THE QUANTUM COHOMOLOGY OF FLAG VARIETIES AND THE PERIODICITY OF THE LITTLEWOOD-RICHARDSON COEFFICIENTS

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Abstract. We give conditions on a curve class that guarantee the vanishing of the structure constants of the small quantum cohomology of partial flag varieties $F(k_1, \dots, k_r; n)$ for that class. We show that many of the structure constants of the quantum cohomology of flag varieties can be computed from the image of the evaluation morphism. In fact, we show that a certain class of these structure constants are equal to the ordinary intersection of Schubert cycles in a related flag variety. As a corollary to the main theorem in [C3], we obtain a Littlewood-Richardson rule for these invariants. Our study also reveals a remarkable periodicity property of the ordinary Littlewood-Richardson coefficients of partial flag varieties.

Contents

1. Introduction

1.1. **Motivating questions.** Let X be a homogeneous variety. A fundamental problem in algebraic geometry, combinatorics and representation theory is to describe the structure constants of the cohomology or, more generally, the quantum cohomology of X with respect to its Schubert basis. In this paper we study conditions that guarantee the vanishing and non-vanishing of the structure constants of the small quantum cohomology ring of partial flag varieties.

Notation 1.1. Let $0 < k_1 < k_2 < \cdots < k_r < n$ be a sequence of strictly increasing positive integers. Let $F(k_1, \dots, k_r; n)$ denote the partial flag variety parameterizing r-tuples

$$V_1 \subset V_2 \subset \cdots \subset V_r \subset V$$

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of linear subspaces of a fixed vector space V of dimension n, where V_i has dimension k_i. For notational convenience we set $k_0 = 0$ and $k_{r+1} = n$. The class of a curve C in $F(k_1, \dots, k_r; n)$ is determined by r non-negative integers $d_1, d_2, \dots, d_r$, where d_i is the Plücker degree of the projection of C to the Grassmannian $G(k_i, n)$. Given any finite increasing sequence of non-negative integers $a_1, \dots, a_r$, we denote the sequence by $a_\bullet$. We denote the number of terms equal to i by l_i and express the sequence also as $0^{l_0}, 1^{l_1}, \dots, a_r^{l_{a_r}}$.

Definition 1.2. *We call a pair of sequences $(k_\bullet, d_\bullet)$ of the same length a flag pair if $0 < k_1 < \cdots < k_r$ are strictly increasing positive integers and $d_1, \dots, d_r$ are non-negative integers.*

The first problem we address is the following.

Problem 1.3. Determine the flag pairs $(k_\bullet, d_\bullet)$ for which there exists a non-zero, three-pointed, genus-zero Gromov-Witten invariant of $F(k_\bullet; n)$ of degree $d_\bullet$ for some n. More generally, determine the triples $(k_\bullet, d_\bullet, n)$ for which there exists a non-zero, three-pointed, genus-zero Gromov-Witten invariant of $F(k_\bullet; n)$ of degree $d_\bullet$.

By Lemmas 3.3 and 3.5, if there exists a non-zero, three-pointed, genus-zero Gromov-Witten invariant of $F(k_\bullet, n_0)$ of degree $d_\bullet$ for some n_0, then there exists a non-zero, three-pointed genus-zero Gromov-Witten invariant of $F(k_\bullet; n)$ of degree $d_\bullet$ for every $n \geq \min(n_0, k_r + d_r)$. Hence, to study the first part of the problem we can take $n = k_r + d_r$. The second part of the problem is more subtle.

In the case of Grassmannians $G(k, n)$ there exists a non-zero, three-pointed Gromov-Witten invariant if and only if $d \leq \min(k, n-k)$ (see [BKT] or [Yo]). As a special case, our main vanishing theorem gives a similar vanishing criterion for two-step flag varieties.

Theorem 1.4. *If there exists a non-zero, three-pointed Gromov-Witten invariant of the two-step flag variety $F(k_1, k_2; n)$ of degree (d_1, d_2), then the following inequalities hold.*

1. $d_1 k_2 + d_2(n - k_1) \leq 2k_1(k_2 - k_1) + 2k_2(n - k_2)$.
2. *If $0 \leq d_1 \leq k_1$, then $d_1 \leq \min(n - k_1, d_2 + k_2 - k_1)$. If $k_1 < d_1$, then $d_1 \leq \min(n - k_1, 2k_1, d_2)$.*
3. *If $0 \leq d_2 \leq n - k_2$, then $d_2 \leq \min(k_2, d_1 + k_2 - k_1)$. If $n - k_2 < d_2$, then $d_2 \leq \min(k_2, 2(n - k_2), d_1)$.*

Note this set of inequalities is invariant under the transformation taking (k_1, k_2) to $(n - k_2, n - k_1)$ and interchanging d_1 and d_2. This transformation reflects the isomorphism between $F(k_1, k_2; n)$ and $F(n - k_2, n - k_1; n)$. We also remark that the inequalities in Theorem 1.4 are sharp. We will see below that if $n \geq k_2 + d_2$, then the inequalities of the theorem precisely determine the range where there exists non-zero invariants.

Example 1.5. As a first non-trivial example, Theorem 1.12 asserts that there are no non-zero, three-pointed, genus-zero invariants of degree $(0, 2)$ of $F(1, 2; 4)$. The dimension of the corresponding Kontsevich moduli space $\overline{\mathcal{M}}_{0,3}(F(1, 2; 4), (0, 2))$ is 11. Since the dimension of $F(1, 2; 4)$ is 5, there are many combinations of Schubert classes whose codimensions add up to 11. None the less, all the invariants vanish. This can be explained geometrically by the fact that the image of the evaluation morphism has dimension only 9. In fact, even the four-pointed invariants of $F(1, 2; 4)$ of degree $(0, 2)$ vanish.

Notation 1.6. Let $\overline{\mathcal{M}}_{0,3}(F(k_\bullet; n), d_\bullet)$ denote the Kontsevich moduli space of three-pointed, genus-zero stable maps of degree $d_\bullet$ to $F(k_\bullet; n)$. $\overline{\mathcal{M}}_{0,3}(F(k_\bullet; n), d_\bullet)$ is equipped with three evaluation morphisms to $F(k_\bullet; n)$. Denote by e the morphism

$$e = ev_1 \times ev_2 \times ev_3 : \overline{\mathcal{M}}_{0,3}(F(k_\bullet; n), d_\bullet) \to F(k_\bullet; n) \times F(k_\bullet; n) \times F(k_\bullet; n).$$

Let $I_{F(k_\bullet; n), d_\bullet}(\lambda, \mu, \nu)$ denote the Gromov-Witten invariant of $F(k_\bullet; n)$ of degree $d_\bullet$ associated to three Schubert classes $\sigma_\lambda, \sigma_\mu$ and σ_ν.

Once we know the range of non-zero invariants, we can concentrate on computing them. While there are many algorithms for computing the structure constants of the small quantum cohomology ring of partial flag varieties (see, for instance, [Ci], [Bu]), there are no known positive algorithms except in the case of Grassmannians ([C4]). In [BKT], Buch, Kresch and Tamvakis provide a beautiful analysis of the case of Grassmannians $G(k,n)$. When $d \leq k$, the evaluation morphism is birational onto its image. The image can be characterized as the closure Ω of the subvariety

$$\Omega^0 = \{ \ (X,Y,Z) \in G(k,n) \times G(k,n) \times G(k,n) \mid \ \dim(X \cap Y \cap Z) = k-d, \ \dim(\overline{XYZ}) = k+d \ \}.$$

The Poincaré dual of the class of Ω is expressed in terms of the Künneth components as

$$\sum_{|\lambda|+|\mu|+|\nu|=k(n-k)+dn} I_{G(k,n),d} \ (\lambda,\mu,\nu) \ \sigma_{\lambda^*} \otimes \sigma_{\mu^*} \otimes \sigma_{\nu^*},$$

where λ^* denotes the dual Schubert cycle to λ. Hence computing the small quantum cohomology of $G(k,n)$ in degree d reduces to computing the (ordinary) cohomology class of Ω. The variety Ω admits a rational map to the two-step flag variety $F(k-d,k+d;n)$ by sending (X,Y,Z) to $(X \cap Y \cap Z, \overline{XYZ})$. Buch, Kresch, Tamvakis show that the intersection of Ω with the pull-back of a Schubert cycle from each factor is equal to the intersection of three Schubert varieties in the two-step flag variety under the correspondence given by this rational map. Consequently, the structure constants of the small quantum cohomology are equal to certain structure constants of the ordinary cohomology of two-step flag varieties. Using this result, in [C4] we obtained a Littlewood-Richardson rule for the small quantum cohomology of Grassmannians.

The quest for a positive algorithm motivates the second problem that we address.

Problem 1.7. Determine the triples $(k_\bullet, d_\bullet, n)$ consisting of flag pairs $(k_\bullet, d_\bullet)$ and n for which the morphism

$$e : \overline{\mathcal{M}}_{0,3}(F(k_\bullet;n), d_\bullet) \to F(k_\bullet;n) \times F(k_\bullet;n) \times F(k_\bullet;n)$$

is birational onto its image. In case e is birational onto its image, provide a Littlewood-Richardson rule for computing the cohomology class of the image.

The following example, also due to Buch, Kresch and Tamvakis, shows that we cannot expect the evaluation morphism to be always birational onto its image.

Example 1.8 (The example of Buch, Kresch and Tamvakis ([BKT]))**.** The Gromov-Witten invariant

$$I_{F(1,2,3,4;5),(2,3,3,2)}(pt,pt,pt) = 2.$$

In particular, the map

$$e : \overline{\mathcal{M}}_{0,3}(F(1,2,3,4;5),(2,3,3,2)) \to F(1,2,3,4;5) \times F(1,2,3,4;5) \times F(1,2,3,4;5)$$

is not birational, but generically two-to-one. Consequently, the Gromov-Witten invariants cannot be computed by calculating the class of the image of e in general. One also needs the degree of e.

This example discouraged efforts to combinatorialize the small quantum cohomology of partial flag varieties in a fashion similar to Grassmannians. The purpose of this paper is to show that under mild assumptions on the triple $(k_\bullet, d_\bullet, n)$ the map e is birational onto its image. Our theorems on birationality and non-vanishing of Gromov-Witten invariants specialize to the following for the two-step flag varieties.

Theorem 1.9. *Let k_1, k_2, d_1, d_2 and n satisfy the inequalities in Theorem 1.4. If $d_1 > k_1$, suppose that $k_2 + d_2 \leq n$. Then there exists a non-zero, three-pointed Gromov-Witten invariant of $F(k_1,k_2;n)$ of degree (d_1,d_2). Furthermore, under the same hypotheses the evaluation morphism is birational onto its image.*

It is natural to ask for a positive, geometric-combinatorial rule for computing the class of the image of e in case e is birational onto its image. Unfortunately, this is a very difficult problem in general. In this paper we solve it under further assumptions on the triple $(k_\bullet, d_\bullet, n)$. We show that in a large subset of the cases when the evaluation morphism is birational onto its image, the invariants may be computed as the intersection of three Schubert varieties in other partial flag varieties. [C3] provides a Littlewood-Richardson rule for these invariants. More importantly, this correspondence leads to fascinating identities among the ordinary Littlewood-Richardson coefficients of flag varieties.

1.2. **The statement of results.** We now state the main results of this paper. Our results are best phrased in terms of the splitting types of the pull-backs of the tautological bundles to the domain curves. We adapt standard terminology from the theory of vector bundles to our case.

Definition 1.10. *Let $(k_\bullet, d_\bullet)$ be a flag pair. A $(k_\bullet, d_\bullet)$-admissible set of sequences is a set*

$$A_\bullet = \{(a_{1,j})_{j=1}^{k_1},\ (a_{2,j})_{j=1}^{k_2}, \dots, (a_{r,j})_{j=1}^{k_r}\}$$

of r sequences of non-negative integers of lengths $k_1, \dots, k_r$, respectively, such that

1. $0 \leq a_{i,j} \leq a_{i,j+1}$ *for every i and every $1 \leq j \leq k_i - 1$;*
2. $a_{i+1,j} \leq a_{i,j}$ *for every $1 \leq i \leq r-1$ and $j \leq k_i$;*
3. $\sum_{j=1}^{k_i} a_{i,j} = d_i$.

