A LITTLEWOOD-RICHARDSON RULE FOR TWO-STEP FLAG VARIETIES

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Abstract. We establish a positive geometric rule for computing the structure constants of the cohomology of two-step flag varieties with respect to their Schubert basis. As a corollary we obtain a quantum Littlewood-Richardson rule for Grassmannians. These rules have numerous applications to geometry, representation theory and the theory of symmetric functions.

Contents

1. Introduction

This paper describes a positive geometric rule for determining the structure constants of the cohomology ring of two-step flag varieties with respect to their Schubert basis. Such combinatorial rules are known as *Littlewood-Richardson rules*. Since the three-pointed Gromov-Witten invariants of a Grassmannian can be calculated as ordinary intersections of Schubert cycles in two-step flag varieties [BKT], we also obtain a quantum Littlewood-Richardson rule for Grassmannians.

The partial flag variety $F(k_1, \dots, k_r; n)$ parameterizes nested sequences of r linear subspaces $V_1^{k_1} \subset \cdots \subset V_r^{k_r}$ of dimensions k_i of a fixed n-dimensional vector space V. Partial flag varieties are fundamental objects in algebraic geometry, combinatorics and representation theory. Consequently their cohomology rings have been studied extensively (see [BGG], [FPi] or [Ful2]). Although there are many presentations for their cohomology rings, there were no proven Littlewood-Richardson rules for flag varieties except in the case of Grassmannians.

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Littlewood-Richardson coefficients exhibit a very rich structure which is best revealed by positive geometric rules. For instance, in recent years, new Littlewood-Richardson rules for the Grassmannians have enabled Klyachko, Knutson, Tao, Woodward and their collaborators to resolve long standing problems such as Horn's Conjecture and the Saturation Conjecture (see [KT] and [KTW]). Vakil using a geometric Littlewood-Richardson rule was able to establish the reality of Schubert calculus [V2].

In the case of Grassmannians there are many Littlewood-Richardson rules in terms of Young tableaux [Ful1], puzzles [KT], and checkers [V1]. A. Knutson conjectured a rule in terms of puzzles (see ([BKT])) for two-step flag varieties and A. Buch extended the conjecture to three-step flag varieties. However, for arbitrary flag varieties, except for multiplying very special classes (e.g. Monk's formula [Ful1]), there is not even a conjectural rule. Even in the case of Grassmannians, the known rules are insufficient and unsatisfactory for answering many natural geometric questions.

In this paper we will first provide a new Littlewood-Richardson rule for Grassmannians (§3, Theorem 3.6). Geometrically, a Littlewood-Richardson rule for $G(k, n)$ can be interpreted as an algorithm for expressing the class of the closure of the locus of k-planes that are spanned by their one-dimensional intersections with k vector spaces $W_1, \dots, W_k$, where W_i are the spans of consecutive basis elements of a fixed ordered basis and $W_i \not\subseteq W_j$ for $i \neq j$. Viewed from this perspective, it is much more natural to study the following generalization.

Problem 1.1. *Find an algorithm for expressing the class of the closure of the locus of k-planes in $G(k, n)$ that intersect k vector spaces, which are expressed as the spans of consecutive basis elements of a given ordered basis, in given dimensions.*

Corollary 3.9 resolves this problem completely. Aside from being a geometrically more natural problem, Problem 1.1 is an essential step for understanding the geometry of partial flag varieties. There are natural projections from the partial flag variety $F(k_1, \dots, k_r; n)$ to the Grassmannians $G(k_i, n)$. The projection of the intersection of two Schubert varieties in $F(k_1, \dots, k_r; n)$ to $G(k_i, n)$ is a variety whose class is computed by Corollary 3.9.

The main result of this paper is a Littlewood-Richardson rule for two-step flag varieties (§4, Theorem 4.6). The strategy that leads to the rule works to compute the classes of subvarieties of other partial flag varieties. However, to keep this paper at a reasonable length, we restrict our discussion to two-step flag varieties. The reader may consult [CV] for an example for three-step flag varieties. In addition to being a step in understanding the geometry of arbitrary partial flag varieties, the study of two-step flag varieties is interesting in its own right. For instance, the work of Buch, Kresch and Tamvakis ([BKT]) has highlighted a beautiful relation between the small quantum cohomology of Grassmannians and the cohomology of two-step flag varieties.

The final result of this paper is a quantum Littlewood-Richardson rule for Grassmannians (Theorem 5.1). The small quantum cohomology rings of Grassmannians have been studied extensively (see [B], [Bu] or [BKT] for references). However, as in the case of flag varieties, there was not a positive combinatorial procedure for finding the quantum Littlewood-Richardson coefficients of Grassmannians. Using the results in [BKT], the problem of computing three-pointed Gromov-Witten invariants turns to a problem of intersecting three Schubert cycles in a two-step flag variety. Hence, our algorithm also solves the quantum Littlewood-Richardson problem.

Our Littlewood-Richardson rules will be in terms of combinatorial objects called Mondrian tableaux. Mondrian tableaux supply a convenient tool for recording the rank table for the intersections of two flags. We start with the intersection of two Schubert varieties defined with respect to two transverse flags. We specialize the flags via codimension one degenerations. In the process we force the intersection of the two Schubert cycles defined with respect to the two flags to break into a union of Schubert cycles. The limits that occur during this process are recorded by Mondrian tableaux.

The principle behind the rule can be stated very succinctly. If a k-dimensional vector space V^k intersects two vector spaces W_1 and W_2 in subspaces of dimension i and $k-i$, respectively, and it does not intersect $W_1 \cap W_2$, then V^k must be contained in the span of W_1 and W_2. Our rule uses this principle repeatedly to resolve the intersection of two Schubert cycles into a union of Schubert cycles.

The idea to study the geometry of flag varieties via degenerations dates back at least to Pieri. Recently R. Vakil proved a geometric Littlewood-Richardson rule for Grassmannians using degenerations [V1]. Our approach is similar to Vakil's. However, even in the case of Grassmannians, by choosing a more natural and canonical degeneration order, we are able to clarify the geometry significantly. The result is a simpler and more efficient rule. In fact, as already mentioned, our methods apply in a much more general setting and give an algorithm to express the classes of a large collection of subvarieties of Grassmannians in terms of Schubert cycles.

Finally, it should be stressed that the algorithms described in this paper do not only compute classes of subvarieties of Grassmannians or two-step flag varieties. They describe the limits of subvarieties of Grassmannians and two-step flag varieties under specializations of the defining flags. The end result of the algorithm is not a collection of cohomology classes, but actual subvarieties.

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2. Preliminaries

In this section we collect well-known facts about flag varieties and the quantum cohomology of Grassmannians. For more detailed discussions the reader can consult [GH] Ch.1 §5, [FPi] and [BKT].

Notation 2.1. Let k_i be an increasing sequence of positive integers. We denote the Grassmannian of k-dimensional subspaces of an n-dimensional vector space by $G(k,n)$. When we interpret this Grassmannian as the parameter space of $(k-1)$-dimensional linear subspaces of $\mathbb{P}^{n-1}$, we will use the notation $\mathbb{G}(k-1,n-1)$.

Let $F(k_1,\dots,k_r;n)$ denote the r-step flag variety of r-tuples of linear subspaces $(V_1,\cdots,V_r)$ of an n-dimensional fixed vector space, where V_i are k_i-dimensional linear spaces and $V_i \subset V_{i+1}$ for all $1 \leq i \leq r-1$. When we would like to consider the flag variety as a parameter space for nested sequences of linear subspaces of projective space, we will use the notation $\mathbb{F}(k_1-1,\dots,k_r-1;n-1)$.

2.1. **The cohomology of flag varieties.** Let $F_\bullet : 0 = F_0 \subset F_1 \subset \cdots \subset F_n = V$ be a fixed complete flag in the n-dimensional vector space V. Given a partition $\lambda : n-k \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 0$, we define the *Schubert variety* Σ_λ in $G(k,n)$ with respect to the flag $F_\bullet$ to be

$$\Sigma_\lambda(F_\bullet) := \{[W] \in G(k,n) \mid \dim(W \cap F_{n-k+i-\lambda_i}) \geq i, \ \forall\ 1 \leq i \leq k\}.$$

We denote the Poincaré dual of the class of Σ_λ by σ_λ. The codimension of Σ_λ in $G(k,n)$ is equal to the *weight* of the partition $|\lambda| = \sum_i \lambda_i$. The classes of Schubert varieties form an additive basis of the homology of $G(k,n)$. The structure constants $c_{\lambda,\mu}^{\nu}$ in the product $\sigma_\lambda \cdot \sigma_\mu = \sum_\nu c_{\lambda,\mu}^{\nu} \sigma_\nu$ are called the *Littlewood-Richardson coefficients.*

Similarly, the cohomology of the r-step flag variety $F(k_1,\dots,k_r;n)$ is generated by Poincaré duals of the classes of Schubert varieties. Schubert varieties are parameterized by permutations ω of length n for which $\omega(i) < \omega(i+1)$ whenever $i \notin \{k_1,\dots,k_r\}$. More explicitly, the Schubert variety $X_\omega(F_\bullet)$ is defined by

$$X_\omega(F_\bullet) := \{\ (V_1,\dots,V_r) \in F(k_1,\dots,k_r;n) \mid \dim(V_i \cap F_j) \geq \#\{\alpha \leq i : \omega(\alpha) > n-j\}\ \forall\ i,j\ \}$$

The Poincaré duals of the classes of all the Schubert varieties form an additive basis for the cohomology of the flag variety. The structure constants with respect to this basis are known as the *Littlewood-Richardson coefficients* for flag varieties.

