dimension by one until Condition (9) is satisfied. In this case, the loop would produce the sequences represented by 11]0]0]0000}00, 11]1]0]000}000 and 11]1]1]00}0000, which satisfies Condition (9). Note however that Condition (7) may now fail to be satisfied, hence needs to be checked again. In Algorithm 5.10, it would have made more sense to swap Steps ii and iii. We write it this way for consistency with the case of flag varieties.

The sequence $(L^b_\bullet, Q^b_\bullet)$ may also fail to satisfy Condition (9). For example, the sequence represented by 3]0000}00}0} fails Condition (9). The loop in Step iii of Algorithm 5.10 increases the dimension of the singular loci and decreases the dimension of the quadrics containing the quadric failing Condition (9) successively. In this case, the loop would produce the sequences represented by 2]0000}0}00} and 1]0000}0}0}0, successively.

The geometric meaning of Step iii in Algorithm 5.10 is as follows. When $r_i = n_j - 1$, by "the variation of tangent spaces", the tangent space to $Q^{r_i}_{d_i}$ is constant along L_{n_j}. Hence, if a linear space intersects L_{n_j}, then it must be contained in this fixed tangent space. Therefore, the subspaces that are contained in $Q^{r_i}_{d_i}$ are already contained in the codimension one quadric cut out on $Q^{r_i}_{d_i}$ by the linear space everywhere tangent to $Q^{r_i}_{d_i}$ along L_{n_j}. The dimension of this quadric is one smaller and its singular locus contains L_{n_j}. Step iii of the Algorithm 5.10 replaces $Q^{r_i}_{d_i}$ with this quadric.

The geometric meaning of Algorithm 5.9 is apparent. Step 1 checks whether a given restriction variety is a Schubert variety. If so, the algorithm stops. Otherwise, we increase the corank of $Q^{r_\kappa}_{d_\kappa}$ by one using Degeneration 5.7. There are two possibilities. Either the linear spaces intersect the new singular locus of $Q^{r_\kappa+1}_{d_\kappa}$ in a vector space of dimension x_κ (this possibility corresponds to the sequence $(L^a_\bullet, Q^a_\bullet)$ and is depicted by D^a) or they intersect the singular locus in a subspace of dimension $x_\kappa + 1$ (this possibility is depicted by the sequence $(L^b_\bullet, Q^b_\bullet)$ and is depicted by D^b). Under the first condition in Step 2, the variety corresponding to $(L^b_\bullet, Q^b_\bullet)$ has smaller dimension than the original variety $V(L_\bullet, Q_\bullet)$. Therefore, the sequence $(L^b_\bullet, Q^b_\bullet)$ does not lead to a component of the flat limit of the Degeneration 5.7. We replace the original sequence by sequences obtained from $(L^a_\bullet, Q^a_\bullet)$. In the second case, $(L^a_\bullet, Q^a_\bullet)$ violates Condition (8), hence the dimension of intersection of the linear spaces with the singular locus $Q^{r'_\kappa}_{d_\kappa}$ has to increase. Therefore, the only possibilities are derived from the sequence $(L^b_\bullet, Q^b_\bullet)$. In the final case, sequences derived from both sequences $(L^a_\bullet, Q^a_\bullet)$ and $(L^b_\bullet, Q^b_\bullet)$ give components of the flat limit of the Degeneration 5.7. This is the geometric branching.

From our description of the two algorithms, it is clear that Algorithm 5.9 and Algorithm 3.9 are the same algorithm, one phrased in terms of admissible sequences and the other in terms of the quadric diagrams representing them. In the rest of the section, we will work with the geometric algorithm.

We will check shortly that Algorithm 5.9 replaces a restriction variety with restriction varieties. Hence, we can apply the algorithm to each of the resulting varieties until the end result is a collection of Schubert varieties. Before proceeding, we urge the reader to work through the examples in the beginning of this section.

Definition 5.11. A degeneration path for V_1 is a sequence of restriction varieties $V_1 \to V_2 \to \cdots \to V_m$ starting with V_1 and ending with a Schubert variety V_m such that V_{i+1} is one of the varieties assigned to V_i by Algorithm 5.9.

Theorem 5.12. *The class of a restriction variety V is equal*

$$[V] = \sum [V_i]$$

where V_i are the restriction varieties produced by Algorithm 5.9. In particular, the coefficient c^μ_λ in

$$[V] = \sum c^\mu_\lambda \sigma^\mu_\lambda$$

is the number of degeneration paths starting with V and ending in a variety with cohomology class σ^μ_λ. Furthermore, the algorithm respects the marking of restriction varieties.

Proof. We prove the theorem in three steps. We first check that Algorithm 5.9 transforms restriction varieties into a collection of restriction varieties of the same dimension. Then we interpret replacing $Q^{r_\kappa}_{d_\kappa}$ by $Q^{r_\kappa+1}_{d_\kappa}$ in Step 1 of Algorithm 5.9

as applying Degeneration 5.7. Using a dimension count, we show that the flat limit is supported along the varieties produced by the algorithm. Finally, we check that the flat limit is reduced at the generic point of each of these varieties. Theorem 5.12 follows. We begin by analyzing each case in the algorithm separately.

• If the sequence $(L_\bullet, Q_\bullet)$ is saturated, then Lemma 4.18 implies that $V(L_\bullet, Q_\bullet)$ is a Schubert variety. In this case, there is nothing further to do. Accordingly, Algorithm 5.9 terminates. From now on we may assume that $(L_\bullet, Q_\bullet)$ is not saturated.

The new sequences $(L^a_\bullet, Q^a_\bullet), (L^b_\bullet, Q^b_\bullet)$ formed in Step 1 may fail to be admissible. However, Conditions (1)-(6) are satisfied for them and for any of the sequences output by Algorithm 5.9. We begin by verifying this for $(L^a_\bullet, Q^a_\bullet)$. Conditions (4) and (5) hold by construction. Since Conditions (1) and (2) hold for $(L_\bullet, Q_\bullet)$ and replacing r_κ by $r_\kappa + 1$ can only increase the left-hand-side of the inequalities, Conditions (1) and (2) also hold for $(L^a_\bullet, Q^a_\bullet)$. The active index κ is chosen so that $Q^{r_\kappa}_{d_\kappa}$ satisfies the strict inequality $d_\kappa + r_\kappa < d_{\kappa-1} + r_{\kappa-1} \leq n$ in Condition (3). Increasing r_κ by one can at worst turn these inequalities into equalities and improves the corresponding inequality for the index κ. Therefore, Condition (3) holds for $(L^a_\bullet, Q^a_\bullet)$. The sequence $(L^a_\bullet, Q^a_\bullet)$ satisfies Condition (6) by the choice of κ. The ranks of the quadrics r_i remain unchanged for indices $i \neq \kappa$. The choice of κ implies that $j - i \leq d_i - d_j = r_j - r_i$ for every $j > i \geq \kappa$ in $(L_\bullet, Q_\bullet)$. Hence, replacing r_κ with $r_\kappa + 1$ ensures the inequality $r'_i - r'_\kappa \geq i - \kappa - 1$ for $i > \kappa$. The inequalities for $r'_\kappa - r'_i$ improve by one for $\kappa > i$. Finally, the second half of Condition (6) is also immediate from the choice of κ. We conclude that Conditions (1)-(6) hold for $(L^a_\bullet, Q^a_\bullet)$.

Next we note that the sequence $(L^b_\bullet, Q^b_\bullet)$ is obtained from $(L^a_\bullet, Q^a_\bullet)$ by replacing the linear space $L_{n_{x'_\kappa+1}}$ with the smaller dimensional linear space $L_{r'_\kappa}$. Replacing a linear space by a smaller dimensional one clearly preserves Conditions (1)-(4) and (6). Since all the quadrics with corank $r_i \leq r'_\kappa$ are singular along $L_{r'_\kappa}$, Condition (5) also holds. Hence, the sequence $(L^b_\bullet, Q^b_\bullet)$ satisfies Conditions (1)-(6).

Finally, we analyze how Algorithm 5.10 affects Conditions (1)-(6). We make the observation that if Condition (9) fails for $(L^a_\bullet, Q^a_\bullet)$, then it fails only for the index κ. If Condition (9) fails for $(L^b_\bullet, Q^b_\bullet)$, then it can only fail for indices $i < \kappa$.

- In Step ii of Algorithm 5.10, the quadric $Q^{r_{k-s}}_{d_{k-s}}$ is replaced by a linear space $L_{d_{k-s}-1}$ of dimension $d_{k-s} - 1$. Conditions (2)-(6) are unaffected by this change. By assumption, we have $d_{k-s} \leq r_{k-s} + 2$. Hence $2n_{s+1} = 2(d_{k-s} - 1) \leq d_{k-s} + r_{k-s} \leq d_{k-s-1} + r_{k-s-1}$. This verifies Condition (1).
- In Step iii of Algorithm 5.10, a quadric $Q^{r_i}_{d_i}$ is replaced by a quadric of corank $r_i + 1$ and ambient dimension $d_i - 1$. Note that all the inequalities in (1)-(3) are invariant under this transformation. Conditions (4) and (5) hold by construction. Condition (9) may fail to be satisfied for the index κ in $(L^a_\bullet, Q^a_\bullet)$ or for some indices $i < \kappa$ in $(L^b_\bullet, Q^b_\bullet)$. In the former case the loop increases the rank r_κ to that of at most $r_{\kappa+1}$ and it is clear that the resulting sequence satisfies Condition (6). If $(L^b_\bullet, Q^b_\bullet)$ violates Condition (9), it either violates it for all $1 \leq i \leq \kappa$ or only for $\kappa - 1$. In the first case, in $(L_\bullet, Q_\bullet)$ we must have $r_1 = r_\kappa = x_\kappa$ in $(L_\bullet, Q_\bullet)$ and the loop produces a sequence that satisfies the same equalities. Else $r_{\kappa-1} = r_\kappa$ in $(L_\bullet, Q_\bullet)$ and by Condition (6) for $(L_\bullet, Q_\bullet)$, $d_{\kappa-1} - 1 = d_\kappa$ and the loop in Step iii,

does not produce any sequences. Hence, the sequences produced in Step iii satisfy Conditions (1)-(6).

We conclude that all the sequences occurring in the algorithm satisfy Conditions (1)-(6).

If Conditions (7)-(9) hold in a sequence for all the indices $i \geq \alpha$, then Algorithm 5.10 does not modify the quadrics with these indices. Hence, every intermediate sequence formed during Algorithm 5.10 also satisfies Conditions (7)-(9) for indices $i \geq \alpha$.

Condition (8) may fail to hold for $(L^a_\bullet, Q^a_\bullet)$. However, since $(L_\bullet, Q_\bullet)$ is admissible, Condition (8) can only fail in $(L^a_\bullet, Q^a_\bullet)$ for the index κ and the right hand side of the inequality can exceed the left hand side by at most 1/2. Moreover, in $(L_\bullet, Q_\bullet)$, we must have the equality

$$x_\kappa = k - \kappa + 1 - \frac{d_\kappa - r_\kappa}{2}.$$

The choice of κ implies that equality holds in Condition (8) for all the indices $i > \kappa$ in the sequence $(L_\bullet, Q_\bullet)$. Since $r_i + d_i = r_\kappa + d_\kappa$ for all $i > \kappa$, we can rewrite the inequality in Condition (8) for the index i as

$$x_i \geq x_\kappa + r_i - r_\kappa + \kappa - i.$$

By Condition (9), $r_{\kappa+1} - r_\kappa - 1 \geq x_{\kappa+1} - x_\kappa$. Hence, we see that equality holds for the index $\kappa + 1$. By induction, it follows that equality holds for all the indices $\kappa \leq i \leq k - s$. Furthermore, $n_{x_\kappa+1} - r_\kappa - 1 = y_{x_\kappa+1} - \kappa$ in $(L_\bullet, Q_\bullet)$. Finally, note that if $x_\kappa = s$, then equality for the index $k - s$ implies that $d_{k-s} = r_{k-s} + 2$ contradicting Condition (7) for $(L_\bullet, Q_\bullet)$. We conclude that if Condition (8) fails for $(L^a_\bullet, Q^a_\bullet)$, then $x_\kappa < s$ and $n_{x_\kappa+1} - r_\kappa - 1 = y_{x_\kappa+1} - \kappa$. Therefore, the cases in Algorithm 5.9 are exhaustive and mutually exclusive. We may assume from now on that the sequence $(L^a_\bullet, Q^a_\bullet)$ satisfies Condition (8). We also conclude that the sequence $(L^b_\bullet, Q^b_\bullet)$ satisfies both Conditions (7) and (8). Since $x_\kappa < s$, Condition (7) has to hold for $(L^a_\bullet, Q^a_\bullet)$. Replacing a linear space with a smaller linear space does not affect Condition (7). Replacing $L_{x'_\kappa+1}$ with $L_{r'_\kappa}$ increases the left hand side of the inequality in Condition (8) by one without affecting the right hand side.

Therefore, $(L^b_\bullet, Q^b_\bullet)$ is either admissible or fails Condition (9). As we observed while verifying Step iii of Algorithm 5.10 preserves Conditions (1)-(6), no new sequences are formed unless Condition (9) fails for all the indices $1 \leq i \leq \kappa - 1$. In this case, any sequence formed in Step iii of Algorithm 5.10 clearly satisfies Condition (9), hence is admissible. Hence, every sequence formed in Step 4 of Algorithm 5.9 is admissible.

Condition (7) may fail to hold for $(L^a_\bullet, Q^a_\bullet)$ or while running Step iii of Algorithm 5.10 on $(L^a_\bullet, Q^a_\bullet)$. This may happen in only one of two ways. The sequence $(L_\bullet, Q_\bullet)$ either has $d_{k-s} = r_{k-s} + 3$ and $\kappa = k - s$; or $d_{k-s} = r_{k-s} + 3 + 2\alpha$, $\kappa = k - s$ and $(L_\bullet, Q_\bullet)$ has α linear spaces of dimensions $r_{k-s}+2, r_{k-s}+3, \dots, r_{k-s}+\alpha+1 = n_s$. By the observation three paragraphs above, $\kappa = k - s$. Hence, either Condition (7) is directly violated for $(L^a_\bullet, Q^a_\bullet)$ or Condition (7) is violated after applying Step iii of Algorithm 5.10 for the index κ α times. The equality $d_{k-s} = r_{k-s} + 3 + 2\alpha$

follows by combining Condition (8)

$$s - \alpha > s + 1 - \frac{d_{k-s} - r_{k-s}}{2}$$

for $(L_\bullet, Q_\bullet)$ with the inequality $d_{k-s} - r_{k-s} - 2\alpha \leq 3$ that expresses that Condition (7) is violated after α-steps. In either of the two cases, Step ii of Algorithm 5.10 outputs admissible sequences.

Finally, if Condition (9) fails for the sequence $(L^a_\bullet, Q^a_\bullet)$, then, as observed above, it fails only for the index κ. Applying Step iii of Algorithm 5.10 either produces a sequence which is admissible or which violates Condition (7). In the latter case, running Step ii of Algorithm 5.10 outputs an admissible sequence. We conclude that all the sequences output by Algorithm 5.9 are admissible. We now analyze the dimensions of the corresponding varieties.