We denote by $l_{i,\alpha} = \#\{\ a_{i,j} \mid a_{i,j} = \alpha\ \}$ the number of integers in the sequence $a_{i,\bullet}$ that are equal to α.

Definition 1.11. *A $(k_\bullet, d_\bullet)$-admissible set of sequences $A_\bullet$ is balanced if $A_\bullet$ minimizes the function*

$$\sum_{i=1}^{r} \sum_{1 \leq l < m \leq k_i} (a_{i,m} - a_{i,l})$$

among the $(k_\bullet, d_\bullet)$-admissible sets of sequences.

Observe that given a flag pair $(k_\bullet, d_\bullet)$, there exists a unique, balanced $(k_\bullet, d_\bullet)$-admissible set of sequences (see Lemma 2.1). We denote this set of sequences by $B_\bullet(k_\bullet, d_\bullet)$.

We also observe that a flag variety $F(k_1, \dots, k_r; n)$ is isomorphic to the dual flag variety $F(n-k_r, n-k_{r-1}, \dots, n-k_1; n)$. Under this isomorphism the curve class $(d_1, \dots, d_r)$ is transformed to the curve class $(d_r, \dots, d_1)$. Consequently, if the Gromov-Witten invariants of $F(k_1, \dots, k_r; n)$ for degree $d_\bullet$ vanish, then the corresponding Gromov-Witten invariants for the dual flag variety must also vanish. We can now state our vanishing theorem for arbitrary partial flag varieties.

Theorem 1.12 (Vanishing). *Let $B_\bullet(k_\bullet, d_\bullet) = \{(a_{i,j})_{j=1}^{k_i} | i = 1, \dots, r\}$ be the set of balanced, admissible sequences associated to the flag pair $(k_\bullet, d_\bullet)$. For simplicity, set $\alpha_i = a_{i,k_i}$. Suppose that there exists a non-zero, three-pointed, genus-zero Gromov-Witten invariant of $F(k_\bullet; n)$ of degree $d_\bullet$. Then the following inequalities have to be satisfied*

1. $\sum_{i=1}^{r} d_i(k_{i+1} - k_{i-1}) \leq 2 \sum_{i=1}^{r} k_i(k_{i+1} - k_i)$;
2. $\alpha_r \leq 2$;
3. $\alpha_{i-1} \leq \alpha_i + 2, \quad$ *for every $2 \leq i \leq r$;*
4. $l_{i-1,\kappa} = l_{i,\kappa}$ *for all $0 \leq \kappa \leq \alpha_i - 2$. If, in addition, $\sum_{j=0}^{\alpha_{i-2}-\alpha_{i-1}} 2^j l_{i-2,\alpha_{i-1}+j} < l_{i-1,\alpha_{i-1}}$, then*
   - $\alpha_{i-1} \leq \alpha_i + 1$;
   - $2l_{i-1,\alpha_i+1} \leq \max(0, l_{i,\alpha_i} - l_{i-1,\alpha_i})$;
   - $2l_{i-1,\alpha_i} \leq \max(0, 2l_{i,\alpha_i} + l_{i,\alpha_i-1} - l_{i-1,\alpha_i-1})$.

Furthermore, the same inequalities must hold for the dual flag variety and the dual flag pair.

For most of the triples $(k_\bullet, d_\bullet, n)$ determined by Theorem 1.12 the evaluation morphism is actually birational onto its image.

Theorem 1.13. *Let $(k_\bullet, d_\bullet)$ be a flag pair whose associated set of balanced, admissible sequences $B_\bullet(k_\bullet, d_\bullet) = \{(a_{i,j})_{j=1}^{k_i} | i = 1, \ldots, r\}$ satisfy the following inequalities:*

(1) $\alpha_r \leq 1$;
(2) $\alpha_{i-1} \leq \alpha_i + 1$, *for every* $2 \leq i \leq r$;
(3) *For every* $2 \leq i \leq r$,
   - $2l_{i-1,\alpha_i+1} \leq \max(0, l_{i,\alpha_i} - l_{i-1,\alpha_i})$ *and*
   - $2l_{i-1,\alpha_i} \leq \max(0, 2l_{i,\alpha_i} + l_{i,\alpha_i-1} - l_{i-1,\alpha_i-1})$

Let $n \geq k_r + d_r$. Then the evaluation morphism

$$e : \overline{\mathcal{M}}_{0,3}(F(k_\bullet; n), d_\bullet) \to F(k_\bullet; n) \times F(k_\bullet; n) \times F(k_\bullet; n)$$

is birational onto its image. In particular, there exists a non-zero, three-pointed, genus-zero Gromov-Witten invariant of $F(k_\bullet; n)$ of degree $d_\bullet$.

We call a Gromov-Witten invariant of a flag variety $F(k_\bullet; n)$ for the class $d_\bullet$ classical if the corresponding evaluation morphism e is birational onto its image. Let $\pi_i : F(k_\bullet; n) \times F(k_\bullet; n) \times F(k_\bullet; n) \to F(k_\bullet; n)$ denote the i-th projection morphism. Let $\Sigma_\lambda, \Sigma_\mu$ and Σ_ν be general Schubert varieties whose codimensions sum to the dimension of the image of e. We call a Gromov-Witten invariant of a flag variety very classical if it is classical and there exists a rational map ϕ from the image of e to a (possibly different) flag variety $F(k'_\bullet, n)$ such that ϕ gives a bijection between the intersection $\pi_1^{-1}(\Sigma_\lambda) \cap \pi_2^{-1}(\Sigma_\mu) \cap \pi_3^{-1}(\Sigma_\nu) \cap Im(e)$ and the intersection of three Schubert varieties in $F(k'_\bullet, n)$. As in [BKT], one can conclude that a large subclass of Gromov-Witten invariants are very classical.

Corollary 1.14 (Very classical Gromov-Witten invariants). *Let $n \geq k_r + d_r$.*

(1) *Let $0 \leq \alpha \leq k_1$. Then the invariants of $F(k_\bullet; n)$ of degree $d_i = k_i - \alpha$ are very classical.*
(2) *Let a be a positive integer. Let $k_i = 2^{i-1}a$ for $1 \leq i \leq r$. Set $d_i = (r - i + 1)k_i$. Then the invariants of $F(k_\bullet; n)$ of degree $d_\bullet$ are very classical.*

An interesting consequence of Corollary 1.14 is a beautiful periodicity property of the ordinary Littlewood-Richardson coefficients of flag varieties. We will give a geometric interpretation of the periodicity property in §6. Here we give the combinatorial description. The Schubert varieties in $F(k_\bullet; n)$ can be indexed by sequences of integers from 1 to $r + 1$, where $k_i - k_{i-1}$ of the digits are i. Let $0 \leq a \leq k_1$ be a non-negative integer. Given a Schubert cycle $\sigma_{\lambda'}$ with $\lambda_1 \leq n - a$ in $G(a, 2n - a)$ define a periodicity map $\pi_{\lambda'}$ from Schubert cycles in $F(k_1 - a, k_2 - a, \ldots, k_r - a; n - a)$ to Schubert cycles in $F(a, 2k_1 - a, 2k_2 - a, \ldots, 2k_r - a; 2n - a)$ as follows. First, replace the sequence $\lambda = \lambda_1, \ldots, \lambda_{n-a}$ by the sequence whose digits are one more than the digits of λ; i.e. $\lambda[1] = \lambda_1 + 1, \ldots, \lambda_{n-a} + 1$. Next replace the sequence corresponding to λ' by $\lambda'[2 \mapsto r + 2]_{|n-a+1}$ obtained by replacing every occurrence of 2 in the sequence corresponding to λ' by $r + 2$ and taking the last n digits. Form the sequence $q(\lambda, \lambda')$ obtained by concatenating the two sequences $\lambda[1]$ and $\lambda'[2 \mapsto r + 2]_{|n-a+1}$. Finally, replace the last $k_r - a$ digits of $q(\lambda, \lambda')$ that are equal to $r + 2$ by the sequence

$$r + 1, \ldots, r + 1, r, \ldots, r, \ldots, 3, \ldots, 3, 2, \ldots, 2,$$

where the last $k_1 - a$ digits are 2, the next to last $k_2 - k_1$ digits are 3 and $k_{i+1} - k_i$ of the digits are $i + 1$ for $2 \leq i \leq r$. Let $\pi_{\lambda'}(\lambda)$ denote the resulting sequence.

For example, let $\lambda = 3, 1, 2, 1, 2, 3$ be the sequence corresponding to a Schubert cycle in $F(2, 4; 6)$. Let

$$2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 1, 2$$

be the sequence in $F(1; 13)$ corresponding to λ'. Then

$$\lambda[1] = 4, 2, 3, 2, 3, 4.$$

$$\lambda'[2 \mapsto 4]_{|7} = 4, 4, 4, 4, 4, 1, 4$$

$$q(\lambda, \lambda') = 4, 2, 3, 2, 3, 4, 4, 4, 4, 4, 4, 1, 4.$$

Finally, the sequence

$$\pi_{\lambda'}(\lambda) = 4,2,3,2,3,4,4,4,3,3,2,1,2$$

is a sequence corresponding to a Schubert cycle in $F(1,5,9;13)$.

Theorem 1.15 (Periodicity of structure coefficients)**.** *Let $0 \leq a \leq k_1$ be a non-negative integer. Let $\sigma_{\lambda'}, \sigma_{\mu'}, \sigma_{\nu'}$ in $G(a, 2n-a)$ be three Schubert cycles whose codimensions sum to $a(2n-2a)$. Suppose in the partitions defining λ', μ' and ν', the first part is at most $n-a$. Let $\sigma_\lambda, \sigma_\mu$ and σ_ν be three Schubert cycles in $F(k_1 - a, \dots, k_r - a; n-a)$ whose codimensions sum to the dimension of the flag variety. Then*

$$c_{\pi_{\lambda'}(\lambda), \pi_{\mu'}(\mu), \pi_{\nu'}(\nu)} = c_{\lambda,\mu,\nu} \cdot c_{\lambda',\mu',\nu'},$$

where $c_{i,j,k}$ denotes the product of the three Schubert cycles $\sigma_i \cdot \sigma_j \cdot \sigma_k$ in their respective flag varieties.

Given a sequence λ corresponding to a Schubert variety in $F(k_1, \dots, k_r; n)$, denote by λt the sequence obtained by adding the trivial tail

$$r+1, \dots, r+1, r, \dots, r, \dots, 1, \dots, 1,$$

where $k_i - k_{i-1}$ of the digits are i. Setting $a = 0$ in Theorem 1.15 we obtain the following corollary.

Corollary 1.16. *Let $\sigma_\lambda, \sigma_\mu$ and σ_ν be three Schubert cycles in $F(k_1, \dots, k_r; n)$ whose codimensions sum to the dimension of the flag variety. Let $\sigma_{\lambda t}, \sigma_{\mu t}$ and $\sigma_{\nu t}$ be the corresponding cycles in $F(2k_1, \dots, 2k_r; 2n)$. Then*

$$c_{\lambda,\mu,\nu} = c_{\lambda t, \mu t, \nu t}.$$

Example 1.17. The first non-trivial example is the equality of $\sigma_1 \cdot \sigma_1 \cdot \sigma_2$ in $G(2,4)$ and $\sigma_{3,2} \cdot \sigma_{3,2} \cdot \sigma_{4,2}$ in $G(4,8)$. This equality holds because both of these products are equal to the Gromov-Witten invariant

$$I_{F(2,4;8),(2,4)}(\sigma_{2,1,2,1,3,3,3,3}, \sigma_{2,1,2,1,3,3,3,3}, \sigma_{1,2,2,1,3,3,3,3}).$$

Remark 1.18. Theorem 1.12 can be generalized from three-pointed invariants to m-pointed invariants by replacing the occurrences of three in the estimates by m. This was already noted for Grassmannians in [C2], where it is proved that assuming $n \geq k + d$, then all m-pointed Gromov-Witten invariants vanish unless

$$d + \frac{m-3}{d} \leq (m-2)k.$$

We will leave it to the reader to make the necessary modifications to extend Theorem 1.12 to m-pointed invariants.