For future reference we note that given a Schubert cycle σ_λ in $G(k,n)$, there is a special Schubert cycle $X_\lambda^{(d)}(F_\bullet)$ in $F(k-d,k+d;n)$ defined by

$$X_\lambda^{(d)}(F_\bullet) := \{\ (V_1,V_2) \mid \dim(V_1 \cap F_{n-i-\lambda_{k-i}}) \geq k-d-i,\ \dim(V_2 \cap F_{n-k+j-\lambda_j}) \geq j\ \}$$

where $1 \leq i \leq k-d$ and $1 \leq j \leq k$.

We need a convenient notation for Schubert varieties of r-step flag varieties. In analogy with the notation for the Grassmannians we will use the notation $\sigma_{\lambda_1,\cdots,\lambda_{k_r}}^{\delta_1,\cdots,\delta_{k_r}}$. The bottom row denotes the usual partition corresponding to the k_r-plane V_r in V treated as a Schubert cycle in $G(k_r,n)$. The numbers δ_i are integers between 1 and r. For a Schubert cycle in $F(k_1,\dots,k_r;n)$, k_1 of the upper indices will be 1 and $k_i - k_{i-1}$ of them will be i. The flag $F_\bullet$ induces a complete flag $G_\bullet$ on the largest vector space V_r. For each j, there exists a smallest i such that

$$\dim(V_i \cap G_j) = \dim(V_i \cap G_{j-1}) + 1.$$

For a Zariski-open subset of the Schubert variety this index will be constant. In that case we write i on top of λ_j. In the case of Grassmannians this notation reduces to the ordinary notation with a sequence of 1s on the top row. For the complete flag variety the top row becomes the permutation defining the Schubert cycle.

Example. Fix a flag $F_1 \subset \cdots \subset F_6$ in V^6. The Schubert cycle $\sigma_{3,2}$ in $G(2,6)$ denotes the two-dimensional subspaces of V^6 that meet F_2 in a line and are contained in F_4. The corresponding special Schubert cycle $X^1_{\sigma_{3,2}}$ has class $\sigma^{2,1,2}_{2,1,0}$ in $F(1,3;6)$. The Schubert cycle $\sigma^{2,1,2}_{2,1,0}$ denotes the pairs of subspaces $V_1 \subset V_2$ where V_1 has dimension one and V_2 has dimension 3. V_2 is required to meet F_2 in dimension one, F_4 in dimension 2 and be contained in F_6. V_1 lies in the intersection of V_2 with F_4.

2.2. **Gromov-Witten invariants.** Let $\overline{M}_{0,m}(\mathbb{G}(k,n),d)$ denote the Kontsevich space of genus zero stable maps to $\mathbb{G}(k,n)$ of Plücker degree d. The Kontsevich space is equipped with m evaluation morphisms,

$$\rho_1, \dots, \rho_m : \overline{M}_{0,m}(\mathbb{G}(k,n),d) \to \mathbb{G}(k,n),$$

where the i-th evaluation morphism maps a stable map to the image of the i-th marked point.

Given m Schubert classes $\sigma_{\lambda_1}, \cdots, \sigma_{\lambda_m}$ in $\mathbb{G}(k,n)$, the Gromov-Witten invariant $I_d(\sigma_{\lambda_1}, \dots, \sigma_{\lambda_m})$ is defined by the formula

$$I_d(\sigma_{\lambda_1}, \dots, \sigma_{\lambda_m}) = \int_{\overline{M}_{0,m}(\mathbb{G}(k,n),d)} \rho_1^*(\sigma_{\lambda_1}) \cup \cdots \cup \rho_m^*(\sigma_{\lambda_m}).$$

Since the three-pointed Gromov-Witten invariants ($m=3$) give the structure constants of the small quantum cohomology ring, they are called the *quantum Littlewood-Richardson coefficients.*

Let $\Sigma_1, \cdots, \Sigma_m$ be general (with respect to the $\mathbb{P}GL(n+1)$ action) Schubert cycles representing the Poincaré duals of the classes $\sigma_{\lambda_1}, \dots, \sigma_{\lambda_m}$, respectively. The following lemma asserts that the Gromov-Witten invariant is equal to the number of rational curves that intersect Σ_i.

Lemma 2.2. *(*[FP] *Lemma 14) The scheme theoretic intersection*

$$\rho_1^{-1}(\Sigma_1) \cap \cdots \cap \rho_m^{-1}(\Sigma_m)$$

is a finite number of reduced points in $M_{0,m}(\mathbb{G}(k,n),d)$. Moreover,

$$I_d(\sigma_{\lambda_1}, \dots, \sigma_{\lambda_m}) = \#\ \rho_1^{-1}(\Sigma_1) \cap \cdots \cap \rho_m^{-1}(\Sigma_m).$$

Remark 2.3. Furthermore, by Kleiman's Transversality Theorem [Kl] one can conclude that the curves contributing to the Gromov-Witten invariants are non-degenerate curves and the restriction of the tautological bundle of $\mathbb{G}(k,n)$ to the curves have balanced splitting (i.e. the degree of any two summands in the Grothendieck decomposition differ by at most one).

2.3. **Scrolls.** Let $r_1 \leq \cdots \leq r_k$ be non-negative integers, not all equal to zero. We let r be the sum $\sum_{i=1}^k r_i$. Let $S_{r_1,\cdots,r_k}$ denote the k-dimensional rational normal scroll in $\mathbb{P}^{r+k-1}$.

To construct it take k rational normal curves of degree $r_1, \dots, r_k$ in $\mathbb{P}^{r+k-1}$ such that the span of any $k-1$ of them is disjoint from the span of the remaining one. Fix an isomorphism between each of these curves and an abstract $\mathbb{P}^1$. $S_{r_1,\cdots,r_k}$ is the union of the $k-1$ planes spanned by the points corresponding under the isomorphism. We allow some of the integers r_i to be zero. In that case we obtain cones over smaller dimensional scrolls. A scroll is *balanced* if $|r_i - r_j| \leq 1$ for $1 \leq i,j \leq k$. It is *perfectly balanced* if all r_i are equal.

Abstractly a scroll is the projectivization of a vector bundle of rank k on $\mathbb{P}^1$. Since any vector bundle E over $\mathbb{P}^1$ is a sum of line bundles, we can express the projectivization as

$$\mathbb{P}E = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(-r_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(-r_k)).$$

The scroll $S_{r_1,\cdots,r_k}$ is the image of $\mathbb{P}E$ under the linear series $\mathcal{O}_{\mathbb{P}E}(1)$.

Scrolls as rational curves in the Grassmannian. Since a scroll $S_{r_1,\cdots,r_k}$ is a family of $k-1$ planes parameterized by a rational curve, it gives rise to a rational curve C in $\mathbb{G}(k-1,r+k-1)$. The curve C is non-degenerate (i.e., does not lie in a subgrassmannian of the form $\mathbb{G}(k-1,s)$ for $s < r+k-1$) and of Plücker degree $r = \sum r_i$.

Conversely, any irreducible, non-degenerate curve C in the Grassmannian $\mathbb{G}(k-1,r+k-1)$ of degree r gives rise to a scroll of degree r in $\mathbb{P}^{r+k-1}$. There are non-isomorphic scrolls of degree r in $\mathbb{P}^{r+k-1}$. The splitting type of the restriction of the tautological bundle of $\mathbb{G}(k-1, r+k-1)$ to C determines the isomorphism type of the scroll (see [C]). We, therefore, obtain the following corollary of Lemma 2.2 and Remark 2.3.

Corollary 2.4. *The Gromov-Witten invariant $I_d(\sigma_{\lambda_1}, \dots, \sigma_{\lambda_m})$, where $\sigma_{\lambda_1}, \dots, \sigma_{\lambda_m}$ are Schubert cycles in $G(k,n)$, is equal to the number of balanced scrolls of degree d and dimension $d+k-1$ in $\mathbb{P}^{n-1}$ containing m fibers satisfying the specified Schubert conditions.*

In [BKT] the authors relate the quantum three-point invariants of degree d for $G(k,n)$ to ordinary intersections in $F(k-d, k+d; n)$. Since we will use their Proposition 1 and Corollary 1, we summarize their results in terms of the geometric point of view offered by Corollary 2.4.

Lemma 2.5. *(*[BKT] *Prop.1) The only non-zero three-point quantum invariant of $G(d, 2d)$ of degree d is*

$$I_d(\sigma_{d,\cdots,d}, \sigma_{d,\cdots,d}, \sigma_{d,\cdots,d}) = 1.$$

By dimension restrictions, the stated invariant is the only non-zero degree d invariant. The invariant is equal to one because there is a unique scroll of degree and dimension d in $\mathbb{P}^{2d-1}$ containing three general $d-1$ dimensional linear spaces A, B, C as fibers. In such a scroll, through every point p of A there is a unique line meeting B and C. This line is $\overline{pB} \cap \overline{pC}$, the intersections of the span of p and B and the span of p and C. This uniquely constructs the scroll.