- The expression in Equation (1) for the dimension of a restriction variety remains unchanged when we replace $Q^{r_\kappa}_{d_\kappa}$ with $Q^{r_\kappa+1}_{d_\kappa}$.
- In Cases 2 and of the Algorithm, we have the equality $n_{x'_\kappa+1} - r'_\kappa = y_{x'_\kappa+1} - \kappa$. Hence, when we replace $L_{n_{x'_\kappa+1}}$ with $L_{r'_\kappa}$ to form $(L^b_\bullet, Q^b_\bullet)$, x_i increases by one for the indices $\kappa \leq i < y_{x_\kappa+1}$. The dimension of the linear space with index $x'_\kappa+1$ decreases by $n'_{x_\kappa+1} - r'_\kappa$. All other terms in the expression in Equation (1) remain unchanged.
- Step iii of Algorithm 5.10 increases x_μ by one and decreases d_μ by one, hence preserves the expression in Equation (1).
- Finally, replacing $Q^{d_{k-s}-2}_{d_{k-s}}$ with $L_{d_{k-s}-1}$ in Step ii of Algorithm 5.10 increases the first sum in Equation (1) by $d_{k-s} - s - 2$. It changes the second sum by $-x_s - d_{k-s} + 2s + 2$. Since we must have $x_s = s$, we conclude that this step also preserves the expression in Equation (1).

Combining these observations, we conclude that every sequence produced by Algorithm 5.9 is admissible and gives rise to a restriction variety of the same dimension as $V(L_\bullet, Q_\bullet)$. The algorithm can be recursively applied to each of the resulting restriction varieties. It is clear that the algorithm must terminate in a collection of Schubert varieties. At each stage of the algorithm, either the corank of a quadric in the sequence increases by at least one or the number of quadrics in the sequence decreases. Since there are finitely many quadrics in the sequence and the corank of the quadrics are bounded above, eventually the sequence must become saturated. Then the resulting varieties are Schubert varieties.

We now analyze Degeneration 5.7 to conclude that the support of the flat limit is the union of restriction varieties replacing $V(L_\bullet, Q_\bullet)$ in Algorithm 5.9. In order to restrict the possible irreducible components of the support of the flat limit, we write down conditions that the linear spaces in the limit have to satisfy. We then observe that these conditions already cut out varieties of dimension equal to the dimension of $V(L_\bullet, Q_\bullet)$. The following observation puts strong restrictions on the support of the flat limit.

Observation 5.13. The linear spaces parameterized by the restriction varieties $V(L_\bullet(t), Q_\bullet(t))$ intersect the linear spaces $L_{n_j}(t)$ in a subspace of dimension at least j and the quadrics $Q^{r_i}_{d_i}(t)$ in a linear space of dimension at least $k - i + 1$. Similarly, they intersect $Q^{r_i,sing}_{d_i}(t)$ in a linear space of dimension at least x_i. Since intersecting a proper variety in at least a given dimension is a closed condition, the linear spaces parameterized by the flat limit $V(0)$ have to intersect the linear spaces $L_{n_j}(0)$ in a subspace of dimension at least j and the quadrics $Q^{r_i}_{d_i}(0)$ in a subspace of dimension at least $k-i+1$. Furthermore, they intersect $Q^{r_i,sing}_{d_i}(0)$ in a subspace of dimension at least x_i.

A quick inspection of the algorithm will reveal that in each of the limits either the linear spaces intersect the vertex of $Q^{r_\kappa+1}_{d_\kappa}(0)$ in a subspace of dimension $x_\kappa+1$ and otherwise remain as unconstrained as possible given Observation 8.3; or the linear spaces continue to intersect $Q^{r_\kappa+1}_{d_\kappa}(0)$ in a subspace of dimension x_κ and only satisfy the constraints imposed by Observation 8.3. A priori in the limit the linear spaces could become more special. However, we claim that these loci have strictly smaller dimension and do not form an irreducible component of the support of the flat limit. We now verify this claim.

Let Y be an irreducible component of the support of the flat limit of Degeneration 5.7. Then we can build a sequence of consisting of k linear spaces and quadrics such that the closure of the locus of linear spaces intersecting the i-th element in the sequence (counting in increasing dimension) in dimension i contains Y. We complete the linear spaces and quadrics in the sequence $(L^a_\bullet, Q^a_\bullet)$ to a set of linear spaces and quadrics whose dimensions increase by one at each stage making sure that Conditions (4) and (5) of Definition 4.2 are satisfied. We then select those linear spaces and quadrics that have a jump in the dimension of intersection with a general linear space parameterized by Y. We thus obtain a set of k linear spaces and quadrics. By construction the closure of the locus X of linear spaces that intersect the i-th one in dimension i contains Y. Observation 8.3 implies that the i-th linear space or quadric in the sequence thus obtained has dimension less than or equal to the i-th linear space or quadric (counting in increasing dimension) in the sequence $(L^a_\bullet, Q^a_\bullet)$. By Proposition 7.21, Equation (1) gives an upper bound on the dimension of X (note that we used the fact that the sequence is admissible in the proof only to deduce the equality).

We now estimate the dimension of X. We obtain the sequence defining X by replacing linear spaces and quadrics in $(L^a_\bullet Q^a_\bullet)$ by smaller dimensional ones in increasing order. We will do this in greater generality in preparation for the discussion of orthogonal flag varieties.

- If we replace a linear space of dimension n'_{i+j} in the $(i+j)$-th position with a linear space of dimension n^*_i in the i-th position not contained in $(L^a_\bullet, Q^a_\bullet)$, then according to Equation (1) the dimension changes as follows. Let y'_{i+j} and y^*_i be the smallest index quadrics containing the corresponding linear spaces in their singular locus. The dimension decreases by $n'_{i+j} - n^*_i + j + y'_{i+j} - y^*_i$. Since Conditions (6) and (9) hold for $(L_\bullet, Q_\bullet)$, we have that $n'_{i+j} - n^*_i + y'_{i+j} - y^*_i \geq 0$. Consequently the decrease in the dimension is at least j with equality when $n'_{i+j} - n^*_i + y'_{i+j} - y^*_i = 0$.
- If we replace the i-th largest quadric in a vector space of dimension d'_i by the $(i+j)$-th largest quadric in a vector space of dimension d^*_{i+j} , then according to Equation (1) the dimension decreases by $d'_i - d^*_{i+j} + x'_i - x^*_{i+j}$. This decrease is at least j with strict inequality unless Condition (9) fails for r'_i.

- Finally, if we replace the quadric $Q_{d'_i}^{r'_i}$ with the linear space $L_{n_j^*}$, then the first sum in Equation (1) changes by $n'_j - s - 1$. The second sum changes by $-d'_i + (k - s - y_j^* - x'_i) + (2s + 2)$. Hence, the total change is

$$-d'_i + n_j^* + k + 1 - x'_i - y_j^*,$$

where $y_{n_j}^*$ denotes the index of the largest dimensional quadric containing $L_{n_j}^*$ in its singular locus. We rewrite this expression as follows:

$$(k - i + 1 - \lfloor \frac{d'_i - r'_i}{2} \rfloor - x'_i) + (n_j^* - \lceil \frac{d'_i + r'_i}{2} \rceil + k - s - y_j^*) + (-k + s + i).$$

The sum in the first parentheses is strictly negative unless Condition (8) is violated or there is equality in Condition (8); otherwise it is zero. The sum in the second parentheses is strictly negative unless $j = s + 1$ and $d'_i + r'_i = d_{k-s} + r_{k-s}$; otherwise it is zero. Finally, the third sum is strictly negative unless $i = k - s$; otherwise it is zero.

Since our degeneration is flat, Y has to have the same dimension as $V(L_\bullet, Q_\bullet)$. Since X contains Y, our dimension calculation puts strong restrictions on X.

First, suppose $x_\kappa = s$ in $(L_\bullet, Q_\bullet)$. Then by Conditions (6) and (9) for $(L_\bullet, Q_\bullet)$, $n'_l - r'_i + y'_l - i > 0$ for every l with $n'_l > r'_i$ in $(L_\bullet^a, Q_\bullet^a)$. Furthermore, Condition (9) holds for $(L_\bullet^a, Q_\bullet^a)$. If $d'_{k-s} - r'_{k-s} > 2$, then replacing any linear space or quadric with a smaller dimensional one strictly decreases the dimension. Note also that $(L_\bullet^a, Q_\bullet^a)$ is admissible. In this case, we conclude that X has to be $V(L_\bullet^a, Q_\bullet^a)$. Since $V(L_\bullet^a, Q_\bullet^a)$ and $V(L_\bullet, Q_\bullet)$ have the same dimension, we conclude that Y has to be a component of $V(L_\bullet^a, Q_\bullet^a)$. If $d'_{k-s} - r'_{k-s} = 2$, then $Q_{d'_{k-s}}^{r'_{k-s}}$ is necessarily reducible consisting of two linear spaces of dimension $d'_{k-s} - 1$. If $2(d'_{k-s} - 1) = n$, then these linear spaces belong to two different connected components. We can therefore replace $Q_{d'_{k-s}}^{r'_{k-s}}$ with either of these linear spaces to obtain two sequences. Note that replacing any other linear space or quadric with a smaller dimensional one strictly decreases the dimension. Hence X has to be the variety corresponding to one of these sequences. Since X has the same dimension as $V(L_\bullet, Q_\bullet)$, we conclude that Y has to be an irreducible component of X. Observe that Algorithm 5.9 selects the sequences corresponding to X.

Next, suppose that $x_\kappa < s$ and $n_{x_\kappa+1} - r_\kappa - 1 > y_{x_\kappa+1} - \kappa$ in $(L_\bullet, Q_\bullet)$. Then the sequence $(L_\bullet^a, Q_\bullet^a)$ is admissible. Furthermore, by our dimension calculation, replacing any linear space or quadric in $(L_\bullet^a, Q_\bullet^a)$ leads to a strictly smaller dimensional locus. We conclude that $X = V(L_\bullet^a, Q_\bullet^a)$ and Y has to be an irreducible component of $V(L_\bullet^a, Q_\bullet^a)$.

Next, suppose that $x_\kappa < s$ and Condition (8) is violated for $(L_\bullet^a, Q_\bullet^a)$ for κ. Note that in that case, there must be an equality in Condition (8) in $(L_\bullet, Q_\bullet)$ for the index κ. Hence, by Conditions (6) and (8), $r_{\kappa-1} < r_\kappa$ in $(L_\bullet, Q_\bullet)$. By the "linear space bound", every linear space of dimension $k - \kappa + 1$ contained in $Q_{d'_\kappa}^{r'_\kappa}$ must intersect the singular locus of this quadric in dimension at least $x_\kappa + 1$. Hence, we can replace the sequence $(L_\bullet^a, Q_\bullet^a)$ with the sequence $(L_\bullet^b, Q_\bullet^b)$. By our dimension calculation, replacing any linear space or quadric by a smaller dimensional one results in a variety of strictly smaller dimension. Hence, we conclude that Y has to be an irreducible component of $V(L_\bullet^b, Q_\bullet^b)$. As we observed above, $(L_\bullet^b, Q_\bullet^b)$ may fail Condition (9) for $i < \kappa$. In that case, by the "variation of tangent spaces", any linear space of dimension $k-i+1$ intersecting $L_{r'_\kappa}$ in dimension x_i is necessarily contained in the quadric everywhere tangent to Q along $L_{r'_\kappa}$. Hence, we can replace the sequence $(L^b_\bullet, Q^b_\bullet)$ as in Step iii of Algorithm 5.10 to obtain an equivalent definition of the same variety (Note that since $r_{\kappa-1} < r_\kappa$ in $(L_\bullet, Q_\bullet)$, the definition of κ ensures that $d_\mu - 1 < d_{\mu-1}$ while running Step iii of Algorithm 5.10).

Finally, suppose that $x_\kappa < s$ and $n_{x_\kappa+1} - r_\kappa - 1 = y_{x_\kappa+1} - \kappa$ in $(L_\bullet, Q_\bullet)$ and $(L^a_\bullet, Q^a_\bullet)$ satisfies Condition (8). Let $i \leq \kappa$ be the smallest index such that $n_{x'_i+1} - r'_i = y_{x'_i+1} - i$ in $(L^a_\bullet, Q^a_\bullet)$. Note that by Conditions (6) and (9) for $(L_\bullet, Q_\bullet)$, there may be such indices precisely when $r_{\kappa-1} = r_\kappa$, $r_i \geq x_\kappa$ and $r_i = r_\kappa - \kappa + i + 1$ in $(L_\bullet, Q_\bullet)$. By our dimension counts, replacing $L_{x'_\kappa+1}$ with $L_{r'_j}$ for an index $i \leq j \leq \kappa$ can result in a sequence that has the same dimension as $V(L_\bullet, Q_\bullet)$. Replacing any other linear space or quadric with a smaller dimensional one, gives a smaller dimensional variety. The rest of the analysis of this case is more subtle. We need to argue that unless $j = \kappa$, these loci do not occur in the limit. For a general linear space $W_t \in V(L_\bullet(t), Q_\bullet(t))$, let $W_{t,j} = Q^{r_j}_{d_j}(t) \cap W_t$ for $i \leq j < \kappa$. The tangent space to $Q^{r_j}_{d_j}$ along $W_{t,j}$ intersects $L_{n_{x_\kappa+1}}$ in a subspace of dimension $r_j + 1$. By semi-continuity, this must be true for every linear space contained in $V(L_\bullet(t), Q_\bullet(t))$ and also in the limit $V(L_\bullet(0), Q_\bullet(0))$. However, the tangent space to $Q^{r_j}_{d_j}$ at a general linear space parameterized by the variety associated to the sequence obtained from $(L^a_\bullet, Q^a_\bullet)$ by replacing $L_{x'_\kappa+1}$ with $L_{r'_j}$ intersects $L_{n_{x_\kappa+1}}$ in dimension $r_j = r'_j$. We conclude that the support of Y cannot equal such a locus. Hence X is the locus associated to one of the sequences $(L^a_\bullet, Q^a_\bullet)$ or $(L^b_\bullet, Q^b_\bullet)$. These sequences may fail to satisfy Condition (9). In that case, Step iii of Algorithm 5.10 replaces them by equivalent varieties unless for $(L^b_\bullet, Q^b_\bullet)$ we have $d_{\kappa-1} - 1 = d_\kappa$. In the latter case, by "the variation of tangent spaces", the $(k-\kappa+2)$-dimensional subspaces of the linear spaces W parameterized by X have to be contained in $Q^{r'_\kappa}_{d'_\kappa}$. In other words, we have to replace $Q^{r'_{\kappa-1}}_{d'_{\kappa-1}}$ by a smaller quadric. By our dimension counts, such a locus has strictly smaller dimension, hence cannot support Y.