Remark 1.19. One can ask given a flag pair $(k_\bullet, d_\bullet)$ satisfying the conclusions of Theorem 1.12 for which triples of Schubert cycles is the invariant $I_{F(k_\bullet;n),d_\bullet}(\sigma_\lambda, \sigma_\mu, \sigma_\nu)$ non-zero. Even for the ordinary cohomology of flag varieties this seems to be a difficult question. While it is possible to give many conditions that guarantee that the invariants vanish and many conditions that guarantee that they do not, the author is unaware of a satisfactory complete characterization of the null-locus of the invariants.

Remark 1.20. It is tempting to ask for other formulae similar to the ones in Theorem 1.15. Given a transformation that takes three Schubert cycles in a flag variety whose intersection is in the top cohomology to three other such Schubert cycles in a different flag variety, we can ask for a relation between the intersection numbers of these cycles. Especially when the procedure involves the embedding of a product of flag varieties into another flag variety, one can expect formulae similar to those in Theorem 1.15 (see Bergeron and Sottile [BS] for related work). Richmond in his thesis has worked out similar formulae for the projection of flag varieties to sub-flag varieties (see [Ri]).

Remark 1.21. It would be interesting to extend the results of this paper to other varieties (See the work of Buch, Kresch, Tamvakis for isotropic Grassmannians and Chaput, Manivel, Perrin [CMP] for (co)-miniscule homogeneous spaces). Similar vanishing and non-vanishing theorems for other homogeneous varieties such as isotropic flag varieties can be obtained by modifying the proofs in this paper. It is also very interesting to study the vanishing and non-vanishing of genus-zero Gromov-Witten invariants for arbitrary rationally connected varieties. It follows from the celebrated work of Kollár, Miyaoka and Mori [KMM] that if X is a uniruled variety, then there exists a non-zero, genus-zero Gromov-Witten invariant (not necessarily three-pointed) where one of the insertions is a point class. Kollár asks if X is a rationally connected variety, whether there exists a non-zero, genus-zero Gromov-Witten invariant where two of the insertions are point classes. A positive answer to this question would have important applications—for instance, it would imply that rational connectivity is a symplectic invariant.

We now describe the organization of the paper. In Section 2 we will provide the necessary background for the cohomology and quantum cohomology of flag varieties and rational scrolls. In Section 3 we will prove a few reduction lemmas. In Section 4 we will prove Theorem 1.12. Section 5 will be devoted to the proof of Theorem 1.13. In Section 6 we will discuss the periodicity property of the structure constants of the cohomology of flag varieties.

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2. Preliminaries

In this section we collect the basic facts that we need about the cohomology of flag varieties, the quantum cohomology of flag varieties and the geometry of rational scrolls.

2.1. **The cohomology of flag varieties.** Let $k_\bullet = 0 < k_1 < \cdots < k_r < n$ be r strictly increasing integers. Recall our convention that $k_0 = 0$ and $k_{r+1} = n$. The homology of the flag variety $F(k_\bullet; n)$ has a $\mathbb{Z}$-basis generated by the classes of Schubert varieties. There are many different notations for Schubert varieties. Here we recall the three commonly used ones for the convenience of the reader.

Schubert classes in $F(k_\bullet; n)$ are parametrized by sequences λ of length n consisting of the integers $1, 2, \ldots, r+1$, where $k_i - k_{i-1}$ of the digits are i. Denote by λ_s the s-th place in the sequence λ. Given a fixed complete flag $F_\bullet$, the Schubert variety $\Sigma_\lambda(F_\bullet)$ is defined by

$$\Sigma_\lambda(F_\bullet) = \{(V_1, \ldots, V_r) \mid \dim(V_i \cap F_j) \geq \#\{s \mid \lambda_s \leq i \text{ for } s \leq j\}.$$

Alternatively, one can parametrize the same data by a pair of sequences λ, δ of lengths k_r, where $\lambda_1 \geq \cdots \geq \lambda_{k_r}$ is the sequence such that the digit in the $(n - k_r + i - \lambda_i)$-th position is less than $r+1$. The digit δ_i records the digit in the $(n - k_r + i - \lambda_i)$-th place. This notation is often more economical than the first notation. In this paper, to keep the notation to a minimum, we will use this notation only for Grassmannians. For Grassmannians $\delta_i = 1$ for all i, so the sequence δ is commonly omitted. This notation is the usual notation for Grassmannians in the literature.

It is also common to parametrize Schubert varieties in $F(k_1, \ldots, k_r; n)$ by permutations ω in $\mathfrak{S}_n$ satisfying $\omega(i) < \omega(i+1)$ unless $i \in \{k_1, \ldots, k_r\}$. We include this notation only for the reader's convenience. In the future we will avoid using it. The correspondence between these parametrizations is straightforward. Given a permutation if $k_i < j \leq k_{i+1}$ place the digit $i+1$ in the spot $\omega(j)$.

For example, the sequence $1, 3, 3, 2, 3, 3, 1, 3, 2$ in $F(2, 4; 9)$ corresponds to $\sigma_{5,3,1,0}^{1,2,1,2}$ in the sequence-pair notation and to the permutation $(2, 7, 5)(3, 4, 9, 8, 6)$ in $\mathfrak{S}_9$

2.2. **The quantum cohomology of flag varieties.** Let X be a smooth, projective variety. Let β denote the homology class of a curve in X. The Kontsevich moduli space $\overline{\mathcal{M}}_{0,m}(X,\beta)$ of genus-zero stable maps paramterizes isomorphism classes of maps

$$f : (C, p_1, \dots, p_m) \to X$$

such that

(1) C is a connected, reduced, at-worst-nodal curve of arithmetic genus zero,
(2) $p_1, \dots, p_m$ are smooth distinct points on C,
(3) $f_*[C] = \beta$ and f is stable (i.e., if f is constant on any irreducible component C_j of C, then the total number of nodes and marked points p_i on C_j is at least three).

The Kontsevich space $\overline{\mathcal{M}}_{0,m}(X,\beta)$ comes equipped with m evaluation morphisms

$$ev_i : \overline{\mathcal{M}}_{0,m}(X,\beta) \to X$$

given by $ev_i((C, p_1, \dots, p_m, f)) = f(p_i)$. Given m cohomology classes of pure dimension $\gamma_1, \dots, \gamma_m$ on X, one defines the Gromov-Witten invariant

$$I_{X,\beta}(\gamma_1, \dots, \gamma_m) = \int_{[\overline{\mathcal{M}}_{0,m}(X,\beta)]^{\text{virt}}} ev_1^*(\gamma_1) \cup \cdots \cup ev_m^*(\gamma_m).$$

When X is a homogeneous variety G/P, the Kontsevich moduli space $\overline{\mathcal{M}}_{0,m}(X,\beta)$ is a smooth, irreducible, Deligne-Mumford stack of the expected dimension

$$c_1(X) \cdot \beta + \dim(X) + m - 3.$$

Furthermore, the Gromov-Witten invariants are enumerative. By Lemma 14 of [FP] they are equal to the number of maps f from an m-pointed $\mathbb{P}^1$ to X such that $f(p_i) \in \Gamma_i$ where Γ_i is a general representative of the Poincaré dual of γ_i. When $m = 3$ and the γ_i are the Schubert basis, the three-pointed Gromov-Witten invariants form the structure constants of the small quantum cohomology ring of X.

If we specialize this discussion to $X = F(k_1, \dots, k_r; n)$ and $\beta = (d_1, \dots, d_r)$, we see that the Kontsevich moduli space $\overline{\mathcal{M}}_{0,3}(F(k_\bullet; n), d_\bullet)$ is irreducible of dimension

$$\sum_{i=1}^{r} k_i(k_{i+1} - k_i) + \sum_{i=1}^{r} d_i(k_{i+1} - k_{i-1}).$$

The structure constants of the small quantum cohomology count irreducible rational curves that intersect general Schubert varieties. We would like to formalize the numerics of the geometry of rational curves in partial flag varieties. Let $f : \mathbb{P}^1 \to F(k_1, \dots, k_r; n)$ be a morphism such that $f_*[\mathbb{P}^1] = (d_1, \dots, d_r)$. The partial flag variety $F(k_1, \dots, k_r; n)$ comes equipped with r tautological bundles $S_1 \subset S_2 \cdots \subset S_r$. By Grothendieck's theorem, the pull-back of S_i by f is (non-canonically) isomophic to a direct sum of line bundles $f^*S_i \cong \mathcal{O}_{\mathbb{P}^1}(-a_{i,1}) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(-a_{i,k_i})$. The minus signs are included so that the integers we consider are always non-negative.

Suppose that the integers $a_{i,j}$ are ordered in increasing order. Note that the total degree of f^*S_i is $-d_i$. Moreover, $a_{i+1,j} \leq a_{i,j}$ since f^*S_{i+1} is a sub-bundle of f^*S_i. Consequently, the set of sequences $(a_{i,j})$ form an admissible set of sequences for the flag pair $(k_\bullet, d_\bullet)$ in the sense of Definition 1.10. Among admissible sets of sequences there is a unique balanced one in the sense of Definition 1.11. This balanced admissible set of sequences will play an essential role.

Lemma 2.1. *The set of $(k_\bullet, d_\bullet)$-admissible sets of sequences contains a unique balanced member.*

Proof. It is easy to give an algorithm for constructing the balanced member among the $(k_\bullet, d_\bullet)$-admissible sets of sequences. Let $d_1 = k_1 q_1 + r_1$, where $0 \leq r_1 < k_1$ is the remainder. Set

$$a_{1,1} = \cdots = a_{1,k_1 - r_1} = q_1, \quad a_{1,k_1 - r_1 + 1} = \cdots = a_{1,k_1} = q_1 + 1.$$

Let $d_2 = k_2q_2 + r_2$, where $0 \leq r_2 < k_2$. If $q_2 < q_1$ or if $q_2 = q_1$ and $k_1 - r_1 \leq k_2 - r_2$, then set

$$a_{2,1} = \cdots = a_{2,k_2-r_2} = q_2, \quad a_{2,k_2-r_2+1} = \cdots = a_{2,k_2} = q_2 + 1.$$

If $q_2 > q_1$ or if $q_1 = q_2$ and $k_1 - r_1 > k_2 - r_2$, let $d_2 - d_1 = (k_2 - k_1)q_2' + r_2'$, where $0 \leq r_2' \leq k_2 - k_1$. Set

$$a_{2,1} = \cdots = a_{2,k_1-r_1} = q_1, \quad a_{2,k_1-r_1+1} = \cdots = a_{2,k_1} = q_1 + 1,$$

$$a_{2,k_1+1} = \cdots = a_{2,k_2-r_2'} = q_2', \quad a_{2,k_2-r_2'+1} = \cdots = a_{2,k_2} = q_2' + 1.$$

Suppose we have constructed the sequences up to $a_{i-1,\bullet}$. We can inductively construct $a_{i,\bullet}$. Let $d_i = k_iq_i + r_i$. If all $a_{i-1,j} > q_i$, set $a_{i,j} = q_i$ for $j \leq k_i - r_i$ and $a_{i,j} = q_i + 1$ for $k_i - r_i < j \leq k_i$. Otherwise, let a_{i-1,j_0} be the integer with the largest index in $a_{i-1,\bullet}$ which is less than or equal to q_i. Set $a_{i,j} = a_{i-1,j}$ for $j \leq j_0$. Let $d_i - \sum_{j=1}^{j_0} a_{i-1,j} = q_i'(k_i - j_0) + r_i'$. If all $a_{i-1,j} > q_i'$ for $j > j_0$, then set $a_{i,j} = q_i'$ for $j_0 < j \leq k_i - r_i'$ and $a_{i,j} = q_i' + 1$ for $j > k_i - r_i'$. Otherwise, let a_{i-1,j_1} be the integer with largest index which is less than or equal to q_i'. Set $a_{i,j} = a_{i-1,j}$ for $j \leq j_1$. Set $d_i - \sum_{j=1}^{j_1} a_{i-1,j} = q_i''(k_i - j_1) + r_i''$ and repeat the process replacing j_0, q_i' and r_i' with j_1, q_i'' and r_i'', respectively. It is clear that the sequence thus constructed is the unique balanced $(k_\bullet, d_\bullet)$-admissible set of sequences. □

Example 2.2. For instance, the balanced set of sequences for $k_\bullet = 2, 4, 6, 8$ and $d_\bullet = 1, 2, 7, 12$ is

$$(0,1), (0,0,1,1), (0,0,1,1,2,3), (0,0,1,1,2,2,3,3).$$

The importance of balanced, admissible set of sequences stems from the fact that the pull-back of the tautological bundles to a rational curve in class $d_\bullet$ contributing to a Gromov-Witten invariant for the flag-variety leads to a balanced, admissible set of sequences.