Lemma 2.6. *(*[BKT] *Cor.1) Let λ, μ, ν be partitions and $d \geq 0$ be an integer satisfying*

(1) $$|\lambda| + |\mu| + |\nu| = k(n-k) + dn.$$

Then the degree d three-point Gromov-Witten invariants of $G(k,n)$ equal the ordinary three-point intersections of special Schubert varieties (see §2.1) in the flag variety $F(k-d, k+d; n)$:

$$I_d(\sigma_\lambda, \sigma_\mu, \sigma_\nu) = \int_{F(k-d,k+d;n)} [X_\lambda^{(d)}] \cup [X_\mu^{(d)}] \cup [X_\nu^{(d)}].$$

By Corollary 2.4 the Gromov-Witten invariant is equal to the number of balanced scrolls of degree d and dimension k in $\mathbb{P}^{n-1}$. Such a scroll is a cone over a perfectly balanced scroll with vertex a linear space of dimension $k-d-1$. It spans a $\mathbb{P}^{k+d-1}$. To each scroll contributing to the Gromov-Witten invariant one associates the pair of linear spaces consisting of the vertex and the span of the cone. This gives the required bijection.

3. Mondrian tableaux and a Littlewood-Richardson rule for Grassmannians

In this section we describe a way to compute the Littlewood-Richardson coefficients of Grassmannians in terms of combinatorial objects called Mondrian tableaux.

Mondrian tableaux. A *Mondrian tableau* associated to a Schubert class $\sigma_{\lambda_1,\cdots,\lambda_k}$ in $G(k,n)$ is a collection of k nested squares centered along the anti-diagonal of an $n \times n$ square such that

- The squares are labeled by integers $1, \dots, k$, where a square with larger index contains all the squares with smaller index.
- The side-length of the j-th square is $n-k+j-\lambda_j$.

Figure 1 depicts two Mondrian tableaux for $\sigma_{2,1}$ in $G(3,6)$.

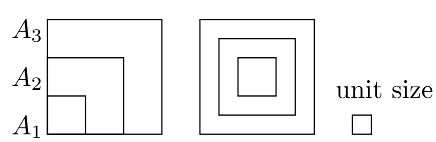

Figure 1. Two Mondrian tableaux associated to $\sigma_{2,1}$ in $G(3,6)$.

Notation 3.1. For the rest of this paper, we will denote combinatorial objects, such as squares in a Mondrian tableau, by capital letters in the math font (e.g., A_i). We will denote the geometric objects represented by the combinatorial objects by the corresponding letter in Roman font (e.g., A_i).

In a Mondrian tableau, a square of side length s represents a vector space of dimension s. The unit squares along the anti-diagonal of an $n \times n$ square represent a basis of the underlying vector space. The vector space represented by a square S is the span of the basis elements corresponding to the unit squares in S. If a square S_1 is contained in another square S_2, then the corresponding linear space S_1 is a subspace of the linear space S_2. A Mondrian tableau depicts the subvariety of $G(k,n)$ where the k-planes are required to intersect the linear spaces S depicted by the squares S in the tableau in dimension at least the number of squares contained in S.

For example, in Figure 1, the Mondrian tableau in the left panel has three squares labeled A_1, A_2, A_3. The corresponding vector spaces $\mathrm{A}_1, \mathrm{A}_2, \mathrm{A}_3$ have dimensions $2, 4$, and 6, respectively. The 3-planes parameterized by the variety represented by this tableau are required to intersect A_i in dimension i. Note that the Mondrian tableau in the second panel of Figure 1 also depicts $\sigma_{2,1}$. However, the defining flag in this panel differs from the first.

3.1. **The game informally.** To multiply two Schubert classes σ_λ and σ_μ in $G(k,n)$ we place the tableau associated to σ_λ (respectively, σ_μ) lower left (respectively, upper right) justified with the $n \times n$ square. We will denote the squares corresponding to λ and μ by A_i and B_j, respectively. Figure 2 shows the initial tableau for the multiplication $\sigma_{2,1,1} \cdot \sigma_{1,1,1}$ in $G(3,6)$.

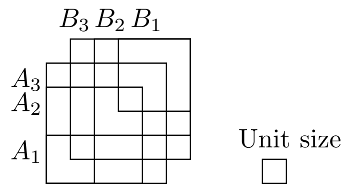

FIGURE 2. The initial tableau for $\sigma_{2,1,1} \cdot \sigma_{1,1,1}$ in $G(3,6)$.

Initially the two Schubert cycles are defined with respect to two transverse flags. $GL(n)$ acts with a dense orbit on the product of two complete flag varieties. For every pair of flags $(F_\bullet, G_\bullet)$ in the dense orbit there is a basis $e_1, \dots, e_n$ of the vector space V such that $F_i =< e_1, \dots, e_i >$ and $G_i =< e_n, e_{n-1}, \dots, e_{n-i+1} >$. The initial tableau depicts the two Schubert varieties defined in terms of such a pair of opposite flags. If the intersection of the two Schubert cycles is non-empty, then the Schubert cycles have to satisfy certain conditions. Once we check that these conditions are satisfied, we will make the flags less transverse via codimension one degenerations and trace the limit of the intersection of the Schubert cycles.

• **The MM (must meet) rule.** We check that A_i intersects B_{k-i+1} in a square of side length at least one for every i between 1 and k. If not, we stop. The Schubert cycles have empty intersection.

In a k-dimensional vector space V^k every i-dimensional subspace (such as $V^k \cap \mathrm{A}_i$) **M**ust **M**eet every $(k-i+1)$-dimensional subspace (such as $V^k \cap \mathrm{B}_{k-i+1}$) in at least a one-dimensional subspace. The intersection of two Schubert cycles is zero if and only if the initial tableau formed by the two cycles does not satisfy the MM rule.

• **The OS (outer square) rule.** We call the intersection of A_k and B_k the **O**uter **S**quare of the tableau. We replace every square with its intersection with the outer square and keep their labels the same. See Figure 3

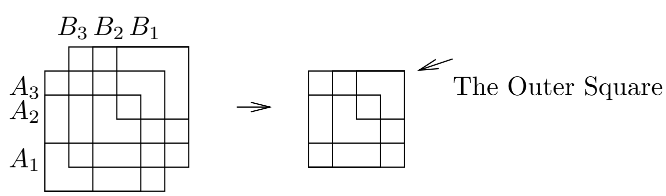

FIGURE 3. The OS rule applied to $\sigma_{2,1,1} \cdot \sigma_{1,1,1}$ in $G(3,6)$.

Since the k-planes are contained in both A_k and B_k, they must be contained in their intersection. Figure 2 shows an example in $G(3,6)$.

• **The S (span) rule.** We check that A_i and B_{k-i} touch or have a common square. If not, we remove the rows and columns between these squares as shown in Figure 4.

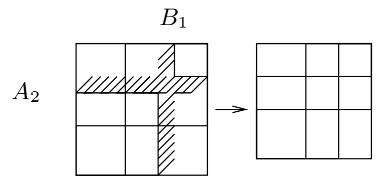

FIGURE 4. The S rule.

This rule corresponds to the fact that a k-dimensional vector space lies in the **S**pan of any two of its subspaces of complimentary dimension whose only intersection is the origin. This rule removes any basis element of V that is not needed in expressing the k-planes parameterized by the intersection of the two Schubert varieties.

Once we have performed these preliminary steps, we will inductively build a new flag (the D flag) by degenerating the two flags (the A and B flags). At each stage of the game we will have a partially built new flag (depicted by D squares that arise as intersections of A and B squares) and partially remaining A and B flags (depicted by squares $A_i, \ldots, A_k$ and B_k, $B_{k-i}, \ldots, B_1$). After nesting the D squares, we will increase the dimension of the intersection of A_i with B_{k-i} by one in order of increasing i. We will depict this move in the Mondrian tableau by sliding A_i anti-diagonally up by one unit. Assuming that there are no squares left justified with A_i, the corresponding degeneration can be described as follows:

Let s be the side-length of A_i and suppose that initially A_i and B_{k-i} intersect in a square of side-length r. There is a family of s-dimensional linear spaces $\mathrm{A}_i(t)$ parameterized by an open subset $0 \in U \subset \mathbb{P}^1$ such that over the points $t \in U$ with $t \neq 0$, the dimension of intersection $\mathrm{A}_i(t) \cap \mathrm{B}_{k-i}$ is equal to r and when $t = 0$, the dimension of intersection $\mathrm{A}_i(0) \cap \mathrm{B}_{k-i}$ is $r+1$. Denoting the basis vectors represented by the unit squares along the diagonal by $e_1, \ldots, e_n$, we explicitly take the family to be

$$\mathrm{A}_i(t) = \text{ the span of } \{(te_1 + (1-t)e_{s+1}, e_2, \ldots, e_s\}.$$

When $t = 1$, we have our original vector space A_i represented by the old position of the square A_i. When $t = 0$, we have the new vector space $\mathrm{A}_i(0)$ represented by the new position of the square A_i. When $t = 0$, the intersection of Schubert varieties defined with respect to the A and B flags either remains irreducible or breaks into two irreducible components. The Littlewood-Richardson rule records these possibilities and can be informally phrased as:

If the k-planes in the limit do not intersect $\mathrm{A}_i(0) \cap \mathrm{B}_{k-i}$*, then they must be contained in their new span.*

Before explaining how to use Mondrian tableaux to give a systematic algorithm for computing Littlewood-Richardson coefficients, we give the example of σ_1^2 in $G(2,4)$. Recall that geometrically σ_1^2 describes the cycle of projective lines in $\mathbb{P}^3$ that intersect two general projective lines. See Figure 5 where we draw the geometric pictures projectively and place the corresponding Mondrian tableau underneath. Briefly, we start by specializing the two skew lines l_1, l_2 to two intersecting lines l_1', l_2. (In the Mondrian tableau this degeneration is depicted by sliding the lower left square corresponding to l_1 up by one unit.) In the limit, lines that intersect both l_1' and l_2 either contain the point of intersection $l_1' \cap l_2$ or are contained in the plane spanned by l_1' and l_2. (These possibilities are depicted by drawing the square corresponding to the point $l_1' \cap l_2$ and restricting the tableau to the square representing the span of l_1' and l_2, respectively.) An easy tangent space calculation shows that each of these limits occur with multiplicity one. Hence we conclude $\sigma_1^2 = \sigma_{1,1} + \sigma_2$.