In order to conclude the proof, we need to verify that the limits all occur and are reduced at the generic point of each of these loci. This is a straightforward local calculation. Let U be the Zariski open set of our family of restriction varieties parameterizing linear spaces $W(t)$ such that $\dim(W(t) \cap Q^{r_\kappa(t)}_{d_\kappa}(t)) = k - \kappa + 1$. Let Z be the family of restriction varieties obtained by applying Degeneration 5.7 to the admissible sequence obtained from $(L_\bullet, Q_\bullet)$ by omitting the quadrics $Q^{r_1}_{d_1}, \dots, Q^{r_\kappa - 1}_{d_{\kappa-1}}$. Then there exists a natural morphism $f : U \to Z$ sending $W(t)$ to $W(t) \cap Q^{r_\kappa(t)}_{d_\kappa}(t)$, which is smooth at the generic point of each of the irreducible components of the fiber of Z at $t = 0$. We may, therefore, assume that $\kappa = 1$. Furthermore, without loss of generality, we may assume that $n = d_\kappa + r_\kappa + 1$ and $x_\kappa = 0$. We will check that the multiplicity is one by exhibiting cycles that intersect $V(L_\bullet, Q_\bullet)$ in one point and exactly one of the potential limits in one point. This will allow us to conclude that each of the limits occur with multiplicity one. There is a Schubert cycle in the class of the variety $V(L^a_\bullet, Q^a_\bullet)$ (respectively, $V(L^b_\bullet, Q^b_\bullet)$) that occurs with coefficient one and does not occur in the class of $V(L^b_\bullet, Q^b_\bullet)$ (respectively, $V(L^a_\bullet, Q^a_\bullet)$). We use the dual of these Schubert cycles for our computation. Note that by our assumptions on κ and n, $d_i + r_i = d_\kappa + r_\kappa = n-1$ for every $i \geq \kappa$. Hence, $n - d_{k-s} + 1 = r_{k-s} + 2$ and $2(r_{k-s}+2) \leq r_{k-s} + d_{k-s} + 1 = n$.

First, suppose $x_\kappa = s(=0)$ and $d_\kappa = r_\kappa + 3$ in $(L_\bullet, Q_\bullet)$. In this case, this is the standard family of a quadric breaking into a union of two linear spaces. Both occur in the limit with multiplicity one. In this case there is nothing to check. Next suppose $x_\kappa = s(=0)$ and $d_\kappa > r_\kappa + 3$ in $(L_\bullet, Q_\bullet)$. Let $\beta_i = n - d_i + 1$. Let S be the Schubert variety defined with respect to a general isotropic flag

$$L_{\beta_1} \subset \cdots \subset L_{\beta_{k-s}}.$$

In case $2\beta_{k-s} = n$, we will always define a second Schubert variety S' by replacing $L_{\beta_{k-s}}$ with $L'_{\beta_{k-s}}$. Note that under our assumptions $V(Q_\bullet)$ is irreducible. Both $V(Q_\bullet)$ and $V(Q_\bullet^\alpha)$ intersect S (and S' when appropriate) in a reduced point. The spans $Span(Q_{d_i}^{r_i})$ and $Span(Q_{d_i}^{r_i'})$ intersect the linear space L_{β_i} in a one dimensional subspace for every $1 \leq i \leq k$. Any k-dimensional linear space contained in $V(Q_\bullet) \cap S$ or $V(Q_\bullet^\alpha) \cap S$ must contain these one-dimensional subspaces. Hence, the k-dimensional linear space is uniquely determined as the $Span((Q_{d_i}^{r_i} \cap L_i), 1 \leq i \leq k)$ or $Span((Q_{d_i}^{r_i'} \cap L_i), 1 \leq i \leq k)$, respectively. By Kleiman's Transversality Theorem [Kl1], we conclude that the intersection of the two varieties consists of a single reduced point. When $2\beta_{k-s} = n$, two general linear spaces in the class $L = L_{\beta_{k-s}}$ intersect in a unique point if $n = 0$ modulo 4 and are otherwise disjoint. A general linear space in the class L and a general linear space in the class $L' = L'_{\beta_{k-s}}$ intersect in a unique point if $n = 2$ modulo 4 and are otherwise disjoint. When $V(Q'_\bullet)$ has two components, repeating the argument with S', we conclude that both components occur with multiplicity one.

Next, suppose that $x_\kappa(=0) < s$ and $(L_\bullet^a, Q_\bullet^a)$ fails to satisfy Condition (8). Let $\alpha_{x_\kappa+1} = \alpha_1 = n - r_\kappa$. Let $\alpha_j = n - n_{j-1}$ for $j > x_{\kappa+1}$. Let $\beta_i = n - d_i + 1$. Let S be the Schubert variety defined with respect to the linear spaces and quadrics

$$L_{\beta_1} \subset \cdots \subset L_{\beta_{k-s}} \subset Q_{\alpha_s}^{n-\alpha_s} \subset \cdots \subset Q_{\alpha_1}^{n-\alpha_1}.$$

Proposition 4.18 implies that S is a Schubert variety. We claim that S intersects both $V(L_\bullet, Q_\bullet)$ and $V(L_\bullet^b, Q_\bullet^b)$ in a unique, reduced point. The linear spaces L_{β_i} intersect the quadrics $Q_{d_i}^{r_i}$ and $Q_{d_i^b}^{r_i^b}$ in unique points. Any k-dimensional linear space in the intersection of S and $V(L_\bullet, Q_\bullet)$ or S and $V(L_\bullet^b, Q_\bullet^b)$ must contain the $(k-s)$-dimensional linear space Λ spanned by these points. In $S \cap V(L_\bullet, Q_\bullet)$, the quadrics everywhere tangent to Λ determine unique points in $Q_{\alpha_j}^{n-\alpha_j} \cap L_{n_j}$ for $j > 0$. In $S \cap V(L_\bullet^b, Q_\bullet^b)$, the quadrics everywhere tangent to Λ determine unique points in $Q_{\alpha_j}^{n-\alpha_j} \cap L_{n_j}$ for $j > 1$ and furthermore the k-plane has to contain the point $L_{r_\kappa^b} \cap Q_{\alpha_1}^{n-\alpha_1}$ (which is contained in the singular locus of all the quadrics). Hence, in both cases, the k-dimensional linear space in the intersection is uniquely determined. We proved above that $n_s = n/2$ in this case. Hence, both $V(L_\bullet, Q_\bullet)$ and $V(L_\bullet^b, Q_\bullet^b)$ are irreducible. Therefore, $V(L_\bullet^b, Q_\bullet^b)$ occurs in the limit with multiplicity one.

Next, suppose $x_\kappa(=0) < s$ and $n_{x_\kappa+1} - r_\kappa - 1 > y_{x_\kappa+1} - \kappa$. In this case, let i_0 denote the smallest index for which equality holds in Condition (8) in $(L_\bullet, Q_\bullet)$. If there is no such index, set $i_0 = 0$ and $r_{i_0} = 0$. For $n_j \leq r_{i_0}$, let $\alpha_j = n - n_j + 1$. For $n_j > r_{i_0}$, set $\alpha_j = n - n_{j-1}$. Next, for each index $i < i_0$, let l_i be the largest positive integer such that $r_i + l_i + 1 = n_{x_i + l_i}$. If there does not exist such l_i, set

$l_i = 0$. Let $\beta_i = n - d_i + l_i + 1$ for $i < i_0$ and let $\beta_i = n - d_i + 1$ for $i \geq i_0$. Let S be the Schubert variety defined by the sequence

$$L_{\beta_1} \subset \cdots \subset L_{\beta_{k-s}} \subset Q^{n-\alpha_s}_{\alpha_s} \subset \cdots \subset Q^{n-\alpha_1}_{\alpha_1}.$$

When $2\beta_{k-s} = n$, define S' by replacing $L_{\beta_{k-s}}$ with $L'_{\beta_{k-s}}$. Note that Proposition 4.18 implies S is a Schubert variety. As in the previous cases, it is straightforward to see that S intersects both $V(L_\bullet, Q_\bullet)$ and $V(L^a_\bullet, Q^a_\bullet)$ in a unique, reduced point. When appropriate, the same holds for S'. We conclude that $V(L^a_\bullet, Q^a_\bullet)$ occurs in the limit with multiplicity one.

Finally, suppose $x_\kappa (= 0) < s$, $n_{x_\kappa+1} - r_\kappa - 1 = y_{x_\kappa+1} - \kappa$ and Condition (8) is satisfied for $(L^a_\bullet, Q^a_\bullet)$. Then duals for $V(L^a_\bullet, Q^a_\bullet)$ and $V(L^b_\bullet, Q^b_\bullet)$ are obtained as in the previous cases. Let S be the Schubert variety defined exactly as in the previous paragraph. Let T be the Schubert variety defined by replacing $\alpha_{x_\kappa+1} = n - n_{x_\kappa+1} + 1$ in the definition of S with $\alpha_{x_\kappa+1} = n - r_\kappa$. Then it is straightforward to see that both S and T intersect $V(L_\bullet, Q_\bullet)$ in a unique reduced point. S (respectively, T) intersects $V(L^a_\bullet, Q^a_\bullet)$ (respectively, $V(L^b_\bullet, Q^b_\bullet)$) in a unique, reduced point and has empty intersection with $V(L^b_\bullet, Q^b_\bullet)$ (respectively, $V(L^a_\bullet, Q^a_\bullet)$). It follows that both limits occur with multiplicity one. Finally, by replacing S with S' and T with T' when appropriate, it is easy to see that in case these varieties are reducible, both components occur with multiplicity one and that the algorithm preserves marking. This concludes the proof of the theorem. □

Remark 5.14. From the analysis in the proof of Theorem 5.12, it follows that at each stage of the degeneration a restriction variety breaks into at most three irreducible components.

6. Applications of Algorithm 5.9

In this section we discuss a couple of immediate applications of Algorithm 5.9. The Introduction discusses other applications.

6.1. **The moduli space of vector bundles on hyperelliptic curves.** There is a beautiful, classical construction that associates to a general pencil of quadric hypersurfaces in $\mathbb{P}^{2g+1}$ a hyperelliptic curve C of genus g. In fact, every smooth hyperelliptic curve of genus g arises this way [GH, §6], [**?**]. We recall the construction for the reader's convenience.

Let Q_1 and Q_2 be general quadric hypersurfaces in $\mathbb{P}^{2g+1}$. Let $tQ_1 + uQ_2$ be the pencil generated by Q_1 and Q_2. Consider the incidence correspondence I parameterizing pairs (Q, C), where Q is a quadric hypersurface contained in the pencil and C is a connected component of the space of g-dimensional projective linear spaces on Q. The incidence correspondence I is irreducible and maps to $\mathbb{P}^1$ by the first projection π_1. When Q is a smooth quadric, the space of g-dimensional projective linear spaces on Q has two connected components. Hence, I is a double cover of $\mathbb{P}^1$. When Q has corank one, then the space of g-dimensional projective spaces has only one component. Hence, π_1 is ramified at the $2g+2$ points in the pencil that are quadrics of corank one. By the Riemann-Hurwitz formula, we conclude that I is a hyperelliptic curve of genus g.

To see that there are $2g+2$ corank one quadrics in a general pencil, observe that the pencil can be identified with a $(2g+2) \times (2g+2)$ symmetric matrix whose entries are linear homogeneous polynomials in t and u. The quadrics of corank one correspond to matrices with zero determinant. Since the determinant is a homogeneous polynomial of degree $2g+2$ in t and u, it will have $2g+2$ roots in $\mathbb{P}^1$. If the pencil is general, these roots will be distinct and the corresponding symmetric matrix will have corank exactly one. Furthermore, it is clear from this description that one can construct a pencil with any $2g+2$ distinct roots. Hence, every smooth hyperelliptic curve of genus g arises via this construction.

Let C be a smooth hyperelliptic curve of genus $g \geq 2$. Let $MV_{2,o}(C_g)$ denote the moduli space of rank two vector bundles with a fixed determinant of odd-degree on C. Realize C as a double cover of a pencil of quadric hypersurfaces in $\mathbb{P}^{2g+1}$. By a celebrated theorem of Desale and Ramanan [**?**], $MV_{2,o}(C_g)$ is isomorphic to the space of $(g-2)$-dimensional projective linear spaces contained in this pencil of quadric hypersurfaces in $\mathbb{P}^{2g+1}$. Equivalently, if Q_1 and Q_2 are two smooth quadric hypersurfaces that generate the pencil, $MV_{2,o}(C_g)$ is isomorphic to the space of $(g-2)$-dimensional projective linear spaces contained in both Q_1 and Q_2.

We can view the space X parameterizing $(g-2)$-dimensional projective linear spaces contained in Q_1 as the orthogonal Grassmannian $OG(g-1, 2g+2)$, which naturally includes in $G(g-1, 2g+2)$. We can also view the space of $(g-2)$-dimensional projective linear spaces contained in Q_2 as a subvariety Y of $G(g-1, 2g+2)$. Of course, Y is isomorphic to X; however, its embedding in $G(g-1, 2g+2)$ differs from that of X by translation with an element of $\mathbb{P}GL(2g+2)$. By Kleiman's Transversality Theorem, X and Y intersect transversally. Therefore, the class of the intersection $Y \cap OG(g-1, 2g+2)$ in $H^*(OG(g-1, 2g+2), \mathbb{Z})$ is the pull-back of the class of Y in $H^*(G(g-1, 2g+2), \mathbb{Z})$ under the map induced by inclusion.

The class of Y in $G(g-1, 2g+2)$ is well-known to be $2^{g-1}\sigma_{g-1,g-2,\ldots,2,1}$. There are several ways of calculating this class. First, it is the top Chern class of the vector bundle $Sym^2(S^*)$ on $G(g-1, 2g+2)$, where S^* denotes the dual of the tautological bundle of $G(g-1, 2g+2)$. Calculating the top Chern class of $Sym^2(S^*)$ is a standard exercise in using the splitting principle. Alternatively, one can use degenerations for a more pleasant calculation. Very briefly, break the quadric into a union of two linear spaces using a general pencil $Q + tL_1L_2$. The flat limit of the space of $(g-2)$-dimensional projective linear spaces contained in Q is the space of $(g-2)$-dimensional projective linear spaces contained in L_1 or L_2 that intersect $Q \cap L_1 \cap L_2$ in $(g-3)$-dimensional projective linear spaces (see [**?**] or [**?**]). Now inductively break $Q \cap L_1 \cap L_2$ into a union of linear spaces using a general pencil. Continuing this process for $(g-1)$ steps, we obtain 2^{g-1} flags of the form

$$\mathbb{P}^{2g} \supset \mathbb{P}^{2g-2} \supset \mathbb{P}^{2g-4} \supset \cdots \supset \mathbb{P}^4,$$

where $\mathbb{P}^{2g-2i}$ is one of the two linear spaces obtained by degenerating the $(2g-2i)$-dimensional quadric. Inductively, the flat limit of Y is the space of $(g-2)$-dimensional projective linear spaces that intersect $\mathbb{P}^{2g-2i}$ in a projective space of dimension $g-2-i$. We conclude that the class of Y is is $2^{g-1}\sigma_{g-1,g-2,\ldots,2,1}$ in the cohomology of $G(g-1, 2g+2)$.