Proposition 2.3. *Let $F_\bullet^1, \dots, F_\bullet^m$ be m general flags. Let $\Sigma_{\lambda_1}(F_\bullet^1), \cdots, \Sigma_{\lambda_m}(F_\bullet^m)$ be m Schubert varieties in $F(k_1, \dots, k_r; n)$ whose codimensions sum to*

$$\sum_{i=1}^{r} k_i(k_{i+1} - k_i) + \sum_{i=1}^{r} d_i(k_{i+1} - k_{i-1}) + m - 3.$$

Let $f : (C, p_1, \dots, p_m) \to F(k_1, \dots, k_r; n)$ be a stable map of degree $d_\bullet = (d_1, \dots, d_r)$ from an m-pointed, connected curve of arithmetic genus zero such that $f(p_i) \in \Sigma_{\lambda_i}(F_\bullet^i)$. Then $C \cong \mathbb{P}^1$ and the set of sequences determined by the isomorphism classes of the duals of the pull-backs of the tautological bundles is the unique balanced admissible set of sequences for the flag pair $(k_\bullet, d_\bullet)$.

Proof. By Lemma 14 of [FP], the Gromov-Witten invariant is equal to

$$\#(ev_1^{-1}(\Gamma_1) \cap \cdots \cap ev_m^{-1}(\Gamma_m)).$$

Furthermore, this intersection is reduced and occurs in $\mathcal{M}_{0,m}(F(k_\bullet; n), d_\bullet)$. By [KP], $\overline{\mathcal{M}}_{0,m}(F(k_\bullet; n), d_\bullet)$ is irreducible. The pull-back of the tautological bundles to the domain $\mathbb{P}^1$ by the stable map leads to a sequence of vector bundles f^*S_i, where f^*S_{i-1} is a subbundle of f^*S_i. Consequently, the sequence of integers determining the isomorphism classes of these bundles form an admissible sequence. By the upper-semi-continuity of cohomology, the locus of stable maps in $\mathcal{M}_{0,m}(F(k_\bullet; n), d_\bullet)$ for which the pull-back of the tautological bundles is not balanced forms a proper subvariety. The locus where the pull-back of the tautological bundles are balanced is Zariski open and non-empty. By Kleiman's Transversality theorem it follows that if the flags $F_\bullet^1, \dots, F_\bullet^m$ are general, then the intersection of $ev_1^{-1}(\Gamma_1) \cap \cdots \cap ev_m^{-1}(\Gamma_m)$ with the locus of maps in $\mathcal{M}_{0,m}(F(k_\bullet; n), d_\bullet)$ for which the pull-back of the tautological bundles is not balanced is empty by dimension considerations. Since the map f in the proposition corresponds to a point in this intersection, the proposition follows. □

2.3. **Rational scrolls.** In this subsection we recall the basic facts about rational scrolls. The reader can consult [C1], [H] or [EH] for more details.

The scrolls $S_{a_1,\cdots,a_k}$. Let $a_1 \leq \cdots \leq a_k$ be a sequence of non-negative integers not all equal to zero. We will reserve the letter d for the sum $d = \sum_{i=1}^k a_i$. We denote the k-dimensional scroll of type $a_1, \ldots, a_k$ in $\mathbb{P}^{r+k-1}$ by $S_{a_1,\cdots,a_k}$. We allow some a_i to be zero. In that case we obtain cones over scrolls of smaller dimension. We say a scroll is *balanced* if $|a_i - a_j| \leq 1$ for all i, j. We say a scroll is *perfectly balanced* if $a_i = a_j$ for every i and j.

Construction 2.4. To construct $S_{a_1,\cdots,a_k}$ fix rational normal curves of degree a_i in general linear spaces $\mathbb{P}^{a_i}$. Choose an isomorphism between each of the rational curves with an abstract $\mathbb{P}^1$. The scroll $S_{a_1,\cdots,a_k}$ is the union of the $(k-1)$-planes spanned by the points corresponding under the isomorphisms.

Construction 2.5. One other construction will be useful. A balanced scroll of dimension k and degree $d \leq k$ can be constructed from three of its fibers f_1, f_2, f_3. To construct $S_{0,\ldots,0,1,\ldots,1} \in \mathbb{P}^{d+k-1}$ take three general linear spaces $\mathbb{P}^{k-1}$ (which will serve as f_1, f_2 and f_3) intersecting in a fixed linear space $v = \mathbb{P}^{k-d-1}$. Such a scroll is a cone over a perfectly balanced scroll of dimension d with vertex v. Intersect the fibers with a $\mathbb{P}^{2d-1}$ complementary to v. The scroll has to intersect this $\mathbb{P}^{2d-1}$ in a perfectly balanced scroll of dimension d. Constructing this scroll and taking its join with v constructs the scroll of degree d and dimension k. In the perfectly balanced scroll in $\mathbb{P}^{2d-1}$ there has to be a line through each point p of the fiber f_1 that intersects the other two fibers f_2 and f_3. This line is uniquely determined as $\overline{pf_2} \cap \overline{pf_3}$. We thus uniquely construct the scroll.

Abstractly a scroll is the projectivization of a vector bundle of rank k on $\mathbb{P}^1$. Hence, we can express the variety as $X = \mathbb{P}E = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(-a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(-a_k))$. If $\pi : X \to \mathbb{P}^1$ is the projection morphism, then the Chow ring of X is generated by the pull-back of the point class F from $\mathbb{P}^1$ and the class $H = \mathcal{O}_{\mathbb{P}E}(1)$ which restricts to the hyperplane class on every fiber of π. The following proposition elucidates the relation between scrolls and projectivization of vector bundles over $\mathbb{P}^1$ (see [EH]).

Proposition 2.6. *The scroll $S_{a_1,\cdots,a_k}$ is the image of $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(-a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(-a_k))$ under the linear series $|H|$.*

More generally, we will consider projections of $S_{a_1,\cdots,a_k}$ along centers disjoint from the scroll. Abusing notation we will denote these projections also by $S_{a_1,\cdots,a_k}$. Note that any rational scroll in projective space is a projection of some $S_{a_1,\cdots,a_k}$.

Subscrolls of $S_{a_1,\cdots,a_k}$. A *subscroll* of $S_{a_1,\cdots,a_k}$ of dimension s is a scroll $S_{b_1,\cdots,b_s} \subset S_{a_1,\cdots,a_k}$ dominating the base of $S_{a_1,\cdots,a_k}$. A subscroll of dimension s has class $H^{k-s} + mH^{k-s-1}F$ for some integer m. The dimension of the space of subscrolls of $S_{a_1,\cdots,a_k}$ may be computed using Riemann-Roch. A one-dimensional subscroll corresponds to a section of the vector bundle. Hence, the dimension of the space of curves of degree b in the class $H^{k-1} + (b-d)H^{k-2}F$ is given by

$$\sum_{i=1}^{k} (\max(-1, b - a_i) + 1) \quad - 1.$$

The dimension of the space of subscrolls of $S_{a_1,\cdots,a_k}$ of type $S_{b_1,\ldots,b_s}$ (assuming that $b_j \geq a_j$ and $s < k$) can now be computed inductively.

Lemma 2.7. *Let $b_j \geq a_j$ and $s < k$ be non-negative integers. The dimension of the space of subscrolls of type $S_{b_1,\ldots,b_s}$ contained in $S_{a_1,\ldots,a_k}$ is given by*

$$\sum_{j=1}^{s} \sum_{i=1}^{k} (\max(-1, b_j - a_i) + 1) - \sum_{j=1}^{s} \sum_{h=1}^{s} (\max(-1, b_j - b_h) + 1). \tag{1}$$

In particular, if $a_j < a_{j+1}$, the scroll $S_{a_1,\ldots,a_k}$ has a unique subscroll of dimension j and degree $\sum_{i=1}^{j} a_i$.

Remark 2.8 (Parallel transport in scrolls)**.** A perfectly balanced scroll of dimension n is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^{n-1}$. Hence, a point determines a unique section of $\mathbb{P}^1$ whose projection to $\mathbb{P}^{n-1}$ is constant. We refer to translating a point along this section as parallel transport. In case the scroll is embedded as $S_{1,\dots,1}$, Construction 2.5 shows how to determine this section given the point and two auxiliary fibers. A similar construction also works for balanced scrolls. A linear space containing the fiber of the minimal subscroll of a balanced scroll can also be uniquely translated to other fibers as linear spaces of the same dimension and containing the corresponding fiber of the minimal scroll.

Remark 2.9 (Balanced subscrolls of balanced scrolls)**.** Let $S_1 = S_{a+1,\dots,a+1}$ be a perfectly balanced subscroll of dimension n_1 of a scroll $S_2 = S_{a,\dots,a}$ of dimension n_2. There exists a unique perfectly balanced subscroll of S_2 of dimension $\min(2n_1, n_2)$ and degree $\min(2n_1, n_2)a$ containing S_1. We may assume that the dimension of S_1 is less than $n_2/2$. Otherwise S_2 is the scroll we want. Take two fibers of S_1. Parallel transporting these fibers to the same fiber and take their span. Then let S_3 be the subscroll of S_2 of dimension $2n_1$ and degree $2n_1$ determined by parallel transporting this span. We claim that S_3 contains S_1 and is the desired scroll. Re-embed S_2 in $\mathbb{P}^{2n_2-1}$ as $S_{1,\dots,1}$. S_1 is re-embedded as $S_{2,\dots,2}$. Since S_1 is covered by conics, it suffices to check the containment when S_1 is a conic. Fix two points on a conic and consider the quadric surface generated by the procedure. We claim that the quadric surface contains the conic. Otherwise, taking a third point on the conic and applying the parallel transport construction would generate $S_{1,1,1}$ containing the conic. ($S_{1,1,1}$ must contain the conic because its span contains the plane of the conic and since S_2 is minimal the intersection of this span with the span of $S_{1,1,1}$ has to equal $S_{1,1,1}$.) The hyperplane section of $S_{1,1,1}$ spanned by the conic and the quadric surface has to consist of the quadric surface and a fiber. If the conic were not contained in the quadric surface, it would be contained in the fiber. This is a contradiction.

Remark 2.10 (The correspondence between scrolls and rational curves in the flag variety)**.** Note that there is a one-to-one correspondence between rational curves $i : \mathbb{P}^1 \to F(k_\bullet; n)$ in the flag variety with $i^* S_j = \mathcal{O}_{\mathbb{P}^1}(-a_{j,1}) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(-a_{j,k_j})$ and nested sequences of scrolls

$$S_{a_{1,1},\dots,a_{1,k_1}} \subset \cdots \subset S_{a_{r,1},\dots,a_{r,k_r}}$$

in $\mathbb{P}^{n-1}$ (here by scroll we mean not just the variety, but also the fibration over $\mathbb{P}^1$). Given a rational curve in the flag variety the projectivization of the pull-backs of the tautological bundles give rise to a nested sequence of scrolls that naturally map to $\mathbb{P}^{n-1}$. Conversely, by the universal property of flag varieties, a nested sequence of scrolls in $\mathbb{P}^{n-1}$ induces a rational curve in the flag variety with the desired properties. In the future we will often use this correspondence without further comment.

3. Basic lemmas

In this section we will prove some basic lemmas required for the proof of Theorem 1.12.

Definition 3.1. *We say that a flag pair $(k_\bullet, d_\bullet)$ is GW-null for n and write $I_3(k_\bullet, d_\bullet, n) = 0$ if there does not exist any non-zero, three-pointed, genus-zero Gromov-Witten invariants of $F(k_\bullet; n)$ of degree $(d_1, \dots, d_r)$.*

It is possible to phrase the condition of being GW-null geometrically in terms of the evaluation morphism.