3.2. **The game.** To multiply two Schubert classes σ_λ and σ_μ in $G(k,n)$ we place the tableau associated to σ_λ (respectively, σ_μ) lower left (respectively, upper right) justified with the $n \times n$ square. We apply the MM, OS and S rules. We need to know the tableaux that correspond to varieties that occur during the degenerations. The admissible Mondrian tableaux characterize the tableaux that occur (see Figure 8 for examples).

Mondrian tableaux. A Mondrian tableau for $G(k,n)$ is a diagram contained in an $n \times n$ square consisting of a collection of squares whose anti-diagonal lies on the anti-diagonal of the $n \times n$ square.

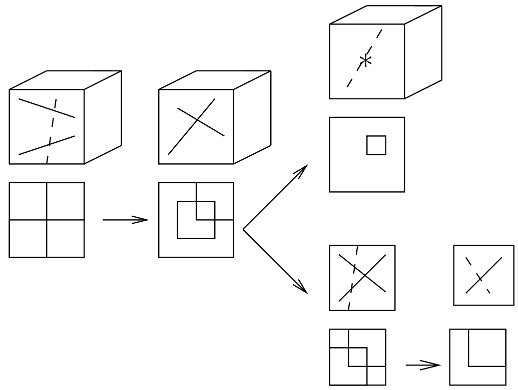

FIGURE 5. Computing $\sigma_1^2 = \sigma_{1,1} + \sigma_2$ in $G(2,4) = \mathbb{G}(1,3)$.

The admissible Mondrian tableaux. A Mondrian tableau is *admissible* for $G(k,n)$ if the squares that constitute the tableau are labeled as indexed A, B or D square such that

(1) The squares $A_k = B_k$ form the outer square. They have side length $m \leq n$ and contain the entire tableau.
(2) The A squares are all nested, distinct, left aligned and strictly contain all the D squares. If the number of D squares is $i-1 < k$, then the A squares are $A_i, A_{i+1}, \dots, A_k$ with the smaller index corresponding to the smaller square. (In particular, the total number of A and D squares is k.)
(3) The B squares are all nested, distinct and right aligned. They are labeled $B_k, B_{k-i}, B_{k-i-1}, \dots, B_1$, where a smaller square has the smaller index. (In particular, the number of B squares equals the number of A squares.) The A and B squares satisfy the MM and S rules. The D squares may intersect B_{k-i}, but none are contained in B_{k-i}.
(4) The D squares are labeled $D_1, \dots, D_{i-1}$. They do not need to be nested; however, there can be at most one unnested D square. An *unnested D square* is a D square that does not contain every D square of smaller index. More precisely, if D_j does not contain all the D squares of smaller index, then it does not contain any of the D squares of smaller index; it is contained in every D square of larger index; and $D_r \subset D_s$ for every $r < s$ as long as r and s are different from j. All the D squares of index lower than j are to the lower left of D_j. D_{j-1} and D_j share a common square or corner. Figure 6 shows a typical configuration of D squares.

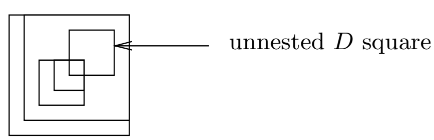

FIGURE 6. A typical configuration of D squares.

(5) Let S_1 and S_2 be any two squares of the tableau. If the number of squares contained in their span but not contained in S_1 is r, then the side-length of S_1 is at least r less than the side-length of their span.

The moves. Let M be an admissible Mondrian tableau with an outer square of side length m.

- If M consists of k nested squares, then M corresponds to a Schubert cycle. The algorithm terminates.
- Otherwise, simplify M as follows. If all the D squares in M are nested, define the *active square* to be the smallest A square A_i. If D_j is the unique unnested D square in M, define the *active square* to be D_{j-1}. Move the active square anti-diagonally up by one unit. Move the D squares that touch the lower left hand corner of the active square also up by one unit. Keep all the remaining squares fixed. Replace M by the following two tableaux unless Tableau 2 is not admissible or if the active square D_{j-1} starts being contained in the unnested square D_j. In the latter two cases, replace M with Tableau 1 only.

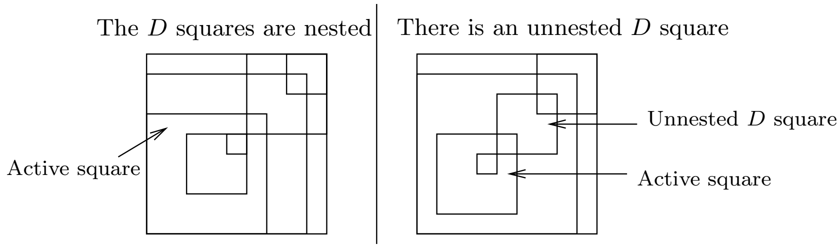

FIGURE 7. Two typical admissible Mondrian tableaux.

Tableau 1. If the active square is A_i, delete A_i and B_{k-i}. Draw their new intersection and label it D_i. Keep their old span as the outer square. If D_i does not intersect or touch B_{k-i-1}, slide all the D squares anti-diagonally up until D_i touches B_{k-i-1}. If the active square is D_{j-1}, delete D_{j-1} and D_j. Draw their new intersection and label it D_{j-1}. Draw their old span and label it D_j. If D_{j-2} does not intersect or touch the new D_{j-1}, slide all the D squares of index $j-2$ anti-diagonally until D_{j-2} touches D_{j-1}. All the remaining squares stay as in M.

Tableau 2. If the active square is A_i, we shrink the outer square by one unit so that it passes along the new boundary of A_i and B_{k-i} and we delete the column and row that lies outside this square. The rest of the squares stay as in M. If the active square is D_{j-1}, we place the squares we move in their new positions and keep the rest of the squares as in M.

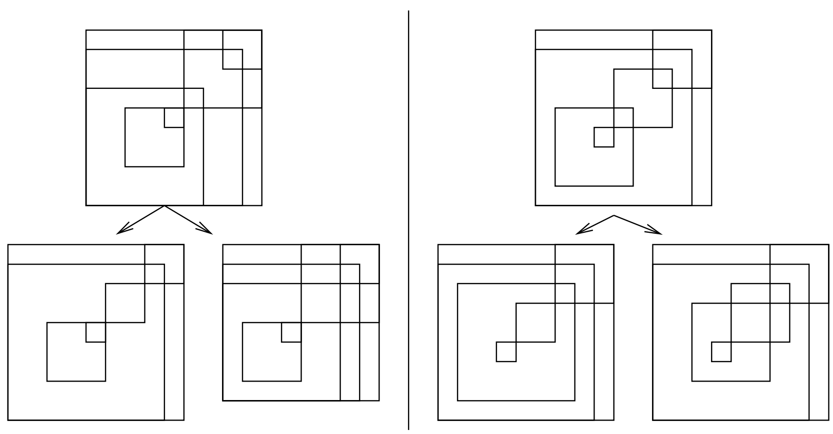

FIGURE 8. Simplifying the Mondrian tableaux in Figure 7.

The two tableaux obtained from M are depicted in Figure 8. Geometrically in the first tableau the k-plane intersects the new intersection $\mathrm{A}_i \cap \mathrm{B}_{k-i}$. In the second tableau, the k-plane lies in the new span of A_i and B_{k-i}.

Remark 3.2. Note that if the active square is A_i, Tableau 2 is not admissible if either A_{i+1} has side-length one larger than A_i or if B_{k-i} has side-length $m-i$ (informally, if A_i or B_{k-i} are as large as possible given A_{i+1} and B_k). If the active square is D_{j-1}, then Tableau 2 is not admissible either if the side-length of D_j is not at least $j-1$ units smaller than the side-length of the span of D_j and D_{j-1} or if D_{j-1} contains D_j as a result of the move (informally, if D_{j-1} and D_j are as large as possible). In these cases we replace M with only Tableau 1.

When to stop. After applying the MM, OS and S rules, the initial tableau is an admissible Mondrian tableau. Similarly, given an admissible tableau the moves clearly give rise to one or two new admissible tableaux. Hence we can apply the moves to each of the tableaux that result from the moves. As we apply the moves, we decrease the number of A and B squares and increase the number of nested D squares in each tableau. We stop applying the algorithm to a tableau when the tableau first consists of k nested squares. A collection of k nested squares corresponds to a Schubert cycle (we read these Schubert cycles as cycles in $G(k, n)$). The algorithm terminates when all the tableaux consist of k nested squares.

Remark 3.3. To facilitate the translation between the final Mondrian tableaux and Schubert cycles, the reader may wish to slide all the squares to the lower left of the outer square until they are all left aligned. However, this loses important information. The final tableau carries more information than just the class of the variety. It records the vector spaces with respect to which the limit Schubert cycle is defined.

Remark 3.4. In a Mondrian tableau the only relevant information is encoded along the anti-diagonal. The particular representation is chosen for notational convenience.