In conclusion, the class of $MV_{2,o}(C_g)$ is 2^{g-1} times the class of the restriction variety associated to the Schubert class $\sigma_{g-1,g-2,\ldots,1}$ in $G(g-1, 2g+2)$. More explicitly, the class of $MV_{2,o}(C_g)$ in $OG(g-1, 2g+2)$ is equal to 2^{g-1} times the class of the restriction variety associated to the admissible sequence

$$Q_5^0 \subset Q_7^0 \subset \cdots \subset Q_{2g-1}^0 \subset Q_{2g+1}^0.$$

Using Algorithm 5.9 the class can be easily computed. Here we give the class for the first few genera.

(1) $[MV_{2,o}(C_2)] = 2\sigma^1$
(2) $[MV_{2,o}(C_3)] = 4\sigma_0^1 + 4\sigma^{3,1}$
(3) $[MV_{2,o}(C_4)] = 16\sigma_2^{3,1} + 16\sigma_{1,0}^1 + 16\sigma_1^{4,1}$
(4) $[MV_{2,o}(C_5)] = 64\sigma_{3,1}^{3,1} + 64\sigma_{2,1,0}^1 + 64\sigma_{2,1}^{5,1} + 32\sigma_{3,0}^{4,1} + 32\sigma_3^{5,4,1} + 32\sigma_{5,0}^{3,1} + 32\sigma_5^{5,3,1} + 32\sigma_{4,0}^{3,2} + 32\sigma_4^{5,3,2}$

More generally, one obtains a recursion in the genus for the class. Suppose that the class of $MV_{2,o}(C_{g-1})$ in $OG(g-3,2g)$ is given by

$$[MV_{2,o}(C_{g-1})] = \sum c_{\lambda,\mu}[\Omega_\lambda^\mu],$$

where Ω_λ^μ is defined with respect to a sequence $(L_\bullet^\lambda, Q_\bullet^\mu)$. Let t be the largest index of a linear space in the sequence such that $n_t = t$. Define a new sequence $(\tilde{L}_\bullet^\lambda, \tilde{Q}_\bullet^\mu)$ by setting $\tilde{L}_{n_j}^\lambda = L_{n_j}^\lambda$ for all $1 \leq j \leq s$ and $\tilde{Q}_{d_{i+1}}^{\mu, r_{i+1}} = Q_{d_i}^{\mu, r_i}$ for $1 \leq i \leq g-3-s$. Set $\tilde{Q}_{d_1}^{\mu, r_1} = Q_{2g+1-t}^t$. Then we have that

$$[MV_{2,o}(C_g)] = 2\sum c_{\lambda,\mu}[V(\tilde{L}_\bullet^\lambda, \tilde{Q}_\bullet^\mu)].$$

Remark 6.1. When $g = 2$, $MV_{2,o}(C_2)$ is a complete intersection of two quadric hypersurfaces in $\mathbb{P}^5$ [GH, §6]. Ciprian Manolescu (in private correspondence) posed the question whether $MV_{2,o}(C_g)$ can be a complete intersection for $g > 2$. In fact, one can ask for a much weaker property. Can $MV_{2,o}(C_g)$ be a complete intersection of ample divisors in $OG(g-1, 2g+2)$? The codimension of $MV_{2,o}(C_g)$ in $OG(g-1, 2g+2)$ is $\frac{g(g-1)}{2}$. The codimension of the Schubert variety $\sigma_g^{g-2,g-3,\dots,2,1}$ is $g+1$. Hence, the sum of the codimensions of these two varieties is $\frac{g^2+g}{2}+1$. If $g > 2$, this is less than the dimension of $OG(g-1, 2g+2)$. Hence, if $MV_{2,o}(C_g)$ were a complete intersection of ample divisors, $\sigma_g^{g-2,g-3,\dots,2,1} \cdot [MV_{2,o}(C_g)] \neq 0$. However, the cup product of these classes is clearly zero since the one-dimensional vector space defining the Schubert variety can be chosen to not be contained in Q_{2g+1}^0 defining the restriction variety. Hence, we conclude that for $g > 2$, $MV_{2,o}(C_g)$ cannot be a complete intersection of ample divisors even in $OG(g-1, 2g+2)$, let alone in $G(g-1, 2g+2)$ or $\mathbb{P}^{\binom{2g+2}{g-1}-1}$.

6.2. **A geometric algorithm for computing the product of arbitrary Schubert cycles.** The pull-back of a Schubert class under the inclusion $j : OG(k,n) \to G(k,n)$ can be expressed as a sum of classes of restriction varieties. Consider a Schubert cycle $\Sigma_{\lambda_1,\dots,\lambda_k}$ defined with respect to a general partial flag

$$F_{n-k+1-\lambda_1} \subset F_{n-k+2-\lambda_2} \subset \cdots \subset F_{n-\lambda_k}.$$

The intersection of this flag with the quadric hypersurface Q leads to the sequence of quadrics

$$Q^0_{n-k+1-\lambda_1} \subset Q^0_{n-k+2-\lambda_2} \subset \cdots \subset Q^0_{n-\lambda_k}.$$

Note that since none of the quadrics are singular, the Conditions (3)-(6) of Definition 4.2 are automatically satisfied. Similarly, since there are no linear spaces in the sequence, Condition (1) is automatically satisfied. However, Condition (2) may be violated. In that case, the corresponding variety is empty and the pull-back is zero. From now on we assume that the sequence satisfies all the conditions in Definition 4.2. If the sequence is admissible, then the pull-back of the Schubert cycle is the class of the corresponding restriction variety. However, the sequence may fail to be admissible and thus the pull-back maybe the sum of classes of restriction varieties. We now describe how to express the pull-back as a sum of these. Since Condition (2) in Definition 4.2 is satisfied, $n-k+i-\lambda_i \geq 2i$ for all i. Suppose that equality holds for $i \leq \alpha$ and the inequality is strict for $i = \alpha+1$. Then the quadric $Q_{n-k+1-\lambda_1}$ consists of two points p_1, p_2. The linear spaces have to contain one of the p_i and be contained in the tangent space to Q along p_i. Then $Q^1_{n-k+2-\lambda_2}$ consists of two lines intersecting at p_i. The linear spaces containing p_i have to contain one of these lines. Continuing we deduce the following proposition.

Proposition 6.2. *Let $\sigma_{\lambda_1,\dots,\lambda_k}$ be a Schubert cycle in $G(k,n)$. Let $j : OG(k,n) \to G(k,n)$ be the natural inclusion. Then*

(1) *$j^*\sigma_{\lambda_1,\dots,\lambda_k} = 0$ unless $n-k-i \geq \lambda_i$ for every $1 \leq i \leq k$.*
(2) *Suppose that $n-k-i = \lambda_i$ for $i = 1,\dots,\alpha$ and $n-k-i > \lambda_i$ for $i = \alpha+1$. Further suppose that if $2k = n$, then $\alpha \neq k$. Let $(L_\bullet, Q_\bullet)$ be the admissible sequence*

$$L_1 \subset L_2 \subset \cdots \subset L_{\alpha-1} \subset L_\alpha \subset Q^\alpha_{n-k+\alpha+1-\lambda_{\alpha+1}} \subset \cdots \subset Q^\alpha_{n-1-\lambda_{k-1}} \subset Q^\alpha_{n-\lambda_k}.$$

Then $j^*\sigma_{\lambda_1,\dots,\lambda_k} = 2^\alpha[V(L_\bullet, Q_\bullet)]$, where $[V(L_\bullet, Q_\bullet)]$ denotes the cohomology class of the restriction variety $V(L_\bullet, Q_\bullet)$. If $2\alpha = 2k = n$, then the class is $2^{\alpha-1}$ times the Poincaré dual of a point.

7. Symplectic restriction varieties

In this section, we interpret admissible symplectic diagrams geometrically. We introduce symplectic restriction varieties and discuss their basic geometric properties.

Recall that Q denotes a non-degenerate skew-symmetric form on a vector space V of dimension n. Let L_{n_j} denote an isotropic subspace of Q of dimension n_j. Let $Q^{r_i}_{d_i}$ denote a linear space of dimension d_i such that the restriction of Q to it has corank r_i. Let K_i denote the kernel of the restriction of Q to $Q^{r_i}_{d_i}$.

Definition 7.1. A *sequence* $(L_\bullet, Q_\bullet)$ is a partial flag of linear spaces $L_{n_1} \subsetneq \cdots \subsetneq L_{n_s} \subsetneq Q^{r_{k-s}}_{d_{k-s}} \subsetneq \cdots \subsetneq Q^{r_1}_{d_1}$ such that

- $\dim(K_i \cap K_h) \geq r_i - 1$ for $h > i$.
- $\dim(L_{n_j} \cap K_i) \geq \min(n_j, \dim(K_i \cap Q^{r_{k-s}}_{d_{k-s}}) - 1)$ for every $1 \leq j \leq s$ and $1 \leq i \leq k-s$.

The main geometric objects of this paper will be sequences satisfying further properties.

Definition 7.2. A sequence is *in order* if

- $K_i \cap K_h = K_i \cap K_{i+1}$, for all $h > i$ and $1 \leq i \leq k-s$, and
- $\dim(L_{n_j} \cap K_i) = \min(n_j, \dim(K_i \cap Q^{r_{k-s}}_{d_{k-s}}))$, for $1 \leq j \leq s$ and $1 \leq i < k-s$.

A sequence $(L_\bullet, Q_\bullet)$ is *in perfect order* if

- $K_i \subseteq K_{i+1}$, for $1 \leq i < k-s$, and
- $\dim(L_{n_j} \cap K_i) = \min(n_j, r_i)$ for all i and j.

Definition 7.3. A sequence $(L_\bullet, Q_\bullet)$ is called *saturated* if $d_i + r_i = n$, for $1 \leq i \leq k-s$.

The next definition is the analogue of Definition 3.24 and is a consequence of the order of specialization.

Definition 7.4. A sequence $(L_\bullet, Q_\bullet)$ is called a *symplectic sequence* if it satisfies the following properties.

(GS1) The sequence $(L_\bullet, Q_\bullet)$ is either in order or there exists at most one integer $1 \leq \eta \leq k-s$ such that

$$K_i \subseteq K_h \text{ for } h > i > \eta \text{ and } K_i \cap K_h = K_i \cap K_{i+1} \text{ for } i < \eta \text{ and } h > i.$$

Furthermore, if $K_\eta \subseteq K_{k-s}$, then

$$\dim(L_{n_j} \cap K_i) = \min(n_j, \dim(K_i \cap Q_{d_{k-s}}^{r_{k-s}})) \text{ for } i < \eta \text{ and}$$

$$\dim(L_{n_j} \cap K_i) = \min(n_j, \dim(K_i \cap Q_{d_{k-s}}^{r_{k-s}}) - 1) \text{ for } i \geq \eta.$$

If $K_\eta \not\subseteq K_{k-s}$, then

$$\dim(L_{n_j} \cap K_i) = \min(n_j, \dim(K_i \cap Q_{d_{k-s}}^{r_{k-s}})) \text{ for all } i.$$

(GS2) If $\alpha = \dim(K_i \cap Q_{d_{k-s}}^{r_{k-s}}) > 0$, then either $i = 1$ and $n_\alpha = \alpha$ or there exists at most one j_0 such that, for $j_0 \neq j > \min(i, \eta)$, $r_j - r_{j-1} = d_{j-1} - d_j$. Furthermore,

$$d_{j_0-1} - d_{j_0} \leq r_{j_0} - r_{j_0-1} + 2 - \dim(K_{j_0-1}) + \dim(K_{j_0-1} \cap Q_{d_{j_0}}^{r_{j_0}})$$

and $K_\eta \not\subset Q_{d_{j_0}}^{r_{j_0}}$.

Remark 7.5. Given a sequence $(L_\bullet, Q_\bullet)$, the basic principles concerning skew-symmetric forms imply inequalities among the invariants of a sequence. The evenness of rank implies that $d_i - r_i$ is even for every $1 \leq i \leq k-s$. The corank bound implies that $r_i - \dim(Q_{d_i}^{r_i} \cap K_{i-1}) \leq d_{i-1} - d_i$. The linear space bound implies that $2(n_s + r_i - \dim(K_i \cap L_{n_s})) \leq r_i + d_i$ for every $1 \leq i \leq k-s$. These inequalities are implicit in the sequence $(L_\bullet, Q_\bullet)$.

Remark 7.6. For a symplectic sequence $(L_\bullet, Q_\bullet)$, the invariants n_j, r_i, d_i together with the dimensions $\dim(L_{n_j}, K_i)$ and $\dim(Q_{d_h}^{r_h} \cap K_i)$ determine the sequence $(L_\bullet, Q_\bullet)$ up to the action of the symplectic group. This will become obvious when we construct these sequences by choosing bases.

Definition 7.7. A symplectic sequence $(L_\bullet, Q_\bullet)$ is *admissible* if it satisfies the following additional conditions:

(GA1) $n_j \neq \dim(L_{n_j} \cap K_i) + 1$ for any $1 \leq j \leq s$ and $1 \leq i \leq k-s$.

(GA2) Let x_i denote the number of isotropic subspaces L_{n_j} that are contained in K_i. Then

$$x_i \geq k - i + 1 - \frac{d_i - r_i}{2}.$$

The translation between sequences and symplectic diagrams. Symplectic sequences can be represented by symplectic diagrams introduced in §3. An isotropic linear space L_{n_j} is represented by a bracket] in position n_j. A linear space $Q_{d_i}^{r_i}$ is represented by a brace } in position d_i such that there are exactly r_i positive integers less than or equal to i to the left of the i-th brace. Finally, $\dim(L_{n_j} \cap K_i)$

and $\dim(Q^{r_h}_{d_h} \cap K_i)$, $h > i$, are recorded by the number of positive integers less than or equal to i to the left of $]^j$ and $\}^h$, respectively.

Example 7.8. $11]200\}0\}00$ records a sequence $L_2 \subset Q^3_5 \subset Q^2_6$, where $L_2 \subset \operatorname{Ker}(Q^2_6)$. In the diagram, there is one bracket that occurs in position 2. There are two braces that occur in positions 5 and 6. We thus conclude that the sequence contains one isotropic subspace of dimension 2 (L_2) and two non-isotropic subspaces of dimensions 5 (Q_5) and 6 (Q_6). There are two integers equal to 1 and one integer equal to 2 in the sequence. Hence, the corank of the restriction of Q to the six (respectively, five) dimensional subspace Q^2_6 (Q^3_5) is two (three). Finally, since every integer to the left of the bracket is equal to one, we conclude that $L_2 \subset \operatorname{Ker}(Q^2_6)$.