Lemma 3.2. *$I_3(k_\bullet, d_\bullet, n) = 0$ if and only if the fibers of the evaluation map*

$$e : \overline{\mathcal{M}}_{0,3}(F(k_\bullet; n), d_\bullet) \to F(k_\bullet; n) \times F(k_\bullet; n) \times F(k_\bullet; n)$$

are positive dimensional.

Proof. Let Ω be the image of the evaluation morphism e. If e is generically finite, then Ω is a projective scheme of dimension

$$D = \sum_{i=1}^{r} k_i(k_{i+1} - k_i) + \sum_{i=1}^{r} d_i(k_{i+1} - k_{i-1}).$$

By the Künneth decomposition, the Poincaré dual ω of the class of Ω can be expressed as

$$\omega = \sum_{|\lambda|+|\mu|+|\nu|=D} c_{\lambda,\mu,\nu}\sigma_{\lambda^*} \otimes \sigma_{\mu^*} \otimes \sigma_{\nu^*},$$

where $|\cdot|$ denotes the codimension of the class and λ^* is the dual of λ. Since Ω is projective at least one of the coefficients $c_{\lambda,\mu,\nu}$ is positive. It follows that if e is generically s-to-one, then

$$I_{F(k_1,\dots,k_r;n),d_\bullet}(\lambda, \mu, \nu) = s \cdot c_{\lambda,\mu,\nu} > 0.$$

Conversely, if e has positive dimensinonal fibers (say with generic fiber dimension δ), then Ω has dimension $D-\delta$. Then $\omega \cup \sigma_{\lambda^*} \otimes \sigma_{\mu^*} \otimes \sigma_{\nu^*} = 0$ for any λ, μ, ν whose codimensions add up to D. A general choice of representatives for the Poincaré duals of $\sigma_\lambda, \sigma_\mu, \sigma_\nu$ is disjoint from Ω. Hence, the three-pointed Gromov-Witten invariants all vanish. □

Lemma 3.3. *If $I_3(k_\bullet, d_\bullet, n) = 0$, then $I_3(k_\bullet, d_\bullet, n-1) = 0$.*

Proof. It is convenient to use the translation between rational curves in flag varieties and rational scrolls in projective space to prove this lemma. If $I_3(k_\bullet, d_\bullet, n) = 0$, then by Lemma 3.2 the fibers of the evaluation morphism e are positive dimensional. In other words, given a triple of flags in the image of e, there is a positive dimensional family of rational scrolls containing those flags as fibers. Every rational scroll in $\mathbb{P}^{n-2}$ is the projection of a rational normal scroll. In particular, it is the projection of a rational scroll from $\mathbb{P}^{n-1}$. If there exists a positive dimensional family of scrolls containing a general point in the image of the evaluation morphism e, then the same will be true for the projection from a general point. The projection of the total space of the one-parameter family from a point p has the same dimension unless the total space is a cone with vertex at p. Since the projection is general, we conclude that the fibers of the evaluation morphism are positive dimensional for $n-1$ as well. By Lemma 3.2 it follows that $I_3(k_\bullet, d_\bullet, n-1) = 0$. □

Remark 3.4. Unfortunately, the converse of Lemma 3.3 is false. For example, there is a unique map of degree 2 to $G(2,4)$ taking three prescribed points on $\mathbb{P}^1$ to three general points. However, there is a two-parameter family of degree 2 maps to $G(2,3)$ taking three prescribed points to three general points. Hence, although $I_{G(2,3),2}(2;3) = 0$, $I_{G(2,4),2}(\sigma_{2,2}, \sigma_{2,2}, \sigma_{2,2}) = 1$. However, if we assume that n is large enough, then the converse becomes true as well.

Lemma 3.5. *If $I_3(k_\bullet, d_\bullet, n_0) = 0$ for some $n_0 \geq k_r + d_r$, then $I_3(k_\bullet, d_\bullet, n) = 0$ for all n.*

Proof. It is again convenient to interpret the rational curves in $F(k_\bullet; n)$ as a nested sequence of rational scrolls in $\mathbb{P}^{n-1}$. A degree d_r variety of dimension k_r can span at most a linear space $\mathbb{P}^{k_r+d_r-1}$ of projective dimension $k_r + d_r - 1$. Since all the other scrolls are contained in the scroll of degree d_r and dimension k_r, they have to be contained in the same span. Consequently, the image of any rational curve of degree $d_\bullet$ in $F(k_1, \dots, k_r; n)$ has to be contained in a sub-flag variety of the form $F(k_1, \dots, k_r; k_r + d_r)$. If $I_3(k_\bullet, d_\bullet, n_0) = 0$ for some $n_0 \geq k_r + d_r$, then by Lemma 3.2 there is a positive dimensional family of scrolls containing any three linear spaces corresponding to a point in the image of the evaluation map e. This will be true for any $n \geq k_r + d_r$ since the scrolls in $\mathbb{P}^{n-1}$ must be contained in $\mathbb{P}^{k_r+d_r-1}$ and it is true in the latter. Hence $I_3(k_\bullet, d_\bullet, n) = 0$ if $n \geq k_r + d_r$. Hence by Lemma 3.3, $I_3(k_\bullet, d_\bullet, n) = 0$ for all n. □

In particular, in order to prove the vanishing of all three-pointed Gromov-Witten invariants for a given flag pair $(k_\bullet, d_\bullet)$, we can take $n \geq k_r + d_r$.

Remark 3.6. We observe that the previous three Lemmas hold for m-pointed Gromov-Witten invariants if we replace 3 by m and e by $ev_1 \times \cdots \times ev_m$.

4. Vanishing theorems

In this section we prove Theorem 1.12. Lemma 3.2 allows us to turn the problem to a dimension count.

Proof of Theorem 1.12. By Lemma 2.1 there exists a unique, balanced, admissible set of sequences $B_\bullet = \{(a_{1,j})_{j=1}^{k_1}, \dots, (a_{r,j})_{j=1}^{k_r}\}$ associated to any flag pair $(k_\bullet, d_\bullet)$. Theorem 1.12 asserts that if $I_3(k_\bullet, d_\bullet, n) \neq 0$, then the balanced, admissible set of sequences associated to $(k_\bullet, d_\bullet)$ have to satisfy various inequalities. The strategy of the proof is as follows. By Lemma 3.2, $I_3(k_\bullet, d_\bullet, n) = 0$ if and only if the evaluation morphism

$$e : \overline{\mathcal{M}}_{0,3}(F(k_\bullet; n), d_\bullet) \to F(k_\bullet; n) \times F(k_\bullet; n) \times F(k_\bullet; n)$$

has positive dimensional fibers. We will show that the fibers of e are positive dimensional if the inequalities in Theorem 1.12 are violated. By Proposition 2.3 any morphism $f : C \to F(k_\bullet; n)$ contributing to a Gromov-Witten invariant for a general choice of Schubert subvarieties of $F(k_\bullet; n)$ has that $C = \mathbb{P}^1$ and the decomposition of the pull-back of the tautological bundles is given by a balanced sequence. We will use the correspondence between rational curves in $F(k_\bullet; n)$ and scrolls to construct one-parameter families of scrolls containing a given triple of linear spaces corresponding to a point in the image of e.

If $\dim(F(k_\bullet; n) \times F(k_\bullet; n) \times F(k_\bullet; n)) < \dim(\overline{\mathcal{M}}_{0,3}(F(k_\bullet; n), d_\bullet))$, then the fibers of e have to be positive dimensional. We conclude that

$$3 \sum_{i=1}^{r} k_i(k_{i+1} - k_i) \geq \sum_{i=1}^{r} k_i(k_{i+1} - k_i) + \sum_{i=1}^{r} d_i(k_{i+1} - k_{i-1});$$

or equivalently

$$2 \sum_{i=1}^{r} k_i(k_{i+1} - k_i) \geq \sum_{i=1}^{r} d_i(k_{i+1} - k_{i-1}).$$

This gives the first inequality in Theorem 1.12.

Recall our convention that $\alpha_i = a_{i,k_i}$. Observe that since the set of sequences is admissible $a_{i+1,j} \leq a_{i,j}$ for all $1 \leq j \leq k_i$. Consequently, $\sum_{\kappa=0}^{\kappa_0} l_{i,\kappa} \leq \sum_{\kappa=0}^{\kappa_0} l_{i+1,\kappa}$. Let κ_0 be the smallest $\kappa \leq \alpha_{i+1} - 2$ such that $l_{i,\kappa} < l_{i+1,\kappa}$. We can modify the sequence $(a_{i+1,j})$ by replacing the smallest index term equal to α_{i+1} by $\alpha_{i+1} - 1$ and the largest index term equal to κ_0 by $\kappa_0 + 1$. The resulting set of sequences is still admissible for $(k_\bullet, d_\bullet)$ and has lower value for the function

$$\sum_{i=1}^{r} \sum_{1 \leq l < m \leq k_i} (a_{i,k} - a_{i,l}).$$

Since the set of sequences is balanced, we conclude that $l_{i,\kappa} = l_{i+1,\kappa}$ for all $0 \leq \kappa \leq \alpha_{i+1} - 2$.

Below we will use the following observation: Let $d_\bullet$ and $d'_\bullet$ be two curve classes in $F(k_\bullet; n)$ such that $d'_i \leq d_i$ for $1 \leq i \leq r$. Let $(X_\bullet, Y_\bullet, Z_\bullet)$ be a triple of flags in the image of the evaluation morphism for the curve class $d'_\bullet$. Then $(X_\bullet, Y_\bullet, Z_\bullet)$ is also in the image of the evaluation morphism for the curve class $d_\bullet$. Furthermore, if $d'_i < d_i$ for at least one i, then the fiber of the evaluation morphism for the class $d_\bullet$ over $(X_\bullet, Y_\bullet, Z_\bullet)$ is positive dimensional. Given a rational curve C in the class $d'_\bullet$ passing through the three points corresponding to $X_\bullet, Y_\bullet$ and $Z_\bullet$, we can attach curves of degree given by the unit vector e_i at any point of C. If we apply this procedure $d_i - d'_i$ times for every i, we obtain a tree of rational curves passing through $X_\bullet, Y_\bullet$ and $Z_\bullet$. Moreover, since we can vary the attachment point, we see that the evaluation morphism is positive dimensional over $(X_\bullet, Y_\bullet, Z_\bullet)$.

We next show that $\alpha_r \leq 2$. Suppose $\alpha_r > 2$. Let u be the largest index, if one exists, such that $\alpha_u < \alpha_r$. Else let $u = 0$ and set S_u to be the empty set. Fix a general point $(X_\bullet, Y_\bullet, Z_\bullet)$ in the image of the evaluation morphism. Let $S_1, \dots, S_r$ be a sequence of balanced scrolls that have these flags as their fibers. We know that $l_{i,\alpha_r-2} = l_{r,\alpha_r-2}$ for $i \geq u$. It follows that $a_{i,j} \geq 2$ for $i > u$ and $j > \sum_{s=0}^{\alpha_r-2} l_{i,s}$ since $\alpha_r - 2 \geq 1$. Choose ordered bases of the linear spaces X_r, Y_r, Z_r such that the first k_i basis elements span X_i, Y_i and Z_i, respectively. Fix the scroll S_u and construct scrolls $S_{u+1}, \dots, S_r$ using Construction 2.4 as follows. Pick the points corresponding to the basis elements of the same index $i > k_u$. Pick general conics containing the three points. Form the scrolls generated by the conics and S_u. We obtain a new sequence of scrolls containing the same fibers but of strictly smaller degree. By the observation in the previous paragraph, we conclude that the fibers of the evaluation morphism have to be positive dimensional. By Lemma 3.2 we conclude that unless $\alpha_r \leq 2$, the invariants must vanish.