Remark 3.5. The labels of an admissible tableau can be recovered from the diagram alone, hence in the sequel we will not label the tableaux. Starting from the upper right one labels the right justified squares other than the outer square as B squares with increasing index starting with B_1. Starting from the upper left one labels the right justified squares as A squares of decreasing index starting with A_k and taking care that the number of A and B squares are equal. The remaining squares are D squares and are indexed in the obvious manner.

The intersection of two Schubert cycles equals the totality of all possible outcomes of a game of Mondrian tableaux starting with the two cycles in an $n \times n$ square. To illustrate how the algorithm works we compute $\sigma_{2,1}^2$ in $G(3, 6)$ (see Figure 9).

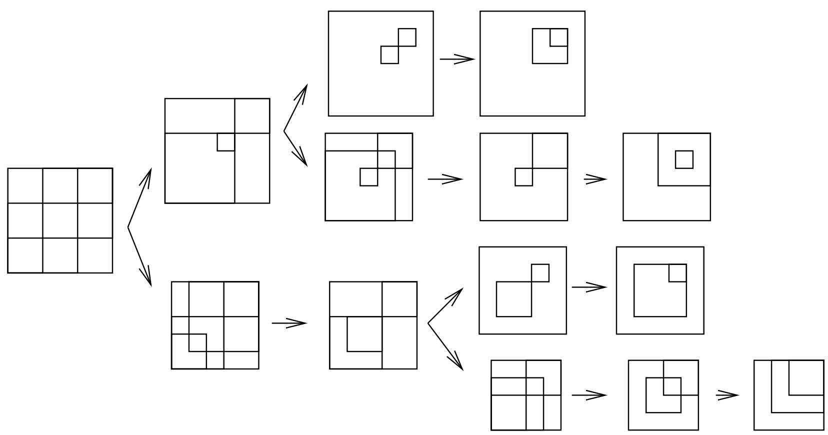

FIGURE 9. The product $\sigma_{2,1}^2 = \sigma_{3,3} + 2\sigma_{3,2,1} + \sigma_{2,2,2}$ in $G(3, 6)$.

Theorem 3.6. *The Littlewood-Richardson coefficient $c_{\lambda,\mu}^{\nu}$ of $G(k, n)$ equals the number of times σ_ν results in a game of Mondrian tableaux starting with σ_λ and σ_μ in an $n \times n$ square.*

Proof. The proof is geometric and has two components. We interpret the moves in the Mondrian tableaux as degenerations of the flags defining the two Schubert varieties. First, by a dimension count we argue that the two limits corresponding to the tableaux in the game are the only set-theoretic limits of our degeneration. Next, we argue that each limit occurs with multiplicity one.

Definition of the relevant subvarieties of the Grassmannian. To every admissible Mondrian tableau M we can associate an irreducible subvariety Σ_M of the Grassmannian. Suppose the smallest A square in M is A_i. Let $AB_r = A_r \cap B_{k-r+1}$ for $i < r \leq k$. We will denote the corresponding vector space by AB_r. If the D squares are not nested, assume that D_j is the D square not containing the squares of smaller index. We will denote the side-length of a square S by $|S|$.

If all the D squares are nested, then consider the k-dimensional subspaces Λ of V that satisfy

(1) $\dim(\Lambda \cap \mathrm{D}_s) = s$ for $1 \leq s < i$,
(2) $\dim(\Lambda \cap \mathrm{A}_i) = i$,
(3) $\dim(\Lambda \cap \mathrm{AB}_r) = 1$ for $i + 1 \leq r \leq k$, and
(4) Λ is spanned by its intersections with A_i and AB_r for $i + 1 \leq r \leq k$.

This locus is an irreducible quasi-projective variety in $G(k,n)$. We will denote its closure by Σ_M (see Figure 10).

If there is an unnested square D_j, then consider the k-dimensional subspaces Λ of V that satisfy

(1) $\dim(\Lambda \cap \mathrm{D}_s) = s$ for $1 \leq s < i, s \neq j$,
(2) $\dim(\Lambda \cap \mathrm{D}_j) = 1$,
(3) $\dim(\Lambda \cap \mathrm{A}_i) = i$,
(4) $\dim(\Lambda \cap \mathrm{AB}_r) = 1$ for $i+1 \leq r \leq k$,
(5) Λ has a j-dimensional subspace spanned by its intersection with D_{j-1} and D_j. Λ is spanned by its intersections with A_i and AB_r for $i+1 \leq r \leq k$.

This locus is also an irreducible quasi-projective subvariety of $G(k,n)$. We will denote its closure by Σ_M.

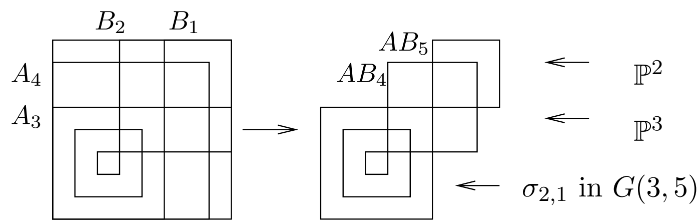

FIGURE 10. The variety associated to a Mondrian tableau.

Note that in both cases it is clear that the varieties defined are irreducible. If there are no unnested D squares, then the variety is described as an open set of a tower of projective bundles over an open set in a Schubert variety in $G(i, |A_i|)$. If D_j is an unnested square, then the variety is realized as an open set in a projective bundle over a subvariety in $G(i, |A_i|)$. The subvariety is irreducible because it maps to an open set in a projective bundle over $G(j-1, |D_{j-1}|)$ with irreducible equi-dimensional fibers. The same description allows us to compute the dimension of these varieties.

Lemma 3.7. *The dimension of the variety associated to a Mondrian tableau is*

$$\sum_{l=1}^{i-1} |D_l| + |A_i| + \sum_{r=i+1}^{r} |AB_r| - \frac{(i-1)i}{2} - k$$

if all the D squares are nested and

$$\sum_{l=1}^{i-1} |D_l| + |A_i| + \sum_{r=i+1}^{k} |AB_k| + j - 1 - \frac{(i-1)i}{2} - k$$

if D_j is unnested.

One can rephrase Lemma 3.7 more geometrically by saying that the dimension of a variety corresponding to a Mondrian tableau is the sum of the side-lengths of the squares D_l, A_i and AB_r for $i+1 \leq r \leq k$ minus the number of containment relations these squares satisfy. When we count the number of containment relations, we always include a square itself among the squares contained in it.

Generalized Mondrian tableaux. We will need a generalization of this dimension estimate both for the proof and the Littlewood-Richardson rule in the case of two-step flag varieties. Let a *generalized Mondrian tableau* for $G(k,n)$ be any Mondrian tableau made up of exactly k squares satisfying the following two properties:

(1) None of the squares are equal to the span of the squares contained in them.
(2) Let S_1 and S_2 be any two squares in the tableau. If the number of squares contained in their span but not contained in S_1 is r, then the side-length of S_1 is at least r less than the side-length of their span.

We can associate an irreducible subvariety of the Grassmannian $G(k,n)$ to a generalized Mondrian tableau. We first define an open subset of the variety by requiring the k-planes to meet the vector spaces represented by each square in dimension equal to the number of squares contained in that square (including itself). We further require the vector subspaces of the k-planes contained in the vector spaces represented by any two squares to only meet along the subspaces represented by squares common to both of the squares and otherwise to be independent. The variety associated to the generalized Mondrian tableau is the closure of the quasi-projective variety parameterizing such k-planes.

To see that the variety associated to a generalized Mondrian tableau MV_k in $G(k,n)$ is irreducible we inductively construct it. If there exists a square containing all the other squares, we can delete this square and we get a generalized Mondrian tableau in $G(k-1,n-1)$ and a variety MV_{k-1} associated to it. The variety MV_k is an open set in a projective bundle over MV_{k-1}, hence by induction it is irreducible. If there does not exist a square containing all the other squares, we take the upper right most square not contained in any other square. If we delete this square, we get a generalized Mondrian tableau in $G(k-1,n)$. The variety MV_k over MV_{k-1} is an open set in a projective bundle. By induction we conclude that these varieties are irreducible. This construction also allows us to compute the dimension of these varieties. The following basic lemma is elementary.

Lemma 3.8. *The dimension of the variety associated to a generalized Mondrian tableau is equal to the sum of the side-lengths of all the squares minus one for every containment relation among the squares where we count a square itself among the squares contained in it.*

We can now proceed with the proof. The algorithm tells us to move A_i if all the D squares are nested or to move D_{j-1} if D_j is unnested. Lemma 3.7 implies that the varieties associated to the Mondrian tableaux that arise as a result of the moves have the same dimension as the variety associated to the Mondrian tableau prior to the move. In Tableau 1 the sum of the side-lengths of the squares increases by one and there is one more containment among the squares compared to the original tableau. In Tableau 2 both the sum of the side-lengths of the squares and the containment relations remain constant.

We need to argue that there are no other possibilities. Here we will treat the case when we move A_i. We need to argue that after the degeneration the limit cycle is supported on the union of the two loci described in the game. The strategy is very simple. The condition of intersecting a linear space in at least a given dimension is a closed condition. Hence, the limiting k-planes continue to satisfy such conditions. A priori there may be other irreducible components of the limit cycle that are supported along loci where the k-plane intersects the vector spaces in the A, B and D flags in special configurations. We will construct varieties associated to generalized Mondrian tableau that have to contain such loci. We will then show that the conditions on these generalized Mondrian tableaux force the dimensions of the associated varieties to be less than the dimension of the original variety. Hence these loci cannot support an irreducible component of the flat limit.