More explicitly, given a symplectic sequence $(L_\bullet, Q_\bullet)$, the corresponding symplectic diagram $D(L_\bullet, Q_\bullet)$ is determined as follows: The sequence of integers begins with $\dim(L_{n_1} \cap K_1)$ integers equal to 1, followed by $\dim(L_{n_1} \cap K_i) - \dim(L_{n_1} \cap K_{i-1})$ integers equal to i, for $2 \leq i \leq k-s$, in increasing order, followed by $n_1 - \dim(L_{n_1} \cap K_{k-s})$ integers equal to 0. The sequence then continues with $\dim(L_{n_j} \cap K_1) - \dim(L_{n_{j-1}} \cap K_1)$ integers equal to 1, followed by $\dim(L_{n_j} \cap K_i) - \max(\dim(L_{n_{j-1}} \cap K_i), \dim(L_{n_j} \cap K_{i-1}))$ integers equal to i in increasing order, followed by $n_j - \max(n_{j-1}, \dim(L_{n_j} \cap K_{k-s}))$ zeros for $j = 2, \ldots, s$ in increasing order. The sequence then continues with $\dim(Q^{r_{k-s}}_{d_{k-s}} \cap K_1) - \dim(L_{n_s} \cap K_1)$ integers equal to 1, followed by $\dim(Q^{r_{k-s}}_{d_{k-s}} \cap K_i) - \max(\dim(Q^{r_{k-s}}_{d_{k-s}} \cap K_{i-1}), \dim(L_{n_s} \cap K_i))$ integers equal to i in increasing order, followed by zeros until position d_{k-s}. Between positions d_i and d_{i-1} ($i > k-s$), the sequence has $\dim(Q^{r_{i-1}}_{d_{i-1}} \cap K_1) - \dim(Q^{r_i}_{d_i} \cap K_1)$ integers equal to 1, followed by $\dim(Q^{r_{i-1}}_{d_{i-1}} \cap K_h) - \max(\dim(Q^{r_i}_{d_i} \cap K_h), \dim(Q^{r_{i-1}}_{d_{i-1}} \cap K_{h-1}))$ integers equal to h in increasing order, for $h \leq i-1$, followed by zeros until position d_{i-1}. Finally, the sequence ends with $n - d_1$ zeros. The brackets occur at positions n_j and the braces occur at positions d_i.

Proposition 7.9. *The diagram $D(L_\bullet, Q_\bullet)$ is a symplectic diagram of type s for $SG(k,n)$. Furthermore, if $(L_\bullet, Q_\bullet)$ is admissible, then $D(L_\bullet, Q_\bullet)$ is admissible.*

Proof. By construction each bracket or brace occupies a position. Since $n_1 < n_2 < \cdots < n_s < d_{k-s} < \cdots < d_1$, a position is occupied by at most one bracket or brace. Since $n_j < d_i$ for every $1 \leq j \leq s$ and $1 \leq i \leq k-s$, every bracket occurs to the left of every brace. By construction, it is clear that $\dim(L_{n_j} \cap K_i)$ and $\dim(Q^{r_h}_{d_h} \cap K_i)$, for $h \geq i$, are recorded by the number of positive integers less than or equal to i to the left of $]^j$ and $\}^h$, respectively. Hence, every integer equal to i occurs to the left of $\}^i$. Finally, the total number of integers equal to zero or greater than i to the left of $\}^i$ is equal to the rank of the restriction of Q to $Q^{r_i}_{d_i}$. Since this rank is necessarily even, the total number of integers equal to zero or greater than i to the left of $\}^i$ is even. This shows that we have a sequence of brackets and braces of type s.

The sequence of brackets and braces is a symplectic diagram. The corank bound implies that $r_i - \dim(Q^{r_i}_{d_i} \cap K_{i-1}) \leq d_{i-1} - d_i$. The left hand side of the inequality is represented by the number of integers equal to i in the sequence. The right hand side of the inequality is equal to the number of integers between $\}^i$ and $\}^{i-1}$. We thus get the inequality $l(i) \leq \rho(i, i-1)$ required by Condition (S1) in Definition 3.24. By the linear space bound, the largest dimensional linear space contained in $Q^{r_i}_{d_i}$ has dimension bounded by $(d_i + r_i)/2$. The invariant r_i is equal to both the number of positive integers less than or equal to i contained to the left of $\}^i$ and $\dim(K_i)$. The span of L_{n_s} and the kernels K_h for $h \geq i$ is an isotropic subspace of $Q^{r_i}_{d_i}$. The dimension of this subspace is denoted by τ_i and is equal to the sum of $p(]^s)$ and the number of positive integers between $]^s$ and $\}^i$. Hence, $2\tau_i \leq p(\}^i) + r_i$ and condition (S2) of Definition 3.24 holds.

If the sequence is in (perfect) order, then the corresponding sequence of brackets and braces is in (perfect) order. Assume the sequence is not in order. The definition of a sequence implies that, for $i < k - s$, there can be at most one i which is not to the left of $\}^{k-s}$. Suppose the sequence satisfies Condition (GS1). Then, there exists an integer η such that for $i > \eta$ those integers that are not to the left of $\}^{k-s}$ are to the immediate left of $\}^{i+1}$. Furthermore, condition (GS1) implies that the positive numbers up to η are in non-decreasing order and η is the only integer violating the order. Thus condition (S3) is satisfied. Finally, condition (GS2) directly translates to condition (S4). We conclude that the sequence of brackets and braces is a symplectic diagram.

If the sequence $(L_\bullet, Q_\bullet)$ is admissible, then the corresponding symplectic diagram is also admissible. Let i be the minimal index such that $L_{n_j} \subset K_i$. If there isn't such an index, let $i = k - s + 1$. If $i > 1$, then condition (GA1) implies that $\dim(L_{n_j} \cap K_{i-1}) \leq n_j - 2$. Hence, the two integers preceding $]^j$ are equal to i (or 0 if $i = k - s + 1$). If $i = 1$, then all the integers preceding $]^j$ are equal to 1. Furthermore, if $n_j = 1$, condition (GA1) implies that $L_{n_j} \subset K_i$ for all $1 \leq i \leq k - s$. We conclude that condition (A1) holds. The invariant x_i is equal to both the number of isotropic subspaces L_{n_j} contained in K_i and the number of brackets such that every integer to the left of it is positive and less than or equal to i. Since $d_i = p(\}^i)$, conditions (A2) and (GA2) are exactly the same. This concludes the proof of the proposition. $\square$

Remark 7.10. Proposition 7.9 also explains the definition of a symplectic diagram in geometric terms. Condition (4) of Definition 3.12 is implied by the evenness of rank and simply states that $d_i - r_i$ has to be even. As discussed in the proof of Proposition 7.9, condition (S1) is a translation of the corank bound and condition (S2) is implied by the linear space bound.

Conversely, we can associate an admissible sequence to every admissible symplectic diagram. By Darboux's Theorem, we can take the skew-symmetric form to be defined by $\sum_{i=1}^m x_i \wedge y_i$. Let the dual basis for x_i, y_i be e_i, f_i such that $x_i(e_j) = \delta_i^j$, $y_i(f_j) = \delta_i^j$ and $x_i(f_j) = y_i(e_j) = 0$. Given an admissible symplectic diagram, we associate $e_1, \dots, e_{p(]^s)}$ to the integers to the left of $]^s$ in order. We then associate $e_{p(]^s)+1}, \dots, e_{r'}$ to the positive integers to the right of $]^s$ and left of $\}^{k-s}$ in order. Let $e_{i_1}, \dots, e_{i_l}$ be vectors that have so far been associated to zeros. Then associate $f_{i_1}, \dots, f_{i_l}$ to the remaining zeros to the left of $\}^{k-s}$ in order. If there are any zeros to the left of $\}^{k-s}$ that have not been assigned a basis vector, assign them $e_{r'+1}, f_{r'+1}, \dots, e_{r''}, f_{r''}$ in pairs in order. Continuing this way, if there is a positive integer between $\}^{i+1}$ and $\}^i$ associate to it the smallest index basis element e_α that has not yet been assigned. Assume that the integers equal to $i+1$ have been assigned the vectors $e_{j_1}, \dots, e_{j_l}$. Assign to the zeros between $\}^{i+1}$ and $\}^i$, the vectors $f_{j_1}, \dots, f_{j_l}$. If there are any zeros between $\}^{i+1}$ and $\}^i$ that have not been assigned a vector, assign them $e_{\alpha+1}, f_{\alpha+1}, \dots, e_\beta, f_\beta$ in pairs until the zeros are exhausted. Let L_{n_j} be the span of the basis elements associated to the integers to the left of $]^j$. Let $Q_{d_i}^{r_i}$ be the span of the basis elements associated to the integers to the left of $\}^i$. We thus obtain a sequence $(L_\bullet, Q_\bullet)$ whose associated symplectic diagram is D.

Example 7.11. To 11]233]0000}00}0}00 we associate the sequence of vectors

$$e_1, e_2, e_3, e_4, e_5, e_6, f_6, e_7, f_7, f_4, f_5, f_3, f_1, f_2.$$

Then L_2 is the span of e_1, e_2, L_5 is the span of e_1 through e_5, Q_9^5 is the span of e_1 through e_7 and f_6, f_7, Q_{11}^3 is the span of e_1 through e_7 and f_4 through f_7. Finally, Q_{12}^2 is the span of Q_{11}^3 and f_3.

To 22]33]0000}00}100}0 we associate the sequence of vectors

$$e_1, e_2, e_3, e_4, e_5, f_5, e_6, f_6, f_3, f_4, e_7, f_1, f_2, f_7.$$

L_2 is the span of e_1, e_2, L_4 is the span of e_1 through e_4, Q_8^4 is the span of e_1 through e_6 and f_5, f_6, Q_{10}^2 is the span of Q_8^4 and f_3, f_4 and Q_{13}^1 is the span of Q_{10}^2 and e_7, f_1, f_2.

Finally, to 22]300]300}00}100}0 we associate the sequence of vectors

$$e_1, e_2, e_3, e_4, e_5, e_6, f_4, f_5, f_3, f_6, e_7, f_1, f_2, f_7.$$

Then L_2 is the span of e_1 and e_2, L_5 is the span of e_1 through e_5, Q_8^4 is the span of e_1 through e_5 and f_4, f_5, Q_{10}^2 is the span of Q_8^4 and f_3, f_6. Finally, Q_{13}^1 is the span of all the vectors but f_7.

Remark 7.12. Notice that equivalent symplectic diagrams correspond to permutations of the basis elements that do not change the vector spaces in $(L_\bullet, Q_\bullet)$.

Remark 7.13. The construction of a symplectic sequence $(L_\bullet, Q_\bullet)$ from a symplectic diagram D is well-defined. By condition (S2), the number of zeros to the left of $]^s$ is less than or equal to the number of zeros between $]^s$ and $\}^{k-s}$. Hence, we can choose vectors $f_{i_1}, \dots, f_{i_l}$ corresponding to the vectors $e_{i_1}, \dots, e_{i_l}$. Similarly, if there does not exist a positive integer between $\}^{i+1}$ and $\}^i$, then by condition (S1), $l(i+1) \leq \rho(i+1, i)$. We can, therefore, associate vectors $f_{j_1}, \dots, f_{j_l}$ to the zeros between $\}^{i+1}$ and $\}^i$. If there exists a positive integer between $\}^{i+1}$ and $\}^i$, then there is only one positive integer between them by condition (S3). If $l(i+1) = \rho(i+1, i)$, then condition (4) is violated. Hence, $l(i+1) < \rho(i+1, i)$ and we can associate vectors $f_{j_1}, \dots, f_{j_l}$ to the zeros between $\}^{i+1}$ and $\}^i$. Thus the construction of the sequence makes sense. It is now straightforward to check that the sequence associated to an admissible symplectic diagram is an admissible sequence. Furthermore, the two constructions are inverses of each other.

We are now ready to define symplectic restriction varieties.

Definition 7.14. Let $(L_\bullet, Q_\bullet)$ be an admissible sequence for $SG(k, n)$. Then the *symplectic restriction variety* $V(L_\bullet, Q_\bullet)$ is the Zariski closure of the locus in $SG(k, n)$ parameterizing

$$\{W \in SG(k,n) \mid \dim(W \cap L_{n_j}) = j \text{ for } 1 \leq j \leq s, \dim(W \cap Q_{d_i}^{r_i}) = k - i + 1$$
$$\text{and } \dim(W \cap K_i) = x_i \text{ for } 1 \leq i \leq k - s\}.$$

Remark 7.15. The geometric reasons for imposing conditions (A1) and (A2) in Definition 3.27 are now clear. Condition (A1) is an immediate consequence of the kernel bound. If $\dim(L_{n_j} \cap K_i) = n_j - 1$ and a linear space of dimension $k - i + 1$

intersects n_j in dimension j and K_i in dimension $j-1$, then the linear space is contained in $L_{n_j}^{\perp}$. Hence, we need to impose condition (A1).

The inequality
$$x_i \geq k - i + 1 - \frac{d_i - r_i}{2}$$
is an easy consequence of the linear space bound. We require the k-dimensional isotropic subspaces to intersect $Q_{d_i}^{r_i}$ in a subspace of dimension $k-i+1$ and to intersect the singular locus of $Q_{d_i}^{r_i}$ in a subspace of dimension x_i. By the linear space bound, any linear space of dimension $k-i+1$ has to intersect the singular locus in a subspace of dimension at least $k-i+1-\frac{d_i-r_i}{2}$, hence the inequality in condition (A2) holds.

Example 7.16. The two most basic examples of symplectic restriction varieties are:

(1) A Schubert variety $\Sigma_{\lambda;\mu}$ in $SG(k,n)$, which is the restriction variety associated to a symplectic diagram $D(\sigma_{\lambda;\mu})$, and
(2) The intersection $\Sigma_{a_\bullet} \cap SG(k,n)$ of a general Schubert variety in $G(k,n)$ with $SG(k,n)$, which is the restriction variety associated to $D(a_\bullet)$.

In general, symplectic restriction varieties interpolate between these two examples.

Lemma 7.17. *A symplectic restriction variety corresponding to a saturated and perfectly ordered admissible sequence is a Schubert variety in $SG(k,n)$. Conversely, every Schubert variety in $SG(k,n)$ can be represented by such a sequence.*

Proof. Let $F_1 \subset \cdots \subset F_1^{\perp} \subset V$ be an isotropic flag. If $\Sigma_{\lambda,\mu}$ is a Schubert variety defined with respect to this flag, then the symplectic restriction variety defined with respect to the sequence $L_{n_j} = F_{\lambda_j}$ and $Q_{d_i}^{r_i} = F_{\mu_{k-i+1}}^{\perp}$ is a saturated and perfectly ordered admissible sequence.

Conversely, suppose that the sequence $(L_\bullet, Q_\bullet)$ is a saturated and perfectly ordered admissible sequence. Since the sequence is saturated, we have that $Q_{d_i}^{r_i} = K_i^{\perp}$. Since the sequence is in perfect order, we have that $\dim(L_{n_j} \cap \mathrm{Ker}(Q_{d_i}^{r_i})) = \min(r_i, n_j)$. Consequently, the set of linear spaces $\{L_{n_j}, \mathrm{Ker}(Q_{d_i}^{r_i})\}$ can be ordered by inclusion, or equivalently, by dimension. Then the resulting partial flag can be extended to an isotropic flag. By condition (GA1) of the definition of an admissible sequence, we have that $n_j \neq r_i + 1$ for any i, j. Hence, the symplectic restriction variety defined with respect to $(L_\bullet, Q_\bullet)$ is the Schubert variety $\Sigma_{\lambda_\bullet;\mu_\bullet}$, where $\lambda_j = n_j$, for $1 \leq j \leq s$, and $\mu_i = r_{k-i+1}$, for $s < i \leq k$. $\square$

Remark 7.18. By Lemma 7.17, the saturated symplectic diagrams in perfect order represent Schubert varieties.