Now we show that $\alpha_i \leq \alpha_{i+1} + 2$. The argument is almost identical. Let t be the smallest index such that $\alpha_t > \alpha_{t+1} + 2$. Let $u < t$ be the smallest index, if there exists one, such that $\alpha_u < \alpha_t$. Then for $u \leq i \leq t$ we have that $l_{i,\kappa} = l_{t,\kappa}$ for every $\kappa \leq \alpha_t - 2$. Furthermore, $l_{u,\alpha_t-1} \leq l_{i,\alpha_t-1}$ for $t > i > u$. Pick a point $(X_\bullet, Y_\bullet, Z_\bullet)$ in the image of the evaluation morphism. Suppose that $S_1 \subset \cdots \subset S_r$ are a sequence of scrolls of degrees $d_1, \dots, d_r$, respectively that have the flags $X_\bullet, Y_\bullet$ and $Z_\bullet$ as fibers. Fix the scrolls $S_{t+1} \subset \cdots \subset S_r$. Note that for $u + 1 \leq i \leq t$ the minimal subscrolls of S_i of dimension $\sum_{\kappa=0}^{\alpha_t-1} l_{i,\kappa}$ and degree $\sum_{\kappa=0}^{\alpha_t-1} \kappa l_{i,\kappa}$ all contain S_u. Pick bases of X_r, Y_r and Z_r such that the first k_i basis elements span X_i, Y_i and Z_i, respectively. On the scroll S_{t+1} there exists a rational section of degree $d \geq \alpha_{t+1} + 2$ passing through three specified points on distinct fibers. By Lemma 2.7 the dimension of the space of such curves is

$$\sum_{i=0}^{\alpha_{t+1}} (\alpha_t + 3 - i) l_{t+1,i} - 1.$$

This is strictly larger than $3(k_{t+1} - 1)$. Furthermore, using the action of the automorphism group of the scroll it is clear that there will be a positive dimensional family of such curves passing through three points in distinct fibers. Using Construction 2.4 construct scrolls spanned by S_u and rational curves of degree $\alpha_t - 1$ passing through the points corresponding to the chosen bases of X_t, Y_t and Z_t. We thus construct a collection of scrolls

$$S_1 \subset \cdots \subset S_u \subset S'_u \subset \cdots S'_t \subset S_{t+1} \subset \cdots \subset S_r$$

having the flags $X_\bullet$, $Y_\bullet$ and $Z_\bullet$ as fibers. Note that by construction $d'_i < d_i$ for $u < i \leq t$. The observation two paragraphs ago now implies that the fibers of the evaluation morphism have to be positive dimensional. Lemma 3.2 then implies that the Gromov-Witten invariants of degree $d_\bullet$ vanish unless $\alpha_t \leq \alpha_{t+1} + 2$.

Now assume that $\sum_{s=0}^{\alpha_{t-1}-\alpha_t} 2^s l_{t-1,\alpha_t+s} < l_{t,\alpha_t}$ Fix a scroll of type S_{t+1} and three of its fibers. We would like to calculate the dimension of the space of a sequence of scrolls $S_1 \subset \cdots \subset S_t$ contained in S_{t+1} and compare it to the dimension of the space of flags they give rise to in the three chosen fibers. In any of these cases fix the scroll S_{t-1} and a scroll S'_t of type

$$a_{t,1}, \dots, a_{t,k_t-l_{t,\alpha_t}}, (\alpha_t)^{(l_{t-1,\alpha_t}+2l_{t-1,\alpha_t+1}+4l_{t-1,\alpha_t+2})}$$

contained in S_{t+1} and containing S_{t-1}. Note that given a scroll $S_{t-1} \subset S_t$, it has to be contained in a scroll of type S'_t contained in S_t by repeated applications of the construction in Remark 2.9. We can now calculate the dimension of the space of scrolls of type S_t contained in S_{t+1} and containing S'_t. To further simplify notation, set $\alpha_{t+1} = \alpha$, $l = l_{t-1,\alpha_t} + 2l_{t-1,\alpha_t+1} + 4l_{t-1,\alpha_t+2}$ and $\alpha_t = \alpha + p$ with $p \geq 0$. By Lemma 2.7 the dimension of the space of such scrolls is given by

$$(l_{t,\alpha+p} - l) \sum_{j=1}^{k_{t+1}} (\alpha + p + 1 - a_{t+1,j}) - (l_{t,\alpha+p} - l) \sum_{j=1}^{k_t} (\alpha + p + 1 - a_{t,j}).$$

This dimension has to be less than or equal to the dimension of the space of three k_t-planes in the designated fibers of S_{t+1} that contain the fibers of S'_t. The latter dimension is at most

$$3(l_{t,\alpha+p} - l)(k_{t+1} - k_t).$$

We thus obtain the inequality

$$(l_{t,\alpha+p} - l)\sum_{j=1}^{k_{t+1}}(\alpha + p + 1 - a_{t+1,j}) - (l_{t,\alpha+p} - l)\sum_{j=1}^{k_t}(\alpha + p + 1 - a_{t,j}) \leq 3(l_{t,\alpha+p} - l)(k_{t+1} - k_t).$$

Since by our simplifying assumption $l_{t,\alpha+p} - l > 0$, we can cancel it from both sides of the inequality. We are left with the inequality

$$\sum_{j=1}^{k_{t+1}}(\alpha + p - 2 - a_{t+1,j}) - \sum_{j=1}^{k_t}(\alpha + p - 2 - a_{t,j}) = \sum_{j=1}^{k_t}(a_{t,j} - a_{t+1,j}) + \sum_{j=k_t+1}^{k_{t+1}}(\alpha + p - 2 - a_{t+1,j}) \leq 0.$$

If $p \geq 2$, then the second sum to the right of the equal sign is non-negative since $\alpha \geq a_{t+1,j}$. On the other hand, the first sum to the right of the equal sign is strictly positive since $a_{t,j} \geq a_{t+1,j}$ for $j < k_t$ and $a_{t,k_t} \geq a_{t+1,k_t} + p$. We thus conclude that the fibers of the evaluation morphism have to be positive dimensional in case $p \geq 2$. If $p = 1$, recalling that $l_{t,\alpha-j} = l_{t+1,\alpha-j}$ for $j > 1$, the same inequality translates to

$$2l_{t,\alpha+1} \leq l_{t+1,\alpha} - l_{t,\alpha}.$$

Finally, if $p = 0$, then we obtain the inequality

$$2l_{t,\alpha} \leq 2_{t+1,\alpha} + l_{t+1,\alpha-1} - l_{t,\alpha-1}.$$

These are the inequalities claimed in the theorem.

Finally, set $k'_i = n - k_{r+1-i}$ and set $d'_i = d_{r+1-i}$. Since $F(k_\bullet; n)$ is isomorphic to $F(k'_\bullet; n)$ and under this isomorphism the curve class $d_\bullet$ is interchanged with $d'_\bullet$, the same inequalities have to hold for the balanced sequence associated with the flag pair $(k'_\bullet, d'_\bullet)$. This concludes the proof of the theorem. □

As observed in the introduction, in the case of two-step flag varieties Theorem 1.12 can be improved.

Proof of Theorem 1.4. We now specialize the calculations in Theorem 1.12 to $r = 2$. By Lemma 3.5 we may assume that $n = k_2 + d_2$. Note that if $t = 1$, the hypotheses of (4) in Theorem 1.12 are vacuously satisfied. We conclude that $\alpha_1 \leq \alpha_2 + 1$. If $\alpha_1 = \alpha_2$, then $2l_{1,\alpha_1} + l_{1,\alpha_1-1} \leq 2l_{2,\alpha_1} + l_{2,\alpha_1-1}$. If $\alpha_1 = \alpha_2 + 1$, then $2l_{1,\alpha_1} \leq l_{2,\alpha_2} - l_{1,\alpha_2}$.

Now we show that $\alpha_2 \leq 1$. Suppose $\alpha_2 = 2$. If $\alpha_1 < 2$, then we can fix the minimal subscroll S'_2 of S_2 of dimension $l_{2,1}$ and degree $l_{2,1}$ (which contains S_1) and three general fibers of S_2. There is a positive dimensional family of such scrolls S_2 by Construction 2.4. There is a two-dimensional family of conics containing three non-collinear points in projective space. Fix a minimal set of ordered points in each of the fibers that span linear spaces complementary to the fibers of S'_2. We can construct a positive dimensional family of scrolls by taking different conics passing through these points in the fibers. In case $\alpha_1 = 2$ the argument is similar. Fix the minimal subscroll S'_2 of S_2 and the intersection of S_1 with it. Note that for a general choice of flags S_1 intersects S'_2 in a scroll S'_1 of type $1^{l_{\alpha_{1,1}}}, 2^{\max(0, l_{\alpha_{1,2}} - l_{\alpha_{2,2}})}$. Fix three general flags of projective dimensions $k_1 - 1$ and $k_2 - 1$ containing three fibers of S'_1 and S'_2, respectively. Fix compatible bases of the complementary linear spaces. We can then construct a positive dimensional family of scrolls containing this configuration by Construction 2.4 as in the previous case. Finally it is clear that we cannot have $\alpha_1 = 3$. Consider the inequality

$$d_1 k_2 + d_2(d_2 + k_2 - k_1) \leq 2k_1(k_2 - k_1) + 2k_2 d_2.$$

We can write $d_1 = 2k_1 + (d_1 - 2k_1)$, where $d_1 - 2k_1 > 0$. Similarly, we can write $d_2 = k_2 + (d_2 - k_2)$, where $d_2 - k_2 > 0$. Rearranging the inequality we obtain

$$2k_1^2 + (d_1 - k_1)k_2 + (d_2 - k_1 - k_2)d_2 \leq 0.$$

But since $l_{1,2}+2l_{1,3} \leq l_{2,2}$, we have that $k_1+(d_1-2k_1) \leq d_2-k_2$. Hence $d_2-k_2-k_1>0$ leading to a contradiction. We conclude that $\alpha_2 \leq 1$.

We can then summarize the inequalities as follows. $\alpha_2 \leq 1$, so $d_2 \leq k_2$. $\alpha_1 \leq 2$. Either $\alpha_1 \leq 1$, so that $d_1 \leq k_1$. In this case the number of zeros in the first sequence has to be less than or equal to the number of zeros in the second sequence. Hence, $k_1-d_1 \leq k_2-d_2$. We also have the inequality on the number of ones in the sequence that translates to $d_1+k_1 \leq d_2+k_2$. If $\alpha_1=2$, then $d_1 \leq 2k_1$. We also have the inequality on the length of two's in the sequence given by $d_1 \leq d_2$. Finally, the same inequalities have to hold for the dual flag variety. These inequalities are summarized in Theorem 1.4. □

5. The proof of Theorem 1.13

In this section we prove Theorem 1.13. Under the numerical assumptions, we will show that given a general point in the image of the evaluation morphism e there is a unique nested sequence of scrolls containing the given linear spaces as fibers. This will show that e is birational onto its image. Consequently there are non-zero, three-pointed Gromov-Witten invariants for the triple $(k_\bullet, d_\bullet, n)$.

Proof of Theorem 1.13. Let $(k_\bullet, d_\bullet)$ be a flag pair whose associated balanced, admissible set of sequences satisfy the inequalities in the hypotheses of the theorem. Suppose $n \geq k_r+d_r$. By Lemma 3.5, it suffices to assume that $n=k_r+d_r$. We will prove the theorem by constructing a sequence of scrolls inductively. The scroll of degree d_r and dimension k_r is easy to construct. Given three linear spaces $f_{r,1}, f_{r,2}$ and $f_{r,3}$ of dimension k_r that intersect in a linear space v_r of dimension k_r-d_r and that span a linear space of dimension k_r+d_r, Construction 2.5 constructs a unique scroll containing their projectivization as fibers. On the other hand, if $f_{1,r}, f_{2,r}$ and $f_{3,r}$ are the k_r-dimensional subspaces of the three flags at a general point of the image of the evaluation morphism, then they span a (k_r+d_r)-dimensional vector space and intersect in a (k_r-d_r)-dimensional vector space. This follows easily from Proposition 2.3 and the description of the corresponding scrolls. Hence the largest scroll can be uniquely constructed by knowing the k_r-dimensional subspaces of the three flags in the image of e.