Reduction: We can assume that the only square that moves is A_i. In the rule if the D squares are left justified with A_i, they are forced to move with A_i. However, we can carry out the same degeneration in stages moving first the smallest D square left justified with A_i. If there are D squares that are left justified with A_i, the Mondrian tableau looks like the tableau in Figure 11.

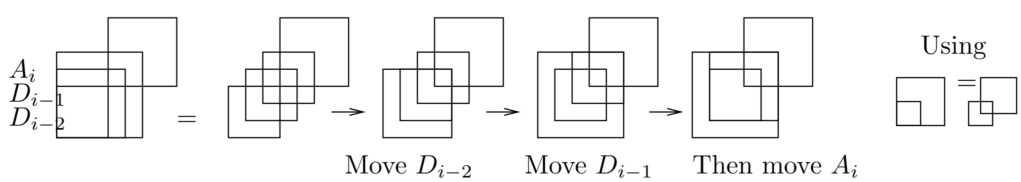

Figure 11. Reduction to the case when we move one square at a time.

Suppose D_j is the smallest D square that abuts the lower left of A_i. The variety defined by such a tableau is equivalent to the variety defined by the tableau where D_{j+1} is one smaller and unnested with respect to D_j. By considering this equivalent generalized Mondrian tableau, what we prove below shows that when we move D_j by one unit, the only limit corresponds to sliding D_j up by one unit in the original tableau. We can thus carry out the step of moving A_i in many steps but where at each stage we move one square. The outcome, of course, is the same as moving A_i and sliding the D squares that touch its lower left up. By this reduction we can assume that at each stage we move only one square. We can further assume that this square is A_i.

Suppose at a general point $[\Lambda]$ of a component of the limit cycle, the linear space Λ does not intersect the intersection A_i and B_{k-i} except for the origin, then Λ must lie in the new span of A_i and B_{k-i}. The intersection of Λ with A_i and B_{k-i} have complementary dimension and since they have trivial intersection, they must span Λ. Since the intersection conditions are closed conditions, Λ must continue to satisfy the incidence conditions required in Tableau 2 with respect to the D squares.

If at a general point on a component of the limit cycle the one-dimensional subspace of the k-plane contained in AB_r remains independent from the $r-1$-dimensional subspace contained in the span of A_i and AB_t for $i+1 \leq t < r$ and all $i+1 \leq r \leq k$, then that component must be contained in the variety represented by the Mondrian tableau described in Tableau 2. Since the variety represented by Tableau 2 is irreducible and of the correct dimension, such a component must be supported on the variety represented by Tableau 2.

If the one-dimensional subspaces contained in AB_r do not remain independent at a general point of a component of a limit cycle, we can build a generalized Mondrian tableau that contains this component. Suppose r is the smallest index for which the one-dimensional subspace contained in AB_r lies in the span of A_i and AB_j for $i+1 \leq j \leq t_1$. We then replace AB_r with its intersection with A_{t_1}. We keep all the AB squares of index smaller than t_1 and larger than r unchanged. We draw the intersection of AB_r and A_{t_1}. We delete the AB_j squares for $t_1 \leq j \leq r$ and draw the squares $A_{j+2} \cap B_{k-j}$ for $t_1 \leq j \leq r$.

If all the remaining one-dimensional linear spaces of the k-plane contained in AB_l for $l > r$ remain independent from the span of the previous ones, then this generalized Mondrian tableau must contain the limit cycle. Otherwise, we repeat the procedure for AB_l. Namely, we replace AB_l with its intersection with the smallest square A_{t_2} that contains the one-dimensional subspace contained in AB_l. We adjust the squares contained between A_{t_2} and A_l by replacing them with the intersection of the same B square passing through the lower left of the square with the A square of one larger index.

We can thus build a generalized Mondrian tableau that contains the hypothetical limit component. The dimension of such a generalized Mondrian tableau is easy to compare to the dimension of Tableau 2. Each time we repeat the procedure we do not change the sum of the side lengths of the squares defining the tableau. On the other hand, if we take the intersection of AB_r and A_{t_1}, we increase the containment relations between the squares by $r-t_1$. By Lemma 3.8 the loci where the linear subspaces of the k-plane contained in AB_r do not remain independent from the span of the previous ones cannot form a component of the limit cycle since they have strictly smaller dimension (see Figure 12). We conclude that if the linear subspaces of the k-plane contained in A_i and B_{k-i} remain independent at a general point of a limit cycle, then this cycle must be supported along the variety associated to Tableau 2.

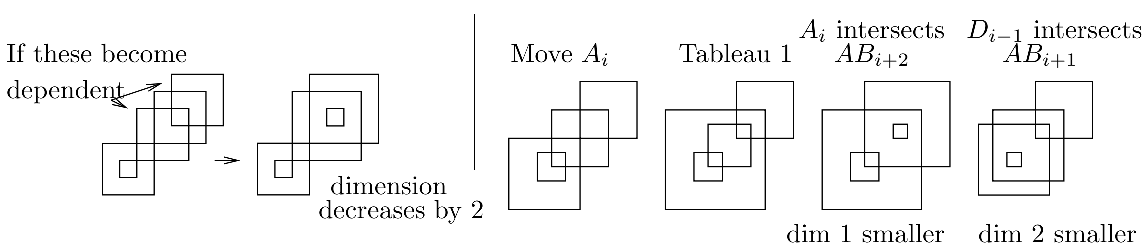

FIGURE 12. The dimension decreases when the subspaces in the AB squares do not remain independent.

We observe that if $\dim \mathrm{A}_{i+1} = \dim \mathrm{A}_i + 1$, then the move forces A_i to contain AB_{i+1}, hence it is not possible for Λ not to meet $\mathrm{A}_i \cap \mathrm{B}_{k_i}$. Similarly when $\dim \mathrm{B}_{k-i} = m-i$, then when we move A_i, Λ must meet $\mathrm{A}_i \cap \mathrm{B}_{k-i}$. In these two cases it is not possible for the linear spaces contained in A_i and B_{k-i} to remain independent. The assumption that the k-plane is still contained in the new span gives smaller dimensional loci. This is the reason for discarding Tableau 2 in the game.

Now we can assume that a general point $[\Lambda]$ of a component of the limit cycle corresponds to an k-plane that intersects $\mathrm{A}_i \cap \mathrm{B}_{k-i}$. A priori there can be many such components. We need to show that any locus of k-planes that meet $\mathrm{A}_i \cap \mathrm{B}_{k-i}$ and satisfy the remaining constraints have dimension at most the dimension of the locus associated to Tableau 1 with equality if and only if the locus coincides with the locus corresponding to Tableau 1.

Note that in the limit any k-plane must still meet the linear spaces that are limits of the linear spaces D_j in dimension at least j. They must meet the limit of A_i in dimension at least i. They must continue to meet AB_r in at least a one-dimensional subspace. Finally, the k-planes must continue to meet the spans of the AB and D in dimension at least equal to the number of AB and D squares contained in each square they span.

Suppose at a general point of a component of the limit cycle the subspaces of the limiting k-planes in D_{i-1} remain independent from the subspace of the k-planes contained in B_{k-i} and the intersection of the k-plane with $\mathrm{A}_i \cap \mathrm{B}_{k-i}$ remains independent from the subspace contained in B_{k-i-1}. Then under our assumptions the limit cycle must be contained in the variety defined by Tableau 1. (By the argument given above we can assume that the linear spaces contained in the AB squares remain independent.) Since the variety defined in Tableau 1 is irreducible of the correct dimension, this component of the limit must be supported on the variety defined by Tableau 1.

We can now assume that at a general point $[\Lambda]$ of a component of the limiting cycle the subspace of the k-plane Λ contained in D_{h_s} becomes dependent with the linear subspaces of Λ contained in B_{t_s} or the k-plane meets the limit subspace contained in A_i along B_j for some $j < k - i$. We can now describe a variety associated to a generalized Mondrian tableau that must contain such a limit. Let h_1 be the smallest index for which $\Lambda \cap \mathrm{D}_{h_1}$ becomes dependent with a linear subspace of Λ contained in a vector space represented by a B square. Then Λ must intersect the intersection of D_{h_1} with this linear space. Let t_1 be the smallest index such that Λ intersects $\mathrm{D}_{h_1} \cap \mathrm{B}_{t_1}$. We then delete D_{h_1} and B_{t_1}. We draw their intersection. We then take the generalized tableau that contains the same D squares other than D_{h_1}, the intersection $D_{h_1} \cap B_{t_1}$, the square A_i and the squares $A_j \cap B_{k-j+2}$ for $k - j + 2 > t_1$ and $j > i$ and $A_j \cap B_{k-j+1}$ for $k - j + 1 < t_1$.

We then proceed to the next D vector space D_{h_2} that contains a linear subspace that meets the vector spaces represented by the B squares in a more specialized way and repeat the construction. If t_2 is the smallest index B square such that Λ meets $\mathrm{D}_{h_2} \cap \mathrm{B}_{t_2}$ in a 2 dimensional subspace for $t_2 > t_1$ or Λ meets $\mathrm{B}_{t_2} \cap \mathrm{D}_{h_2}$ in a one-dimensional subspace for $t_2 < t_1$, we draw the intersection of D_{h_2} and B_{t_2} and take the intersections of the remaining A and B squares so that the smallest A square meets the largest B square, the next largest A square the next smallest B square, etc. We continue the construction for all the indices. Any k-dimensional linear space that lies in a limit cycle and satisfies the rank conditions must be contained in the generalized Mondrian tableau we constructed. See Figure 12.