Next, we show that the intersection of a general Schubert variety Σ with the symplectic Grassmannian $SG(k,n)$ (when non-empty) is a restriction variety.

Lemma 7.19. *Let Σ be the Schubert variety defined with respect to a general partial flag $F_{a_1} \subset \cdots \subset F_{a_k}$. Then $\Sigma \cap SG(k,n) \neq \emptyset$ if and only if $a_i \geq 2i-1$ for $1 \leq i \leq k$.*

Proof. Suppose $a_i < 2i-1$ for some i. If $[W] \in \Sigma \cap SG(k,n)$, then $W \cap F_{a_i}$ is an isotropic subspace of $Q \cap F_{a_i}$ of dimension at least i. Since F_{a_i} is general, the corank of $Q \cap F_{a_i}$ is 0 or 1 and equal to a_i modulo 2. By the linear space bound, the largest dimensional isotropic subspace of $Q \cap F_{a_i}$ has dimension less than or equal to $i-1$. Therefore, W cannot exist and $\Sigma \cap SG(k,n) = \emptyset$.

Conversely, let $a_i = 2i-1$ for every i. Then $G_1 = F_1$ is isotropic, $G_2 = F_1^{\perp}$ in F_3 is the unique two-dimensional isotropic subspace of $Q \cap F_3$ containing G_1. By induction, we see that $G_i = G_{i-1}^{\perp}$ is the unique subspace of dimension i isotropic with respect to $Q \cap F_{2i-1}$ that contains G_{i-1}. Continuing this way, we construct a unique isotropic subspace W of dimension k contained in $\Sigma \cap SG(k,n)$. If $a_i \geq 2i-1$, the vector space W just constructed is still contained in $\Sigma \cap SG(k,n)$, hence this intersection is non-empty. □

Lemma 7.20. *Let Σ be the Schubert variety defined with respect to a general partial flag $F_{a_1} \subset \cdots \subset F_{a_k}$ such that $a_i \geq 2i-1$. Then $\Sigma \cap SG(k,n) = V(D(a_\bullet))$.*

Proof. Let $a_i = 2i-1$, then since F_{a_i} is general, the restriction of Q to F_{a_i} has a one-dimensional kernel K_i. By the linear space bound, any i-dimensional isotropic subspace W contained in F_{a_i} contains K_i. For each j such that $a_j > 2j-1$, recall that u_j is the number of $i < j$ such that $a_i = 2i-1$ and v_j is the number of $i > j$ such that $a_i = 2i-1$. Let K be the span of one-dimensional kernels K_i for each $a_i = 2i-1$. Then $\dim(K) = u$ and any k-dimensional subspace W contained in $\Sigma \cap SG(k,n)$ contains K. For j such that $a_j > 2j-1$, let $G_{j+v_j} = \mathrm{Span}(F_{a_j}, K) \cap K^{\perp}$. The dimension of G_{j+v_j} is $a_j - u_j + v_j$. The corank of the restriction of Q to G_{j+v_j} is $u + \delta(a_j)$, where $\delta(a_j) = 0(1)$ if a_j is even (odd). Furthermore, any isotropic linear space contained in $\Sigma \cap SG(k,n)$ intersects G_{j+v_j} in a subspace of dimension at least $j+v_j$. From this description and the definition of $V(D(a_\bullet))$, it is now clear that $\Sigma \cap SG(k,n) = V(D(a_\bullet))$. □

Proposition 7.21. *Let $(L_\bullet, Q_\bullet)$ be an admissible sequence. Then $V(L_\bullet, Q_\bullet)$ is an irreducible subvariety of $SG(k,n)$ of dimension*

$$\dim(V(L_\bullet, Q_\bullet)) = \sum_{j=1}^{s}(n_j - j) + \sum_{i=1}^{k-s}(d_i - 1 - 2k + 2i + x_i). \tag{2}$$

Proof. The proof is by induction on k. When $k=1$, if the sequence consists of an isotropic linear space L_{n_1}, then the corresponding symplectic restriction variety is $\mathbb{P}L_{n_1}$ hence it is irreducible of dimension $n_1 - 1$. If the sequence consists of one non-isotropic subspace $Q_{d_1}^{r_1}$, then the corresponding symplectic restriction variety is also projective space of dimension $d_1 - 1$. In both cases, the varieties are irreducible of the claimed dimension. This proves the base case of the induction.

If the sequence does not contain any skew-symmetric forms, then the corresponding restriction variety is isomorphic to a Schubert variety in the ordinary Grassmannian $G(k,n)$. In that case, it is well known that Schubert varieties are irreducible and have dimension $\sum_{j=1}^{k}(n_j - j)$ [**?**].

Observe that omitting $Q_{d_1}^{r_1}$ from an admissible sequence $(L_\bullet, Q_\bullet)$ for $SG(k,n)$ gives rise to an admissible sequence $(L'_\bullet, Q'_\bullet)$ for $SG(k-1,n)$. There is a natural surjective morphism $f : V^0(L_\bullet, Q_\bullet) \to V^0(L'_\bullet, Q'_\bullet)$ that sends a vector space W to $W \cap Q_{d_2}^{r_2}$ (or $W \cap L_{n_{k-1}}$ if $s = k-1$). By induction, $V(L'_\bullet, Q'_\bullet)$ is irreducible of dimension $\sum_{j=1}^{s}(n_j - j) + \sum_{i=2}^{k-s}(d_i - 1 - 2k + 2i + x_i)$. The fibers of the morphism f over a point W' correspond to choices of isotropic k-planes W that contain W' and are contained in $Q_{d_1}^{r_1}$. This is a Zariski dense open subset of projective space of dimension $d_1 - 2(k-1) - 1 + x_1$. Hence, by the Theorem on the Dimension of Fibers [S, I.6.7], $V(L_\bullet, Q_\bullet)$ is irreducible of the claimed dimension. This concludes the proof of the proposition. □

8. The geometric explanation of the combinatorial game

In this section, we will prove the combinatorial rule by interpreting it geometrically. The transformation from an admissible diagram D to D^a records a one-parameter specialization of the restriction variety $V(D)$. The algorithm describes the flat limit of this specialization.

The specialization. We now explain the specialization. There are several cases depending on whether D is in order and whether $l(\kappa) < \rho(\kappa, \kappa - 1) - 1$ or not. In the previous section, given an admissible quadric diagram D, we associated an admissible sequence by defining each of the vector spaces $(L_\bullet, Q_\bullet)$ as a union of basis elements that diagonalize the skew-symmetric form Q. All our specializations will replace exactly one of the basis elements $v = e_u$ or $v = f_u$ for some $1 \leq u \leq m$ with a vector $v(t) = e_u(t)$ or $v(t) = f_u(t)$ varying in a one-parameter family. For $t \neq 0$, the resulting set of vectors will be a new basis for V, but when $t = 0$ two of the basis elements will become equal. Since each linear space in $(L_\bullet, Q_\bullet)$ is a union of basis elements, we get a one-parameter family of vector spaces $(L_\bullet(t), Q_\bullet(t))$ by replacing every occurrence of the vector v with $v(t)$ for $t \neq 0$. Correspondingly, we have a one-parameter family of restriction varieties $V(L_\bullet(t), Q_\bullet(t))$. Since these varieties are projectively equivalent as long as $t \neq 0$, we obtain a flat one-parameter family. Our task is to describe the limit when $t = 0$.

In case (1)(i), D is not in order, η is the unique integer violating the order, and ν is the leftmost integer equal to $\eta + 1$. Suppose that under the translation between symplectic diagrams and sequences of vector spaces, e_u is the vector associated to η and e_v is the vector associated to ν. Then consider the one-parameter family obtained by changing e_v to $e_v(t) = te_v + (1-t)e_u$ and keeping every other vector fixed. When the set of basis elements spanning a vector space L_{n_j} or $Q_{d_i}^{r_i}$ contains e_v, $L_{n_j}(t)$ or $Q_{d_i}^{r_i}(t)$ is the span of the same basis elements except that e_v is replaced with $e_v(t)$. Otherwise, $L_{n_j}(t) = L_{n_j}$ or $Q_{d_i}^{r_i}(t) = Q_{d_i}^{r_i}$.

In case (1)(ii), D is not in order, η is the unique integer violating the order, $i > \eta$ does not occur in the sequence to the left of η and ν is the leftmost integer equal to $i + 1$. Let e_u be the vector associated to η and let e_v be the vector associated to ν. Consider the one-parameter family obtained by changing f_v to $f_v(t) = tf_v + (1-t)e_u$. When the set of basis elements spanning a vector space L_{n_j} or $Q_{d_i}^{r_i}$ contains f_v, $L_{n_j}(t)$ or $Q_{d_i}^{r_i}(t)$ is the span of the same basis elements except that f_v is replaced with $f_v(t)$. Otherwise, $L_{n_j}(t) = L_{n_j}$ or $Q_{d_i}^{r_i}(t) = Q_{d_i}^{r_i}$.

In case (2)(i), D is in order and $l(\kappa) < \rho(\kappa, \kappa - 1) - 1$. Suppose that e_v is the vector associated to ν, the leftmost $\kappa + 1$. Let e_u and f_u be two vectors associated to zeros between $\}^\kappa$ and $\}^{\kappa-1}$. These exist since $l(\kappa) < \rho(\kappa, \kappa-1) - 1$. Consider the one-parameter specialization replacing f_v with $f_v(t) = tf_v + (1-t)e_u$. When the set of basis elements spanning a vector space L_{n_j} or $Q_{d_i}^{r_i}$ contains f_v, $L_{n_j}(t)$ or $Q_{d_i}^{r_i}(t)$ is obtained by replacing f_v with $f_v(t)$. Otherwise, $L_{n_j}(t) = L_{n_j}$ or $Q_{d_i}^{r_i}(t) = Q_{d_i}^{r_i}$.

In case (2)(ii)(a), D is in order and $l(\kappa) = \rho(\kappa, \kappa - 1) - 1$. Let ν be the leftmost integer equal to κ and suppose that e_v is the vector associated to ν. Let e_u be the vector associated to the $\kappa - 1$ following $\}^\kappa$. Then let $e_v(t) = te_v + (1-t)e_u$. When the set of basis elements spanning a vector space L_{n_j} or $Q_{d_i}^{r_i}$ contains e_v, $L_{n_j}(t)$ or $Q_{d_i}^{r_i}(t)$ is obtained by replacing e_v with $e_v(t)$. Otherwise, $L_{n_j}(t) = L_{n_j}$ or $Q_{d_i}^{r_i}(t) = Q_{d_i}^{r_i}$.

Finally, in case (2)(ii)(b), D is in order, $l(\kappa) = \rho(\kappa, \kappa-1) - 1$ and there does not exist an integer equal to κ to the left of κ. Let e_v be the vector associated to ν, the leftmost integer equal to $\kappa+1$ and let e_u be the vector associated to $\kappa-1$ to the right of $\}^\kappa$. Then let $f_v(t) = tf_v + (1-t)e_u$. When the set of basis elements spanning a vector space L_{n_j} or $Q_{d_i}^{r_i}$ contains f_v, $L_{n_j}(t)$ or $Q_{d_i}^{r_i}(t)$ is obtained by replacing f_v with $f_v(t)$. Otherwise, $L_{n_j}(t) = L_{n_j}$ or $Q_{d_i}^{r_i}(t) = Q_{d_i}^{r_i}$.

The flat limits of the vector spaces are easy to describe. If L_{n_j} or $Q_{d_i}^{r_i}$ does not contain the vector v, then $L_{n_j}(t) = L_{n_j}$ and $Q_{d_i}^{r_i}(t) = Q_{d_i}^{r_i}$ for all $t \neq 0$. Hence, the flat limit $L_{n_j}(0) = L_{n_j}$ and $Q_{d_i}^{r_i}(0) = Q_{d_i}^{r_i}$. Similarly, if L_{n_j} or $Q_{d_i}^{r_i}$ contains both of the basis elements spanning $v(t)$, then $L_{n_j}(t) = L_{n_j}$ and $Q_{d_i}^{r_i}(t) = Q_{d_i}^{r_i}$ for all $t \neq 0$. Then in the limit $L_{n_j}(0) = L_{n_j}$ and $Q_{d_i}^{r_i}(0) = Q_{d_i}^{r_i}$. A vector space changes under the specialization only when it contains the vector with coefficient t and does not contain the vector with coefficient $(1-t)$. In this case, in the limit $t = 0$, the flat limit $L_{n_j}(0)$ or $Q_{d_i}^{r_i}(0)$ is obtained by replacing in L_{n_j} or Q_{d_i} the basis element with coefficient t with the basis element with coefficient $(1-t)$.

Notice that in each of these cases, the set of limiting vector spaces is depicted by the symplectic diagram D^a. In case (1)(i), if η is between $\}^a$ and $\}^{a-1}$ and ν is between $]^b$ and $]^{b+1}$ (respectively, between $]^s$ and $\}^{k-s}$), the vector spaces L_{n_j} for $j \leq b$ (respectively, $j \leq s$) and $Q_{d_i}^{r_i}$ for $i < a$ are unaffected. In all the other vector spaces, e_v is replaced by e_u. The effect on symplectic diagrams is to switch η and ν as in the definition of D^a. In case (1)(ii), assume that η is between $\}^i$ and $\}^{i-1}$. The linear spaces other than $Q_{d_i}^{r_i}$ remain unchanged under the degeneration. In $Q_{d_i}^{r_i}$ the vector f_v is replaced by e_u. Note that this increases the corank of the restriction of Q to $Q_{d_i}^{r_i}(0)$ by two since now both vectors e_u and e_v in the kernel. This has the effect of changing ν to i and a zero between $\}^{i+1}$ and $\}^i$ to η as in the definition of D^a. In case (2)(i), all the vector spaces but $Q_{d_\kappa}^{r_\kappa}$ remain unchanged. The degeneration replaces f_v in $Q_{d_\kappa}^{r_\kappa}$ by e_u. This increases the corank of the restriction of Q to $Q_{d_\kappa}^{r_\kappa}(0)$ by two since both e_u and e_v are now contained in the kernel of the restriction. The corresponding symplectic diagram is obtained by changing ν and a zero between $\}^{\kappa+1}$ and $\}^\kappa$ to κ as in the definition of D^a. The cases (2)(ii)(a) and (b) are analogous to the cases (1)(i) and (1)(ii), respectively.

For the rest of the paper, we use the specialization just described.