Next we would like to construct the scroll S_{r-1}. Since this is our basic construction we explain it in detail. Pick a linear space v_{r-1} of dimension $l_{r-1,0}$ in v_r. Next pick general linear spaces w_i of dimension $l_{r-1,0}+l_{r-1,1}$ in f_i containing v_{r-1} and spanning a linear space of dimension $l_{r-1,0}+2l_{r-1,1}$. Note that there is a unique scroll S'_{r-1} of type $0,\dots,0,1,\dots,1$ with $l_{r-1,0}$ zeros and $l_{r-1,1}$ ones contained in S_r. By Construction 2.5 there is a unique such scroll containing w_i as fibers. But since this scroll is covered by lines that intersect S_r in three points, it must be contained in S_r. (Note that since S_r is a non-degenerate variety of minimal degree, any line containing three collinear points on S_r is automatically contained in S_r.) Furthermore, any subscroll of S_r that has the same type as S_{r-1} has a unique subscroll of the same type as S'_{r-1}. Hence the three linear spaces of dimension k_{r-1} occurring in the image of the evaluation morphism must have subspaces of the form w_i. Next pick three linear spaces y_i of dimension $l_{r,0}+l_{r-1,1}+2l_{r-1,2}$ in $f_{r,i}$ containing w_i and v_r such that they span a linear space of dimension $l_{r,0}+2l_{r-1,1}+4l_{r-1,2}$. Note that we are able to choose such linear spaces because $l_{r-1,1}+2l_{r-1,2} \leq l_{r,1}$ by assumption. There is a unique scroll $\tilde{S}_{r-1}$ of type $0,\dots,0,1,\dots,1$ where there are $l_{r,0}$ zeroes and $l_{r-1,1}+2l_{r-1,2}$ ones contained in S_r and containing the three linear spaces y_i. Now let $f_{r-1,i}$ be three general linear spaces in y_i containing w_i of dimension k_{r-1}. Then the scroll S_{r-1} is the linear section of $\tilde{S}_{r-1}$ contained in the span of $f_{r-1,i}$. Conversely, by Remarks 2.8 and 2.9 the three linear spaces $f_{r-1,i}$ determine the scroll $\tilde{S}_{r-1}$. Hence given linear spaces in the image of the evaluation morphism e, we have uniquely constructed a pair of scrolls $S_{r-1} \subset S_r$.

Continue by descending induction on i to construct the scrolls. Suppose that we have constructed the scroll S_{i+1}. We would like to discuss how to construct the scroll S_i. The construction is essentially the same as the last step. Suppose the type of S_{i+1} is $a_{i+1,1},\dots,a_{i+1,k_{i+1}}$. To simplify notation set $a_{i+1,k_{i+1}}=\alpha$. Since the sequence is balanced, if $a_{i,k_i} \leq \alpha-2$, then S_i has to coincide with the unique minimal subscroll S'_{i+1} of type $a_{i+1,1},\dots,a_{i+1,k_{i+1}-l_{i+1,\alpha-1}-l_{i+1,\alpha}}$. In particular, in this case the linear spaces V_i have to coincide with the fibers of the subscroll S'_{i+1} of this type in the three fibers. In this case S_i is automatically constructed. It is equal to S'_{i+1} and there is nothing further to do. Even when $a_{i,k_i} > \alpha - 2$, then the subscroll S'_{i+1} has to be the minimal subscroll of both S_i and S_{i+1} of dimension and degree equal to the dimension and degree of S'_{i+1}. In S_{i+1} fix a scroll S''_{i+1} of type $\alpha-1, \dots, \alpha$, where $l_{i+1,\alpha-1}$ of the digits are $\alpha - 1$ and $l_{i+1,\alpha}$ of the digits are α, complementary to S'_{i+1}. Then the scroll S_i we would like to construct has to intersect S''_{i+1} in a scroll $\tilde{S}_i$ of type $a_{i,k_{i+1}-l_{i+1,\alpha-1}-l_{i+1,\alpha+1}}, \dots, a_{i,k_i}$. Conversely, given any such scroll in S''_{i+1}, we can uniquely construct S_i by taking the fiber-wise joins of the scrolls S'_{i+1} and $\tilde{S}_i$. Now we are reduced to the case of the previous paragraph. S''_{i+1} admits a morphism to $\mathbb{P}^{l_{i+1,\alpha-1}+2l_{i+1,\alpha}-1}$ with image $S_{0,\dots,0,1,\dots,1}$ with $l_{i+1,\alpha-1}$ zeros and $l_{i+1,\alpha}$ ones. The image of $\tilde{S}_i$ is a scroll of type $0, \dots, 0, 1 \dots, 1, 2, \dots, 2$, where $l_{i,\alpha-1}$, $l_{i,\alpha}$ and $l_{i,\alpha+1}$ of the digits are zero, one and two, respectively. The previous paragraph determines the conditions that the fibers of such a scroll has to satisfy. Furthermore, it uniquely constructs the scroll from three of its fibers given that the fibers satisfy these conditions. This concludes the inductive construction and the proof of the theorem. □

One can relax the assumptions of Theorem 1.13 slightly as in the following Corollary. Theorem 1.9 is also a special case of the Corollary.

Corollary 5.1. *Let $(k_\bullet, d_\bullet)$ be a flag pair whose associated balanced set of admissible sequences satisfy the assumptions of Theorem 1.13. Suppose $n \geq k_i + d_i$. Further assume that $a_{j,k_j} \leq 1$ for every $j \geq i$ and that $l_{j,1} \leq l_{j-1,1}$ for $i < j \leq r$. Then the evaluation morphism e is birational onto its image.*

Proof. The proof of Theorem 1.13 already constructs uniquely the scrolls of dimension less than or equal to k_i given a point in the image of the evaluation morphism. Note that the scrolls of dimension larger than k_i are also uniquely determined by the image of the evaluation morphism. They are cones over perfectly balanced scrolls. The vertex is determined by the intersection of the linear spaces. Note that they are cones over perfectly balanced subscrolls of the k_i-dimensional scroll complementary to the vertex. Hence, they are uniquely determined by their fibers. □

Note that the proof of Theorem 1.13 explicitly computes many three-pointed Gromov-Witten invariants of partial flag varieties. Assuming the hypotheses of Theorem 1.13, any time we can determine uniquely a point in the image of the evaluation morphism, we can conclude that the corresponding Gromov-Witten invariant is one. We state the following important cases. Note that these calculations are generalizations of well-known classical facts. For example, the fact that given three pairwise skew lines in space there exists a unique hyperboloid containing them was most likely known as far back as Antiquity. Corollary 5.2.4 gives a vast generalization of this fact.

Corollary 5.2. *Theorem 1.13 computes the following Gromov-Witten invariants.*

(1) *Let $0 \leq \alpha \leq k_1$. Let $d_i = k_i - \alpha$. Let σ_λ denote the cycle corresponding to the sequence*

$$1^{k_1-\alpha}, 2^{k_2-k_1}, \dots, (r-1)^{k_{r-1}-k_{r-2}}, r^{k_r-k_{r-1}}, 1^\alpha, (r+1)^{d_r}.$$

Let σ_μ denote the pull back of the point class from $G(k_r, k_r + d_r)$ by the natural projection. Finally, let σ_ν denote the pull-back of the class corresponding to the sequence $1^{k_r-\alpha}, 2^{d_r}, 1^\alpha$ from $G(k_r, k_r + d_r)$. Then $I_{F(k_1,\dots,k_r;k_r+d_r),d_\bullet}(\lambda, \mu, \nu) = 1$.

(2) *More generally, let $0 \leq \alpha \leq k_1$ and set $d_i = k_i - \alpha$. Let λ_i for $i = 1, 2$ denote the cycles*

$$1^{s_{i,1}}, 2^{s_{i,2}}, \dots, (r-1)^{s_{i,r-1}}, r^{k_r-k_{r-1}}, (r-1)^{k_{r-1}-k_{r-2}-s_{i,r-1}}, \dots, 1^{k_1-s_{i,1}}, (r+1)^{d_r},$$

where $s_{1,j} + s_{2,j} = k_j - k_{j-1}$ if $j > 1$ and $s_{1,1} + s_{2,1} = k_1 - \alpha$. Let σ_ν denote the same cycle as above. Then $I_{F(k_1,\dots,k_r;k_r+d_r),d_\bullet}(\lambda_1, \lambda_2, \nu) = 1$.

(3) *Suppose $2k_1 \geq k_r$. Set $d_i = k_r$ for all i. Let σ_{λ_i} be the Schubert cycle in $F(k_1, \dots, k_r; 2k_r)$ corresponding to the sequence*

$$1^{s_{i,1}}, 2^{s_{i,2}}, \dots, (r-1)^{s_{i,r-1}}, r^{k_r-k_{r-1}}, (r-1)^{k_{r-1}-k_{r-2}-s_{i,r-1}}, \dots, 1^{k_1-s_{i,1}}, (r+1)^{k_r}.$$

Suppose that $\lambda_1, \lambda_2, \lambda_3$ *are three Schubert cycles with* $\sum_{1 \leq i \leq 3, j \leq t} s_{i,j} = k_r + k_t$ *for every* $1 \leq t \leq r-1$. *Then* $I_{F(k_1,\dots,k_r;k_r+d_r),d_\bullet}(\lambda_1, \lambda_2, \lambda_3) = 1$.

(4) *Let* $0 < a$ *be a positive integer. Set* $k_i = 2^{i-1}a$ *and* $d_i = (r+1-i)2^{i-1}a$. *Then*

$$I_{F(k_\bullet;2^{r+1}a),d_\bullet}(pt, pt, pt) = 1,$$

where pt *denotes the Poincaré dual of the point class.*

6. Very classical Gromov-Witten invariants

In the previous section we saw that a large class of the three-pointed Gromov-Witten invariants of $F(k_\bullet; n)$ of degree $d_\bullet$ are classical. In particular, if $n \geq 2k_2$, then every three-pointed Gromov-Witten invariant of $F(k_1, k_2; n)$ is classical. In this section we turn our attention to determining which among these are very classical; i.e., can be computed as ordinary Littlewood-Richardson coefficients of flag varieties in a natural way.

Let σ_λ be a Schubert cycle in $F(k_1, \dots, k_r; n)$ associated to the sequence λ. Given an integer $0 \leq \alpha \leq k_1$, set $d_i = k_i - \alpha$. Define the quantum cycle $q_\alpha(\lambda)$ associated to λ and α to be the Schubert cycle of $F(\alpha, k_1 + d_1, \dots, k_r + d_r; n)$ associated to the sequence obtained by λ as follows. First, replace λ by $\lambda[1]$ obtained by adding one to each digit of λ. Then replace the last α twos by one. Finally, replace the last $k_1 - \alpha$ digits that are equal to $r+2$ in the resulting sequence by 2, the next $k_2 - k_1$ digits that are $r+2$ by 3, the next $k_3 - k_2$ digits that are $r+2$ by 4, etc. The resulting sequence is $q_\alpha(\lambda)$. For example, suppose $1, 3, 2, 1, 3, 3, 2, 2, 3, 3$ be a sequence in $F(2, 5; 10)$. Let $\alpha = 1$. The corresponding sequence $q_1(\lambda)$ in $F(1, 3, 9; 10)$ is given by $2, 4, 3, 1, 3, 3, 3, 3, 3, 2$.

The following is a more precise statement of the first part of Corollary 1.14.

Theorem 6.1. *Let* $0 \leq \alpha \leq k_1$ *be an integer. Set* $d_i = k_i - \alpha$. *Let* $n \geq k_r + d_r$. *The three-pointed Gromov-Witten invariant* $I_{F(k_1,\dots,k_r;n),d_\bullet}(\lambda, \mu, \nu)$ *of* $F(k_1, \dots, k_r; n)$ *is equal to* $q_\alpha(\lambda) \cdot q_\alpha(\mu) \cdot q_\alpha(\nu)$ *in* $F(\alpha, k_1 + d_1, \dots, k_r + d_r; n)$.