The dimension of such a tableau is easy to compare to the dimension of the original tableau. Each time we repeat the procedure we do not change the sum of the side-lengths of the squares except when we take the intersection of A_i with one of the B squares in which case we increase the sum of the side-lengths by one. However, when we take the intersection of D_s and B_{k-i-l} we increase the number of containment relations among the squares by $l + 1$. Hence the variety associated to the resulting generalized Mondrian tableau has strictly smaller dimension unless the only intersection occurs between A_i and B_{k-i} and the remaining linear spaces remain independent. In fact, we obtain a precise estimate for the dimension of the loci of k-planes that meet the D and B squares in more specialized ways.

From this description it follows that except for the locus associated to Tableau 1, the k-planes that meet $\mathrm{A}_i \cap \mathrm{B}_{k-i}$ along a more specialized locus cannot form a component of the limit cycle because they have strictly smaller dimension. We conclude that the limit cycle is supported along the union of the varieties described by Tableaux 1 and 2.

There remains to show that each of the limits occur with multiplicity one. This easily follows from a local calculation. As a first step we can reduce the calculation to the case when there is only one AB square and the rest of the squares are D squares or A_i. We achieve this reduction by considering the map that sends a k-plane Λ to its intersection to A_{i+1}. At a general point of both limits this is a smooth morphism since it is the projection map of a tower of projective bundles. It suffices to prove that the multiplicity is one in that case since the general case follows by pulling back the multiplicity by the smooth morphism. We can further restrict to the case of two dimensional linear spaces by taking the quotient of $\Lambda \cap \mathrm{A}_{i+1}$ with $\Lambda \cap \mathrm{D}_{i-1}$. This morphism is also smooth at general points of the loci we are interested in. Now the calculation reduces to a product of two Pieri classes in a Grassmannian of two planes, where we know by Pieri's formula for that case that the multiplicities are one. It follows that both limits occur with multiplicity one.

The case when we move D_{j-1} is almost identical. The reader can think of that case as a special case of the previous case where the degeneration is taking place in a Grassmannian of j planes and there is only one A and one B square. An argument similar to the argument just given shows that at a general point of a limit component the linear subspace of Λ contained in D_{j-1} remains independent from the subspaces contained in the B square. Consequently, we are reduced to the previous case. We leave the details to the reader. This completes the proof. □

3.3. **A generalized Littlewood-Richardson rule for Grassmannians.** A closer examination of the proof of Theorem 3.6 reveals that we obtained a Littlewood-Richardson rule for expressing the class of any variety defined by a generalized Mondrian tableau as a sum of Schubert cycles. The previously known Littlewood-Richardson rules do not apply in this generality. In many geometric situations varieties do not occur as intersection of Schubert varieties, but as varieties of k-planes satisfying certain intersection conditions with respect to vector spaces expressible as the span of elements of a fixed basis. This flexibility that Mondrian tableaux offer is the main advantage of this rule over previously known rules.

The algorithm. *Step 1.* Given any generalized Mondrian tableau replace it with a tableau where no two squares are left justified (as in the reduction step in the proof of Theorem 3.6). More precisely, among any two left-justified squares replace the larger by a square one unit smaller and no longer left justified with the smaller square. Repeat until no two squares are left justified.

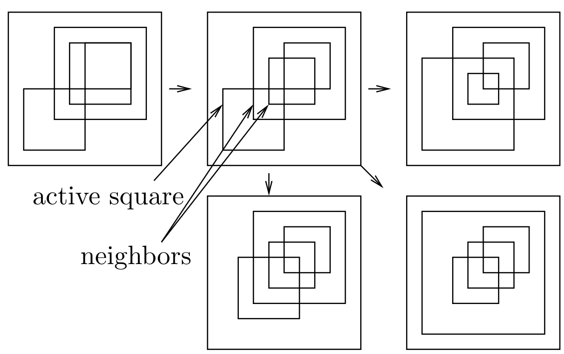

FIGURE 13. An example of the algorithm for generalized Mondrian tableaux.

Step 2. If all the squares are nested, the algorithm terminates. The tableau corresponds to the class of a Schubert variety. Otherwise, consider the lower left most square S. If S contains every square which does not contain S, proceed to the next lower left most square. Designate the lower left most square that fails this property the active square.

Step 3. Replace the tableau with a tableau where, among the squares not containing the active square, no two squares to the northeast of the active square are right justified by shrinking the larger square by one unit.

Let S be the active square. After Step 3, a square N is a *neighbor* of S if the following hold:

(1) N intersects or touches S, but is not contained in S and does not contain S.
(2) If the southwest corner of a square S' is between the southwest corners of N and S, then either S' contains N or is contained in S.

Step 4. Move the active square anti-diagonally up by one unit and replace the tableau by the following tableaux. If after the move the active square contains its smallest neighbor, replace the active square with its old span with its smallest neighbor. Otherwise, for each neighbor of the active square, draw a tableau where we delete the neighbor and the active square, draw the old span and the new intersection of these two squares and keep everything else the same. Finally, as long as the largest neighbor does not contain the active square and condition 2 defining generalized Mondrian tableaux hold, we place the active square in its new position.

We apply the algorithm to every tableau we obtain until all the squares in each are nested. Figure 13 gives an example. A corollary of the dimension counts in the proof of Theorem 3.6 is the following.

Corollary 3.9. *The coefficient of a Schubert cycle σ_λ in the class of a generalized Schubert variety is the number of times a Mondrian tableau corresponding to σ_λ occurs as a result of running the algorithm described above.*

Remark 3.10. The Littlewood-Richardson rule given by Corollary 3.9 differs from the one given by Theorem 3.6. The Littlewood-Richardson rule for the two-step flag varieties will follow the rule given in Theorem 3.6 more closely.

4. Painted Mondrian Tableaux and a Littlewood-Richardson rule for two-step flag varieties

In this section we obtain a Littlewood-Richardson rule for two-step flag varieties. Recall that $F(k_1, k_2; n)$ denotes the two-step flag variety that parameterizes pairs of vector spaces $V_1 \subset V_2$ of dimensions k_1 and k_2, respectively, of a fixed n-dimensional vector space V. We preserve the notation from §2.1 and §3.

Painted Mondrian tableaux associated to Schubert cycles. Let C_1 and C_2 be two colors ordered by their indices. A *painted Mondrian tableau* associated to the Schubert cycle $\sigma^{\delta_1,\dots,\delta_{k_2}}_{\lambda_1,\dots,\lambda_{k_2}}$ in $F(k_1, k_2; n)$ is a nested sequence of k_2 squares centered along the anti-diagonal such that

- The squares are indexed by a pair of integers (i, j), where i and j are the number of squares (inclusive) of color C_1 and the total number of squares (inclusive), respectively, contained in the square.
- The square with second index j has side-length $n - k_2 + j - \lambda_j$ and color C_{δ_j}.

See Figure 14 for an example. In illustrations we will set C_1 = red and C_2 = black.

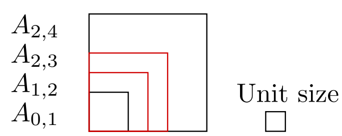

Figure 14. A painted Mondrian tableau associated to $\sigma^{2,1,1,2}_{1,1,1,0}$ in $F(2, 4; 6)$.

In the painted Mondrian tableau associated to $\sigma^{\delta_1,\dots,\delta_{k_2}}_{\lambda_1,\dots,\lambda_{k_2}}$, k_1 of the squares have color C_1 and $k_2 - k_1$ of the squares have color C_2. If the (i, j)-th square is of color C_1, then the preceding square has index $(i - 1, j - 1)$. If the (i, j)-th square has color C_2, then the preceding square has index $(i, j - 1)$. We will often refer to the square of index (i, j) and color C_1 as the *i-th square of color C_1*. We will refer to the square of index (i, j), irrespective of its color, as the *j-th square*. When we do not wish to specify one of the indices, we will place a $*$ instead of that index. If we forget the colors, we obtain the Mondrian tableau associated to the Schubert variety $\sigma_{\lambda_1,\dots,\lambda_{k_2}}$ in $G(k_2, n)$.

The translation between painted Mondrian tableaux and the geometry of the two-step flag variety is straightforward. A square of side-length s in a painted Mondrian tableau represents a vector space of dimension s. The pairs of vector spaces (V_1, V_2) parameterized by the variety associated to a painted Mondrian tableau have to satisfy certain intersection conditions with respect to the vector spaces represented by the squares in the tableau. The vector space V_2 is required to intersect a flag element in dimension equal to the number of squares contained in the square representing that flag element. The vector space V_1 is required to intersect a flag element in dimension equal to the number of squares of color C_1 contained in the square representing that flag element.

The game. To multiply two Schubert cycles σ^δ_λ and σ^ν_μ in $F(k_1, k_2; n)$ we place the painted Mondrian tableaux corresponding to σ^δ_λ (respectively, σ^ν_μ) lower left justified (respectively, upper right justified) with an $n \times n$ square. We denote the squares at the lower left by $A_{i,j}$ and the ones at the upper right by $B_{i,j}$. The initial configuration corresponds to the intersection of Schubert varieties defined with respect to transverse flags.

There is a simple criterion (the MM rule) to check whether the intersection of two Schubert varieties is empty.