Example 8.1. For concreteness, consider the restriction variety associated to 200}000}00 in $SG(2,8)$ parameterizing isotropic subspaces that intersect $A = \mathrm{Span}(e_1, e_2, f_2)$ and are contained in $B = \mathrm{Span}(e_i, f_i)$, $1 \leq i \leq 3$. The first specialization is given by $tf_2 + (1-t)e_3$. In the limit, $A_1 = A(0) = \mathrm{Span}(e_1, e_2, e_3)$ and $B(0) = B$. This changes the diagram to 000]000}00. The corresponding restriction variety parameterizes linear spaces that intersect $A(0)$ and are contained in B. The next specialization is given by $tf_1 + (1-t)e_4$. In the limit, $A_1(0) = A_1$ and $B_1 = B(0) = \mathrm{Span}(e_1, e_2, e_3, e_4, f_2, f_3)$. This changes the diagram to 100]100}00. The corresponding restriction variety parameterizes linear spaces that intersect A_1 and are contained in B_1. The final specialization is given by $te_2 + (1-t)e_4$. In the limit, $A_2 = A_1(0) = \mathrm{Span}(e_1, e_4, e_3)$ and $B_1(0) = B_1$. This changes the diagram to 110]000}00. The flat limit of the restriction varieties has two components. The linear spaces may intersect $\mathrm{Span}(e_1, e_4)$, in which case we get the restriction variety associated to the diagram 11]0000}00. Otherwise, by the kernel bound, the linear spaces have to be contained in $A_2^\perp$. In this case, we get the restriction variety associated to the diagram 111]00}000. The reader should convince themselves that this is precisely the outcome of Algorithm 3.39.

We are now ready to state and prove the main geometric theorem.

Theorem 8.2. *(The Geometric Branching Rule) The flat limit of the specialization of $V(D)$ is supported along $\bigcup V(D_i)$, where $V(D_i)$ is a symplectic restriction variety associated to a diagram D_i obtained by running Algorithm 3.39 on D. Furthermore, the flat limit is generically reduced along each $V(D_i)$. In particular, the equality*

$$[V(D)] = \sum [V(D_i)]$$

holds between the cohomology classes of symplectic restriction varieties.

proof of theorem 3.43 assuming theorem 8.2. By Proposition 3.49, Algorithm 3.39 replaces each admissible symplectic diagram by one or two admissible symplectic diagrams. Hence, the algorithm can be repeated. By Proposition 3.50, after finitely many steps, the algorithm terminates leading to a collection of saturated admissible symplectic diagrams in perfect order. By Lemma 7.17, each of these diagrams represent a Schubert variety. Therefore, Theorem 3.43 is an immediate corollary of Theorem 8.2. ☐

Proof of Theorem 8.2. The proof of Theorem 8.2 has two steps. First, we interpret the algorithm as the specialization described in the beginning of this section. Let $V(D)$ denote the initial symplectic restriction variety. Let $V(D(t))$ denote the one-parameter family of restriction varieties described in the specialization and let $V(D(0))$ be the flat limit at $t = 0$. We show that $V(D(0))$ is supported along the union of restriction varieties $V(D_i)$, where D_i are the admissible symplectic diagrams derived from D via Algorithm 3.39. In the second step, we verify that the support of the flat limit contains each $V(D_i)$ and the flat limit is generically reduced along each $V(D_i)$. This suffices to prove the theorem.

We now analyze the specialization to conclude that the support of $V(D(0))$ is the union of symplectic restriction varieties $V(D_i)$. The proof is by a dimension count. In order to restrict the possible irreducible components of $V(D(0))$, we find conditions that the linear spaces parameterized by $V(D(0))$ have to satisfy. We then observe that these conditions already cut out the symplectic varieties $V(D_i)$ and that each $V(D_i)$ has the same dimension as $V(D)$. The following observation puts strong restrictions on the support of the flat limit.

Observation 8.3. The linear spaces parameterized by $V(D(t))$ intersect the linear spaces $L_{n_j}(t)$ (respectively, $Q^{r_i}_{d_i}(t)$) in a subspace of dimension at least j (respectively, $k-i+1$). Similarly, they intersect $\mathrm{Ker}(Q^{r_i}_{d_i}(t))$ in a linear space of dimension at least x_i. Since intersecting a proper variety in at least a given dimension is a closed condition, the linear spaces parameterized by $V(D(0))$ have to intersect the linear spaces $L_{n_j}(0)$ (respectively, $Q^{r_i}_{d_i}(0)$) in a subspace of dimension at least j (respectively, $k-i+1$). Furthermore, they intersect $\mathrm{Ker}(Q^{r_i}_{d_i}(0))$ in a subspace of dimension at least x_i.

Let Y be an irreducible component of $V(D(0))$. We can construct a sequence of vector spaces $F_{u_1} \subset \cdots \subset F_{u_k}$ such that the locus Z parameterizing linear spaces with $\dim(W \cap F_{u_j}) \geq j$ contains Y. We have already seen that the linear spaces $L_{n_j}(0)$ and $Q^{r_i}_{d_i}(0)$ are the linear spaces recorded by the symplectic diagram D^a.

Let $z_1, \dots, z_n$ be the ordered basis of V obtained by listing the basis elements associated to D^a from left to right. Let F_u be the linear space spanned by the basis elements $z_1, \cdots, z_u$. Let $F_{u_1} \subset \cdots \subset F_{u_k}$ be the jumping linear spaces for Y, that is the linear spaces of the form F_u such that $\dim(W \cap F_u) > \dim(W \cap F_{u-1})$ for the general isotropic space W parameterized by Y. Observation 8.3 translates to the inequalities $u_j \leq n_j$ for $j \leq s$ and $u_i \leq d_{k-i+1}$ for $s < i \leq k$. Hence, we can obtain a sequence depicting the linear spaces $F_{u_1}, \dots, F_{u_k}$ by moving the braces and brackets in the diagram D^a to the left one at a time. By the proof of Proposition 7.21, Equation (2) gives an upper bound on the dimension of the locus Z (note that we used the fact that the sequence is admissible in the proof only to deduce the equality).

We now estimate the dimension of Z. Let $(L^a_\bullet, Q^a_\bullet)$ denote the linear spaces depicted by the diagram D^a. We obtain the sequence defining Z by replacing linear spaces in $(L^a_\bullet, Q^a_\bullet)$ by smaller dimensional ones.

- If we replace a linear space $L^a_{n_i}$ of dimension n_i in $(L^a_\bullet, Q^a_\bullet)$ with a linear space F_{u_i} not contained in $(L^a_\bullet, Q^a_\bullet)$ but containing $L^a_{n_{i-1}}$, then according to Equation (2) the dimension changes as follows. Let y^a_i be the index of the smallest index linear space $Q^{r_l}_{d_l}$ such that $L^a_{n_i} \subset K_l$. Similarly, let y^u_i be the smallest l such that $F_{u_i} \subset K_l$. The left sum in Equation (2) changes by $u_i - n^a_i$. The quantities x_l increase by one for $y^u_i \leq l < y^a_i$. Hence, the sum on the right increases by $y^a_i - y^u_i$. Hence, the total change in dimension is $u_i - n^a_i + y^a_i - y^u_i$. By condition (S4) of Definition 3.24 for D^a and condition (A1) for D, in D^a, there is at most one missing integer among the positive integers to the left of the brackets and the two integers preceding all brackets but possibly $]^{x_{\nu-1}+1}$ are equal. We conclude that if we move any bracket to the left except for $]^{x_{\nu-1}+1}$, we strictly decrease the dimension. Furthermore, if we move $]^{x_{\nu-1}+1}$ to the left, we strictly decrease the dimension unless in D we have the equality $p(]^{x_{\nu-1}+1}) - \pi(\nu) - 1 = y_{x_{\nu-1}+1} - \nu$, so that the decrease in the position resulting by shifting the bracket in D^a is equal to the increase in the number of linear spaces $Q^{r_l}_{d_l}$ containing F_{u_i} in their kernel.
- If we replace the linear space $Q^{r_i,a}_{d_i}$ of dimension d^a_i in $(L^a_\bullet, Q^a_\bullet)$ with a non-isotropic linear space $F_{u_{k-i+1}}$ of dimension d^u_i containing $Q^{r_{i-1},a}_{d_{i-1}}$, then, by Equation (2), the dimension changes as follows. Let x^u_i be the number of linear spaces that are contained in the kernel of the restriction of Q to $F_{u_{k-i+1}}$. Then the dimension changes by $d^u_i - d^a_i - x^a_i + x^u_i$. We have that $d^u_i - d^a_i - x^a_i + x^u_i \leq 0$ with strict inequality unless the number of linear spaces contained in the kernel of $F_{u_{k-i+1}}$ increases by an amount equal to $d^a_i - d^u_i$. The latter can only happen if condition (A1) is violated for the diagram so that increasing the dimension of the kernel by one can increase the number of linear spaces contained in the kernel.
- Finally, if we replace the linear space $Q^{r_{k-s},a}_{d_{k-s}}$ of dimension d^a_{k-s} in $(L^a_\bullet, Q^a_\bullet)$ with an isotropic linear space $F_{u_{s+1}}$ containing L_{n_s}, then the first sum in Equation (2) changes by $u_{s+1} - s - 1$. The second sum changes by $-d^a_{k-s} + y^u_{s+1} - x^a_{k-s} + (2s+1)$, where y^u_{s+1} denotes the number of non-isotropic subspaces containing $F_{u_{s+1}}$ in the kernel of the restriction of Q.

Hence, the total change is

$$-d^a_{k-s} + u_{s+1} - x^a_{k-s} + y^u_{s+1} + s.$$

If $x^a_{k-s} = s - j < s$, then $y^u_{s+1} = 0$. Since by the linear space bound $u_{s+1} + j + 1 \leq d_{k-s}$, we conclude that the dimension strictly decreases. If $x_{k-s} = s$, then the change is strictly negative unless $r_{k-s} = d_{k-s}$ and $d_{k-s} = u_{s+1}$.

The dimension count shows that $V(D)$ and $V(D^a)$ have the same dimension. When $p(]^{x_{\nu-1}+1}) - \pi(\nu) - 1 = y_{x_{\nu-1}+1} - \nu$ in D, $V(D^b)$ and $V(D^a)$ have the same dimension. Furthermore, Step 2 of Algorithm 3.35 and Algorithm 3.36 preserve the dimension of the variety. By Equation (2), Step 1 of Algorithm 3.35 also preserves the dimension. If condition (A2) is violated for D^a for the index i, then by Proposition 3.49, we have that $2x_i = 2k - 2i - d_i + r_i$. On the other hand, the operation in Step 1 of Algorithm 3.35 changes the left sum in Equation (2) by $r_i + (s - x_i) - s - 1 = r_i - x_i - 1$, since it adds a new bracket of size r_i and increases the positions of the brackets with index $x_i + 1, \dots, s$. It changes the left sum by $-d_i + 1 - x_i + 2k - 2(k-s) + 2(k-s-i)$ since it removes the brace with index i and increases the positions and x_l for the braces with indices $l = i+1, \dots, k-s$. We conclude that the change in dimension is $r_i - 2x_i - d_i + 2k - 2i = 0$. We conclude that every variety $V(D_i)$ associated to $V(D)$ by Algorithm 3.39 has the same dimension as $V(D)$.

We can now determine the support of the flat limit of the specialization. Since in flat families the dimension of the fibers are preserved, Y has the same dimension as $V(D)$. Hence, our dimension calculation puts very strong restrictions on Z. First, suppose that either $x_{\nu-1} = s$ or $p(\}^{x_{\nu-1}+1}) - \pi(\nu) - 1 > y_{x_{\nu-1}+1} - \nu$ in D. If D^a is admissible, then by our dimension counts, replacing an isotropic or non-isotropic linear space in $(L^a_\bullet, Q^a_\bullet)$ with a smaller dimensional linear space produces a strictly smaller dimensional locus. We conclude that the general linear space parameterized by Y satisfies exactly the rank conditions imposed by $(L^a_\bullet, Q^a_\bullet)$. Hence, Y is contained in $V(D^a)$. Since both are irreducible varieties of the same dimension, we conclude that $Y = V(D^a)$. If D^a is not admissible, then it either violates condition (A1) or (A2). If D^a fails condition (A2), then $x_i < k - i + 1 - \frac{d_i - r_i}{2}$ for some i. Since the linear spaces parameterized by Y have to intersect $Q^{r_i}_{d_i}$ in a subspace of dimension $k-i+1$, by the linear space bound, we conclude that these linear spaces have to intersect K_i in a subspace of dimension at least $x_i + 1$. In D^a, there is only one integer i that is not in the beginning non-decreasing part of the sequence of integers. Geometrically, the linear spaces $L^a_{n_j}$ or $Q^{r_j,a}_{d_j}$ either contain or are contained in K_i or intersect K_i in a codimension one linear space. Let $F_{a_1} \subset F_{a_2} \subset \cdots \subset F_{a_l}$ be a partial flag such that F_{a_h} intersects M in a codimension one subspace of M. Let $M = G_{a_0+1} \subset G_{a_1+1} \subset \cdots \subset G_{a_l+1}$ be the partial flag where G_{a_h+1} is the span of F_{a_h} and M for $h \geq 1$. The locus of linear spaces of dimension $x_i + l + 1$ that intersect F_{a_h} in a subspace of dimension at least $x_i + h$ and intersect M in a subspace of dimension at least $x_i + 1$ is equivalent to the locus of linear spaces that intersect the vector spaces G_{a_h+1} in subspaces of dimension at least $x_i + 1 + h$. Notice that the diagram D^c formed in Step 1 of the Algorithm 3.35 depicts the linear spaces

$$L_{n_1}, \dots, L_{n_{x_i}}, K_i, \mathrm{Span}(K_i, L_{n_{x_i+1}}), \cdots, \mathrm{Span}(K_i, Q^{r_{i+1}}_{d_{i+1}}), Q^{r_{i-1}}_{d_{i-1}}, \cdots, Q^{r_1}_{d_1}.$$

Hence, by the linear space bound Y must be contained in $V(D^c)$. By Proposition 3.49, D^c is an admissible symplectic diagram. Hence, $V(D^c)$ is an irreducible variety that has the same dimension as Y. We conclude that $Y = V(D^c)$. On the other hand, if D^a satisfies condition (A2) but fails condition (A1), then it fails it for the bracket with index $x_{\nu-1}+1$ and the index ν. By the kernel bound, any linear space that intersects $L_{n_{x_{\nu-1}+1}}$ in a subspace away from the kernel of Q restricted to $Q^{r_{\nu-1}}_{d_{\nu-1}}$ has to be contained in $L^{\perp}_{n_{x_{\nu-1}+1}}$. The latter vector space is depicted in a symplectic diagram by changing ν to $\nu-1$ and shifting $\}^{\nu-1}$ one unit to the right as in Step 2 of Algorithm 3.35. This argument applies as long as condition (A1) fails for the resulting sequence. We conclude that Y has to be contained in $V(D^c)$. Since Y and $V(D^c)$ are irreducible varieties of the same dimension, we conclude that $Y = V(D^c)$.