Proof. Set D equal to the dimension of the Kontsevich moduli space $\overline{\mathcal{M}}_{0,3}(F(k_\bullet; n), d_\bullet)$. By Theorem 1.13 the evaluation morphism e gives a birational map from the Kontsevich space to the subvariety of $F(k_\bullet; n) \times F(k_\bullet; n) \times F(k_\bullet; n)$ defined as the closure of the following variety:

$$\Omega = \{(X_\bullet, Y_\bullet, Z_\bullet) \in F(k_\bullet; n) \times F(k_\bullet; n) \times F(k_\bullet; n) | \dim(X_i \cap Y_i \cap Z_i) = \alpha, \dim(\overline{X_i Y_i Z_i}) = 2k_i - \alpha\}.$$

Consequently the class of this variety is given by

$$[\Omega] = \sum_{|\lambda|+|\mu|+|\nu|=D} I_{F(k_\bullet;n),d_\bullet}(\lambda, \mu, \nu) \sigma_{\lambda^*} \otimes \sigma_{\mu^*} \otimes \sigma_{\nu^*}.$$

Given three general representatives of $\Sigma_\lambda(F_\bullet)$, $\Sigma_\mu(G_\bullet)$ and $\Sigma_\nu(H_\bullet)$ in $F(k_\bullet; n)$, their inverse images by the three projections will intersect Ω in $I_{F(k_\bullet;n),d_\bullet}(\lambda, \mu, \nu)$ points. Given one of these points $(X_\bullet, Y_\bullet, Z_\bullet)$ of intersection, we obtain a point in $\Sigma_{q_\alpha(\lambda)}(F_\bullet) \cap \Sigma_{q_\alpha(\mu)}(G_\bullet) \cap \Sigma_{q_\alpha(\nu)}(H_\bullet)$ in $F(\alpha, 2k_1 - \alpha, \dots, 2k_r - \alpha; n)$ by sending the point $(X_\bullet, Y_\bullet, Z_\bullet)$ to the sequence of flags

$$(X_1 \cap Y_1 \cap Z_1, \overline{X_1 Y_1 Z_1}, \overline{X_2 Y_2 Z_2}, \dots, \overline{X_r Y_r Z_r}).$$

It is clear by construction that this point is in the desired intersection. Conversely, given a point in the intersection $\Sigma_{q_\alpha(\lambda)}(F_\bullet) \cap \Sigma_{q_\alpha(\mu)}(G_\bullet) \cap \Sigma_{q_\alpha(\nu)}(H_\bullet)$ in $F(\alpha, 2k_1 - \alpha, \dots, 2k_r - \alpha; n)$, take the intersection of V_i, $i > 1$, with the flag element F_s, where s is the minimal place in $q_\alpha(\lambda)$ for which the number of digits less than or equal to i but larger than 1 in positions less than or equal to s is $k_i - \alpha$. Let $X_\bullet$ be defined by setting $X_{i-1} = \overline{V_1 V_i}$. Since the flags are general X_i has dimension k_i. Define $Y_\bullet$ and $Z_\bullet$ similarly. It is clear that this procedure produces a point of intersection of Ω with the inverse images of $\Sigma_\lambda(F_\bullet)$, $\Sigma_\mu(G_\bullet)$ and $\Sigma_\nu(H_\bullet)$ by the natural projection. Furthermore, these operations give a one-to-one correspondence between the two intersections. Suppose that the point in Ω constructed from

$$(X_1 \cap Y_1 \cap Z_1, \overline{X_1 Y_1 Z_1}, \overline{X_2 Y_2 Z_2}, \dots, \overline{X_r Y_r Z_r})$$

differs from $(X_\bullet, Y_\bullet, Z_\bullet)$. Then we can assume that there are two flags $X_\bullet$ and $X'_\bullet$ that lie in $\Sigma_\lambda(F_\bullet)$. Let F_s be the smallest index where the intersections of F_s with $X_\bullet$ and $X'_\bullet$ are different. There is a one-parameter family of flags varying this intersection and keeping the spans and the intersections the same (as these are already determined by $Y_\bullet, Z_\bullet$ and $Y'_\bullet, Z'_\bullet$, respectively). This contradicts the finiteness of the intersection of Ω with the inverse images of the three Schubert varieties. The theorem follows. □

This theorem can be used to prove relations among structure constants of different partial flag varieties. We begin by proving Corollary 1.16 to clarify the geometric idea. Given a Schubert cycle σ_λ in $F(k_1, \dots, k_r; n)$, let $\sigma_{\lambda t}$ be the Schubert cycle in $F(2k_1, 2k_2, \dots, 2k_r; 2n)$ obtained by adding to the sequence λ the trivial tail

$$r+1, \dots, r+1, r, \dots, r, \dots, 1, \dots, 1,$$

where there are $k_i - k_{i-1}$ digits i. Let σ_λ, σ_μ and σ_ν be three Schubert cycles in $F(k_1, \dots, k_r; n)$ whose codimensions sum to the dimension of $F(k_1, \dots, k_r; n)$. Recall that Corollary 1.16 asserts that

$$c_{\lambda,\mu,\nu} = c_{\lambda t, \mu t, \nu t}.$$

Proof of Corollary 1.16. Consider the partial flag variety $F(k_1, k_2, \dots, k_r, n; 2n)$. Given the Schubert cycle σ_λ, let $\sigma_{\lambda q}$ be the Schubert cycle obtained by adding a tail of n $r+2$'s. For example, if $\lambda = 1, 2, 1, 3$, then $\lambda q = 1, 2, 1, 3, 4, 4, 4, 4$. Consider the Gromov-Witten invariant

$$I_{F(k_1,\dots,k_r;n),(k_1,k_2,\dots,k_r,n)}(\sigma_{\lambda q}, \sigma_{\mu q}, \sigma_{\nu q}).$$

By Theorem 6.1 this Gromov-Witten invariant is equal to $c_{\lambda t, \mu t, \nu t}$ in $F(2k_1, \dots, 2k_r; 2n)$. On the other hand, the Gromov-Witten invariant counts the number of scrolls of degrees $k_1, k_2, \dots, k_r, n$. Since we require the scroll of degree n to contain three linear spaces of projective dimension $n-1$ in $\mathbb{P}^{2n-1}$, that scroll is uniquely determined by Construction 2.5. Notice that this scroll is perfectly balanced. In other words, the scroll abstractly is the projectivization of a trivial bundle. By Remark 2.8, we can parallel transport any linear space in one fiber to a linear space in another fiber uniquely. Note that the subscrolls of this scroll counted by the Gromov-Witten invariant in question are also perfectly balanced. Hence they are determined by a unique fiber by parallel transport. The fibers of the subscrolls have to satisfy the conditions imposed by the parallel transport of the conditions imposed by the three Schubert cycles $\sigma_\lambda, \sigma_\mu, \sigma_\nu$. Hence, the Gromov-Witten invariant is also equal to $c_{\lambda,\mu,\nu}$ in $F(k_1, \dots, k_r; n)$. We conclude that $c_{\lambda,\mu,\nu} = c_{\lambda t, \mu t, \nu t}$. □

Proof of Theorem 1.15. The proof of Theorem 1.15 is almost identical. Let λ', μ' and ν' be three Schubert cycles in the Grassmannian $G(a, 2n-a)$. Let λ, μ and ν be three Schubert cycles in $F(k_1 - a, k_2 - a, \dots, k_r - a, n - a)$. Consider the three-pointed Gromov-Witten invariant of $F(k_1, \dots, k_r, n; 2n-a)$ of degrees $d_i = k_i - a$ for $1 \leq i \leq r$ and $d_{r+1} = n - a$ corresponding to the following cycles. Remove the first $(n-a)$ digits of λ', μ' and ν'. By our assumption that the first 1 occurs at position greater than $n-a$ means that this removes only the initial $n-a$ twos in the sequences. Replace each of the two's in the truncated sequences by $r+2$. Label the resulting sequences by λ'', μ'' and ν'', respectively. Concatenate λ, λ'', μ, μ'' and ν, ν'' to obtain $\tilde{q}(\lambda, \lambda')$, $\tilde{q}(\mu, \mu')$ and $\tilde{q}(\nu, \nu')$. Consider the Gromov-Witten invariant

$$I_{F(k_1,\dots,k_r,n;2n-a),(d_1-a,d_2-a,\dots,d_r-a,n-a)}(\tilde{q}(\lambda, \lambda'), \tilde{q}(\mu, \mu'), \tilde{q}(\nu, \nu')).$$

By Theorem 6.1 this Gromov-Witten invariant is equal to

$$\pi_{\lambda'}(\lambda) \cdot \pi_{\mu'}(\mu) \cdot \pi_{\nu'}(\nu)$$

in $F(a, 2k_1 - a, 2k_2 - a, \dots, 2k_r - a; 2n - a)$. On the other hand, the Gromov-Witten invariant counts scrolls. The vertex of the scroll is determined by the cycles λ', μ', ν' in $G(a, 2n-a)$. There are $c_{\lambda',\mu',\nu'}$ choices for the vertex. Once we fix the vertex, the scroll of dimension n is uniquely determined by Construction 2.5. The vector bundle corresponding to this scroll is no longer trivial. Hence, we can only transport linear conditions from one fiber to another up to the ambiguity introduced by the vertex. Since the vertices of all the scrolls are equal, we can take a linear section of the scroll complementary to the vertex. This linear section of the scroll is perfectly balanced. Constructing the sections of the scrolls uniquely constructs them. Now we are reduced to the previous case since we are trying to construct a sequence of scrolls in a perfectly balanced scroll with three fibers satisfying the Schubert conditions imposed by λ, μ, ν in $F(k_1 - a, k_2 - a, \dots, k_r - a, n - a)$. There are $c_{\lambda,\mu,\nu}$ such scrolls. Consequently,

$$c_{\pi_{\lambda'}(\lambda),\pi_{\mu'}(\mu),\pi_{\nu'}(\nu)} = c_{\lambda,\mu,\nu} \cdot c_{\lambda',\mu',\nu'}.$$

□

Finally, the second part of Corollary 1.14 is easy. Any non-zero invariant of $F(a, 2a, \dots, 2^{r-1}a; n)$ of degree $d_\bullet$, where $d_i = (r - i + 1)k_i$ and $n \geq 2^r a$, gives rise to a non-zero invariant of $G(2^{r-1}a; n)$. The classes that lead to non-zero invariants must have as insertions classes corresponding to sequences λ, where if we remove the digits equal to $r + 1$ from λ, we obtain the sequence $1^{k_1}, 2^{k_2-k_1}, \dots, r^{k_r-k_{r-1}}$. Otherwise, by considering the dimension of the Kontsevich moduli space, we see that the codimension imposed on the k_r-planes would be larger than $k_r(n-k_r)+k_r n$. This is impossible. Now it is immediate by[BKT] that these invariants may be computed as the ordinary intersection numbers of Schubert varieties in $G(2^r a, n)$.

We conclude with some problems that remain to be studied.

Let $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_r$ be a weakly increasing sequence of non-negative integers such that $\alpha_i \leq k_i$. Similarly, let $\beta_1 \leq \beta_2 \leq \cdots \leq \beta_r$ be a collection of weakly increasing positive integers such that $\beta_i \geq k_i$ and $n \geq \beta_r$. Define $\Sigma(\alpha_\bullet, \beta_\bullet)$ as the closure of the following locus in $F(k_\bullet; n) \times F(k_\bullet; n) \times F(k_\bullet; n)$

$$\Sigma(\alpha_\bullet, \beta_\bullet)^\circ = \{(X_\bullet, Y_\bullet, Z_\bullet | \dim(X_i \cap Y_i \cap Z_i) = \alpha_i, \text{ and } \dim(\overline{X_i Y_i Z_i}) = \beta_i, \ 1 \leq i \leq r \ \}.$$

Problem 6.2. Find a positive, geometric rule for computing the class of $\Sigma(\alpha_\bullet, \beta_\bullet)$.

In particular, when $d_i = k_i - \alpha_i$ and $\beta_i = 2k_i - \alpha_i$, the resulting variety is the image of the evaluation morphism from $\overline{\mathcal{M}}_{0,3}(F(k_\bullet; n), d_\bullet)$. This problem is likely amenable to the techniques of [C3]. Note that the variety $\Sigma(\alpha_\bullet, \beta_\bullet)$ admits a rational map to an incidence variety in a product of flag varieties $F(\alpha_\bullet; n) \times F(\beta_\bullet; n)$. The calculation can be carried out in the latter variety. Of course, one can ask more generally for a positive rule for computing the class of the image of e. The proof of Theorem 1.13 gives a description of this image.

Problem 6.3. When the evaluation morphism e is generically finite, determine its degree.

In particular, this degree divides the greatest common divisor g of the Gromov-Witten invariants of degree $d_\bullet$. It would be interesting to know when the degree is equal to g.

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