Now suppose that $x_{\nu-1} < s$ and $p(\}^{x_{\nu-1}+1}) - \pi(\nu) - 1 = y_{x_{\nu-1}+1} - \nu$ in D. Then, by our dimension count, replacing the linear space $L_{n_{x_{\nu-1}+1}}$ by a linear space $F_{u_{x_{\nu-1}+1}}$ corresponding to a bracket of the form

$$\cdots a \; a+1 \ldots \nu-1 \; \nu \; \nu+2 \ldots \nu+l-1 \; \nu+l \; \nu+l] \cdots \rightarrow$$

$$\cdots a] a+1 \ldots \nu-1 \; \nu \; \nu+2 \ldots \nu+l-1 \; \nu+l \; \nu+l \cdots$$

produces a locus Z that has the same dimension as Y. Replacing any other linear space results in a smaller dimensional locus. However, unless $F_{u_{x_{\nu-1}+1}} = \mathrm{Ker}(Q^{r_\nu}_{d_\nu}) \cap L_{n_{x_{\nu-1}+1}}$ not all linear spaces parameterized by Z can be in the flat limit. Observe that $W^{\perp}(t)$ intersects $L_{n_{x_{\nu-1}+1}} \cap \mathrm{Ker}(Q^{r_a}_{d_a})$ in a subspace of dimension at least $\pi(a)+1$ for every $W(t) \in V(D(t))$. By upper semi-continuity, the same has to hold of the flat limit at $t=0$. Hence, unless $a = \nu$, we obtain a smaller dimensional variety. We conclude that $Y \subset V(D^b)$. If D^b is admissible, then both varieties are irreducible of the same dimension and we conclude that $Y = V(D^b)$. If D^b is not admissible, then by Proposition 3.49, D^b satisfies condition (A2) but fails condition (A1). Furthermore, it fails condition (A1) only for the bracket $\cdots a \; \nu] \cdots$. By the kernel bound, the linear spaces parameterized of dimensions $k-a, k-a+1, \ldots, k-\nu+2$ contained in $Q^{r_{a+1}}_{d_{a+1}}, \ldots, Q^{r_{\nu-1}}_{d_{\nu-1}}$, respectively, are contained in $(L_{n_{x_{\nu-1}+1}} \cap \mathrm{Ker}(Q^{r_a}_{d_a}))^{\perp}$ in $Q^{r_{a+1}}_{d_{a+1}}, \ldots, Q^{r_{\nu-1}}_{d_{\nu-1}}$. Algorithm 3.36 replaces the linear spaces $Q^{r_{a+1}}_{d_{a+1}}, \ldots, Q^{r_{\nu-1}}_{d_{\nu-1}}$ with $(L_{n_{x_{\nu-1}+1}} \cap \mathrm{Ker}(Q^{r_a}_{d_a}))^{\perp}$ in $Q^{r_{a+1}}_{d_{a+1}}, \ldots, Q^{r_{\nu-1}}_{d_{\nu-1}}$, respectively. Hence, Y is contained in $V(D^c)$. Finally, if during the process two braces occupy the same position, then the resulting locus Z has strictly smaller dimension by our dimension counts so does not lead to a locus Z containing Y. Since in all other cases Y and $V(D^c)$ are irreducible varieties of the same dimension, we conclude that $Y = V(D^c)$. This completes the proof that the support of the flat limit of the specialization is contained in the union of $V(D_i)$, where D_i are the admissible symplectic diagrams associated to D by Algorithm 3.39.

Finally, there remains to check that each of the irreducible components occur with multiplicity one. This is an easy local calculation. The point here is that taking the option D^a at each stage of the algorithm leads to a Schubert variety. Similarly, taking the option D^b at all allowed places in the algorithm leads to a Schubert variety. The classes of these two Schubert varieties occur in the class of

$V(D)$ with multiplicity one. Therefore, by intersecting $V(D)$ with the dual of these Schubert varieties, we can tell the multiplicity of $V(D^a)$ and $V(D^b)$.

First, in each of the five cases we can assume that $\eta = 1$. Let U be the Zariski open set of our family of restriction varieties parameterizing linear spaces $W(t)$ such that $\dim(W(t) \cap Q_{d_\eta}^{r_\eta(t)}(t)) = k - \eta + 1$. Let Z be the family of symplectic restriction varieties obtained by applying the specialization to the admissible sequence $(L'_\bullet, Q'_\bullet)$ (represented by D') obtained from $(L_\bullet, Q_\bullet)$ by omitting the linear spaces $Q_{d_1}^{r_1}, \ldots, Q_{d_{\eta-1}}^{r_{\eta-1}}$. Then there exists a natural morphism $f : U \to Z$ sending $W(t)$ to $W(t) \cap Q_{d_\eta}^{r_\eta(t)}(t)$, which is smooth at the generic point of each of the irreducible components of the fiber of Z at $t = 0$. The fibers f over $W' \in Z$ is the linear spaces of dimension k that contain W' and satisfy the appropriate rank conditions with respect to the linear spaces $Q_{d_1}^{r_1}, \ldots, Q_{d_{\eta-1}}^{r_{\eta-1}}$. Notice that running Algorithm 3.39 on D' results in the same outcome as running in D and removing the braces with indices $i < \eta$. Hence, we can do the multiplicity calculation for the family Z. We may, therefore, assume that $\eta = 1$.

In all the cases, the argument is almost identical with very minor variations. We will give it in the hardest case, case (2)(i), and leave the minor modifications in the other cases to the reader. In case (2)(i), by a similar argument, we may further assume that $\kappa = 1$, $d_\kappa + r_\kappa = n - 2$, $x_\kappa = 0$ and $s \leq 1$. The most interesting case is when $s = 1$ and $2d_{k-s} \geq n$. Let y_1 be the minimal index l such that L_{n_1} is contained in $\mathrm{Ker}(Q_{d_l}^{r_l})$. We will check that the multiplicities are one by finding a cycle that intersects $V(D)$ in one point and exactly one of the limits in one point. If D^a is admissible, then consider the Schubert variety Σ defined with respect to a general isotropic flag with the following invariants

$$\lambda_i = n - d_i + 2 \text{ for } \kappa = 1 \leq i \leq l - 1, \; \lambda_i = n - d_i + 1 \text{ for } l \leq i \leq k - 1,$$
$$\text{and } \mu_k = n - n_1 + 1.$$

If D^a satisfies condition (A2) but not (A1), change the definition of λ_1 so that $\lambda_1 = n - d_1 + 2$. If D^a fails condition (A2), change the definition of Σ so that

$$\lambda_i = n - d_{i+1} + 1 \text{ for } 1 \leq i \leq l - 2, \; \lambda_i = n - d_{i+1} \text{ for } l - 1 \leq i \leq k - 2,$$
$$\text{and } \mu_{k-1} = n - n_1, \mu_k = n - r_\kappa + 1.$$

By Kleiman's Transversality Theorem [Kl1], it is immediate that both $\Sigma \cap V(D)$ and $\Sigma \cap V(D^a)$ consist of a single reduced point, whereas $\Sigma \cap V(D^b)$ is empty. Since Σ requires the k-plane to be contained in a linear space of dimension $n - n_1 + 1$ and $V(D^b)$ requires the linear space to intersect a linear space of dimension less than n_1, these conditions cannot be simultaneously satisfied for general choices of linear spaces. Hence, $\Sigma \cap V(D^b)$ is empty. On the other hand, the intersection $L_{n_1} \cap F_{\mu_k}^{\perp}$ consists of a one-dimensional vector space W_1 and $Q_{d_i}^{r_i} \cap F_{\lambda_i}$ consist of one-dimensional linear spaces contained in $W_1^{\perp}$ when $l \leq i \leq k - 1$ and two-dimensional linear spaces not contained in $W_1^{\perp}$ when $1 \leq i \leq l - 1$. Since any linear space contained in $V(D) \cap \Sigma$ or $V(D^a) \cap \Sigma$ must intersect all these linear spaces in one-dimensional subspaces, we conclude that the k-dimensional linear space satisfying conditions imposed by $V(D)$ and Σ or $V(D^a)$ and Σ are uniquely determined. It follows that the multiplicity of $V(D^a)$ is one.

Similarly, if $p(]^1) - \pi(2) - 1 = y_1 - 2$, then D^b is admissible. Let Ω be the Schubert variety defined with respect to a general isotropic flag with the following invariants:

$$\lambda_i = n - d_i + 1 \text{ for } 1 \leq i \leq k-1, \quad \mu_k = r_1.$$

By Kleiman's Transversality Theorem [Kl1], it is immediate that both $\Omega \cap V(D)$ and $\Omega \cap V(D^b)$ consist of a single reduced point, whereas $\Omega \cap V(D^a)$ is empty. The conditions imposed by Ω and $V(D^a)$ cannot be simultaneously satisfied, hence $\Omega \cap V(D^a)$ is empty. On the other hand, $F_{\lambda_i} \cap Q_{d_i}^{r_i}$ by construction are one-dimensional subspaces that need to be contained in any W contained in $\Omega \cap V(D)$ or $\Omega \cap V(D^b)$. These determine $(k-1)$-dimensional subspace W' of W. $L_{n_1}^b \cap F_{\mu_1}^\perp$ is also a one-dimensinonal subspace Λ that needs to be contained in W. Since $\Lambda \subset (W')^\perp$, this uniquely constructs $W \in \Omega \cap V(D^b)$. Similarly, $L_{n_1} \cap F_{\mu_1}^\perp$ is a y_1-dimensional linear space. However, the intersection of this linear space with $(W')^\perp$ is one-dimensional and must be contained in W. This uniquely constructs W in $V(D) \cap \Omega$. We leave the minor modifications necessary in the other cases to the reader (see [C3] for more details in the orthogonal case). This concludes the proof of the theorem. □

References

[Be] E. Bertini. *Introduzione alla geometria proiettiva degli iperspazi* Enrico Spoerri, Pisa, 1907, MR1548022, JFM 49.0484.08.

[BL] S. Billey and V. Lakshmibai. *Singular loci of Schubert varieties.* Birkhäuser, Boston, MA, 2000, MR1782635, Zbl 0959.14032.

[BH] A. Borel and A. Haefliger, La classe d'homologie fondamental d'un espace analytique, *Bull. Soc. Math. France*, **89** (1961), 461–513.

[Br] R. Bryant. *Rigidity and quasi-rigidity of extremal cycles in Hermitian symmetric spaces.* Princeton University Press Ann. Math. Studies AM–153, 2005.

[C1] I. Coskun, Rigid and non-smoothable Schubert classes, *J. Differential Geom.*, **87** no. 3 (2011), 493–514.

[C2] I. Coskun, Rigidity of Schubert classes in orthogonal Grassmannians, to appear Israel J. Math.

[C3] I. Coskun, Restriction varieties and geometric branching rules, *Adv. Math.*, **228** no. 4 (2011), 2441–2502.

[C4] I. Coskun, Symplectic restriction varieties and geometric branching rules, to appear in Clay Proceedings.

[C5] I. Coskun, Degenerations of surface scrolls and Gromov-Witten invariants of Grassmannians, *J. Algebraic Geom.*, **15** (2006), 223–284.

[D] Del Pezzo. Sulle superficie di ordine n immerse nello spazio di $n+1$ dimensioni, *Nap. rend.*, **XXIV**, 1885, 212–216, JFM 19.0841.01.

[EH] D. Eisenbud, and J. Harris. On varieties of minimal degree (a centennial account) Algebraic geometry, Bowdoin 1985 *Proc. Sympos. Pure Math.* **46** no. 1, Amer. Math. Soc., Providence, RI, 1987, MR0927946, Zbl 0646.14036.

[EH] David Eisenbud and Joe Harris. *The geometry of schemes*, volume 197 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, 2000.

[Ful1] W. Fulton. *Young tableaux*, volume 35 of *London Mathematical Society Student Texts.* Cambridge University Press, Cambridge, 1997.

[Ful2] W. Fulton. *Intersection theory*, volume 2 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics.* Springer-Verlag, Berlin, second edition, 1998.

[GH] P. Griffiths and J. Harris. *Principles of Algebraic Geometry.* Wiley Interscience, 1978.

[H] J. Harris. *Algebraic geometry*, volume 133 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, 1995.

[HM] J. Harris and I. Morrison. *Moduli of curves.* Springer-Verlag, 1998.

[Ha] R. Hartshorne. *Algebraic geometry.* Springer-Verlag, New York, 1977. Graduate Texts in Mathematics, No. 52.

[HRT] R. Hartshorne, E. Rees, and E. Thomas. Nonsmoothing of algebraic cycles on Grassmann varieties. *Bull. Amer. Math. Soc.* **80** no. 5 (1974), 847–851, MR0357402, Zbl 0289.14011.

[Hi] H. Hironaka. A note on algebraic geometry over ground rings. The invariance of Hilbert characteristic functions under the specialization process. *Illinois J. Math.* **2** (1958), 355–366, MR0102519, Zbl 0099.15701.

[Ho1] J. Hong. Rigidity of smooth Schubert varieties in Hermitian symmetric spaces. *Trans. Amer. Math. Soc.* **359** (2007), 2361–2381, MR2276624, Zbl 1126.14010.

[Ho2] J. Hong. Rigidity of singular Schubert varieties in $Gr(m, n)$. *J. Differential Geom.* **71** no. 1 (2005), 1–22, MR2191767, Zbl 1137.14311.

[IM] A. Iliev, and L. Manivel, The Chow ring of the Cayley plane, *Compositio Math.* **141** (2005), 146–160.

[Kl1] S. L. Kleiman. The transversality of a general translate. *Compositio Math.* **28**(1974), 287–297.

[Kl2] S. L. Kleiman. Problem 15: Rigorous foundation of Schubert's enumerative calculus. In *Mathematical developments arising from Hilbert problems (Proc. Sympos. Pure Math., Northern Illinois Univ., De Kalb, Ill., 1974)*, pages 445–482. Proc. Sympos. Pure Math., Vol. XXVIII. Amer. Math. Soc.,
Providence, R. I., 1976.

[KL] S. L. Kleiman and D. Laksov. Schubert calculus. *Amer. Math. Monthly* **79**(1972), 1061–1082.

[KT] A. Knutson and T. Tao. Puzzles and (equivariant) cohomology of Grassmannians. *Duke Math. J.* **119**(2003), 221–260.

[K] J. Kollár. *Rational curves on algebraic varieties*, volume 32 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics.* Springer-Verlag, Berlin, 1996.

[Ko] J. Kollár. Flatness criteria. *J. Algebra* **175** (1995), 715–727, MR1339664, Zbl 0853.14008.

[LS] V. Lakshmibai and C.S. Seshadri. Singular locus of a Schubert variety. *Bull. Amer. Math. Soc.* vol. **11** no. 2 (1984), 363–366, MR0752799, Zbl 0549.14016.

[NS] S.I. Nikolenko, and N.S. Semenov, Chow ring structure made simple, arxiv: 0606335v1.pdf

[R] C. Robles, Schur flexibility of cominuscule Schubert varieties, preprint.

[RT] C. Robles and D. The, Rigid Schubert varieties in compact Hermitian symmetric spaces, *Selecta Math.*, **18** no. 2 (2012), 717–777.

[S] I.R. Shafarevich, *Basic Algebraic Geometry I*, Springer-Verlag, 1994.

[V1] R. Vakil. A geometric Littlewood-Richardson rule. *to appear Ann. of Math.*

[V2] R. Vakil. Schubert induction. *to appear Ann.of Math.*

[W] M. Walters. Geometry and uniqueness of some extreme subvarieties in complex Grassmannians. *Ph. D. thesis,* University of Michigan, 1997.