• **The MM (must meet) rule.** For $1 \leq i \leq k_1$, we check whether the i-th A square of color C_1 intersects the $(k_1 - i + 1)$-th B square of color C_1 in a square of side-length at least one. If not, we stop. The intersection of the two Schubert cycles is zero. We then check whether the j-th A square $A_{*,j}$ intersects the $(k_2 - j + 1)$-th B square B_{*,k_2-j+1} in a square of side-length at least one for $1 \leq j \leq k_2$. If not, we stop. The intersection of the Schubert cycles is zero.

Geometrically, the vector space V_1 intersects the vector spaces represented by the first pair of squares in dimensions i and $k_1 - i + 1$, respectively. Since the dimension of V_1 is k_1, these two vector spaces must intersect in at least a one-dimensional subspace. Similarly, the vector space V_2 intersects the vector spaces represented by the second pair of squares in dimensions j and $k_2 - j + 1$, respectively. Since the dimension of V_2 is k_2, these two vector spaces must meet in at least a one-dimensional subspace. Note that the intersection of two Schubert varieties is empty if and only if the MM rule is not satisfied.

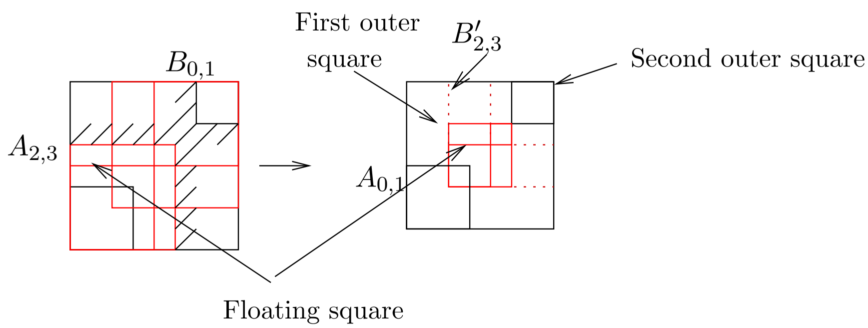

FIGURE 15. An application of the OS and S rules.

• **The OS (outer square) rule.** We take the intersection of A_{k_1,k_2} with B_{k_1,k_2} and call their intersection the second **O**uter **S**quare. We replace every square with their intersection with the second outer square and keep their labels the same. More generally, in a Mondrian tableau the *second outer square* will be the span of all the squares in the tableau. We then replace every A square of color C_1 with their intersection with the largest B square of color C_1 and every B square of color C_1 with their intersection with the largest A square of color C_1. We will refer to the span of all the squares of color C_1 in the tableau as the *first outer square* (see Figure 15). Note that the first and second outer squares do not need to be distinct.

Remark 4.1. If the largest B square of color C_1 is not the second outer square, the A squares of color C_1 are replaced by smaller squares. We refer to such smaller squares as *floating squares*. However, by the A square we will continue to refer to the square whose north and east sides are extended to the second outer square (see Figure 15).

• **The S (span) rule.** Suppose $A_{*,j}$ does not intersect or touch B_{*,k_2-j}, then we remove the the rows and columns that are between $A_{*,j}$ and B_{*,k_2-j} (see Figure 15).

The fact that two complementary dimensional linear subspaces of a vector space with trivial intersection span the vector space justifies this rule.

The Littlewood-Richardson rule for the two-step flag variety will be similar to the Littlewood-Richardson rule for the Grassmannian. We will move the A squares anti-diagonally up by one unit in a specified order. These moves correspond to degenerations of the flags identical to the ones in §3. The varieties arising as the limits of the degenerations will satisfy conditions with respect to a partially built flag (denoted by D squares) and partially remaining flags (denoted by the remaining A and B squares). Since instead of one vector space we are parameterizing a pair of vector spaces, the description of the limits will be slightly more complicated. We will also have to be more careful with our degeneration order. However, the principle underlying the Littlewood-Richardson rule can still be stated very simply as follows:

If V_1 does not intersect the intersection of complementary dimensional constraints on V_1, then V_1 lies in their span. If, in addition, V_2 does not intersect the intersection of complementary dimensional constraints, then V_2 lies in their span.

Before delving into the technical details of the rule we give a few examples that highlight the new features of the two-step rule. We strongly urge the reader to study these examples with paper and pencil in hand before proceeding.

4.1. **Examples.** In this subsection we give some examples of the Littlewood-Richardson rule for two-step flag varieties. These examples illustrate the novel geometric features of the two-step flag varieties. Once one understands these examples well, the general rule is simply a matter of bookkeeping.

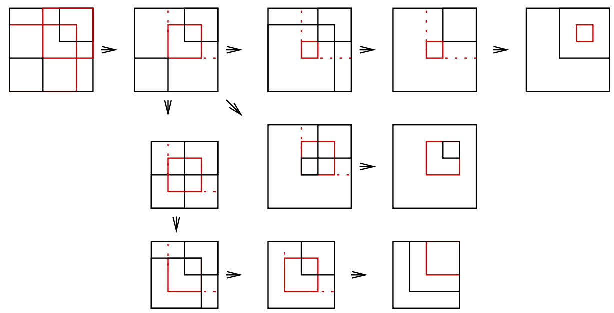

FIGURE 16. The calculation $\sigma^{2,1,2}_{1,0,0} \cdot \sigma^{2,1,2}_{1,1,0} = \sigma^{1,2,2}_{2,1,0} + \sigma^{2,1,2}_{2,2,0} + \sigma^{1,2,2}_{1,1,1}$ in $F(1,3;5)$.

4.1.1. *Example 1.* In the first example we compute $\sigma^{2,1,2}_{1,0,0} \cdot \sigma^{2,1,2}_{1,1,0}$ in $F(1,3;5)$. We first check that the Must Meet rule is satisfied. Then we apply the Outer Square rule. The intersection of $A_{1,3}$ and $B_{1,3}$ already form the second outer square. We replace the two squares $A_{1,2}$ and $B_{1,2}$ with their intersection to form the first outer square.

We now start the degenerations. The order will be similar to the Grassmannian case. We will eliminate A and B squares starting with the smallest A square and we will form nested D squares. There will be some new features that we will demonstrate in these examples. We start by degenerating the A-flag starting with $\mathrm{A}_{0,1}$. We depict this by sliding $A_{0,1}$ anti-diagonally up by one unit. We replace the initial tableau with three new tableaux.

- Tableau 1: We draw in C_1 the new intersection of $A_{0,1}$ with the first square in C_1 to its northeast (here $A_{1,2} \cap B_{1,2}$). We delete $A_{0,1}$ and the square in C_1 and draw their old span in C_2. We keep all the other squares as in the original tableau. This tableau is depicted to the right of the initial tableau.
- Tableau 2: We draw in C_2 the new intersection of $A_{0,1}$ with the first square to its northeast. Here whether we delete the two squares depends on their color. We keep those of color C_1 and delete those of color C_2. In general (but not for this example), this case might require adjusting the tableau so that MM and S rules are satisfied by the sub-tableau to the northeast of the new intersection. We keep all other squares as in the original tableau. This tableau is depicted to the southeast of the initial tableau.
- Tableau 3: We remove from the initial tableau the row and column to the south and west of the new position of $A_{0,1}$. This tableau is depicted below the initial tableau.

Geometrically, the first tableau depicts the limit where V_1 intersects the new $\mathrm{A}_{0,1} \cap \mathrm{B}_{1,2}$. The second tableau depicts the limit where V_2 intersects the new intersection of $\mathrm{A}_{0,1} \cap \mathrm{B}_{1,2}$. The third tableau corresponds to the limit where both V_1 and V_2 are contained in the new span of $\mathrm{A}_{0,1}$ and $\mathrm{B}_{1,2}$. This step already encapsulates the rule. At each stage we will replace a given tableau with three new tableaux unless the variety corresponding to one or more of the tableaux has strictly smaller dimension. Informally, the variety associated to a painted Mondrian tableau is the closure of the quasi-projective variety parameterizing pairs of vector spaces V_1, V_2 where

- V_2 intersects vector spaces represented by each square in dimension equal to the number of squares (excluding those that are the spans of other squares) contained in that square; and
- V_1 intersects the subspace of V_2 contained in a vector space corresponding to a square of color C_1 in dimension equal to the number of squares of color C_1 in that square.

We discard the tableaux corresponding to varieties of smaller dimension. As we will explain below, whether one of the three tableaux will have smaller dimension is determined by the combinatorics of the original tableau.

We continue simplifying each of the resulting tableaux. In the first tableau, the red square of side-length one is $D_{1,1}$. We simplify this tableau by moving $A_{1,2}$. The move does not increase the length of intersection of $A_{1,2}$ with any square of side-length C_1. Hence, we do not get the possibility described by Tableau 1. The variety corresponding to Tableau 3, has strictly smaller dimension. Hence we replace this tableau with only the option described by Tableau 2.

The second tableau is the most interesting. The side-length one square of color C_2 is a filler. We will define and give more examples of fillers below. Fillers hold the key to understanding the geometry of flag varieties. We will always move fillers before we move the next A square. In general when we move a filler, we replace the tableau by the Tableaux 2 or 3. In this example, Tableau 3 would correspond to a variety of smaller dimension. Hence, we only get one possibility.

The third tableau is simplified in a similar fashion. We conclude that $\sigma^{2,1,2}_{1,0,0} \cdot \sigma^{2,1,2}_{1,1,0} = \sigma^{1,2,2}_{2,1,0} + \sigma^{2,1,2}_{2,2,0} + \sigma^{1,2,2}_{1,1,1}$.

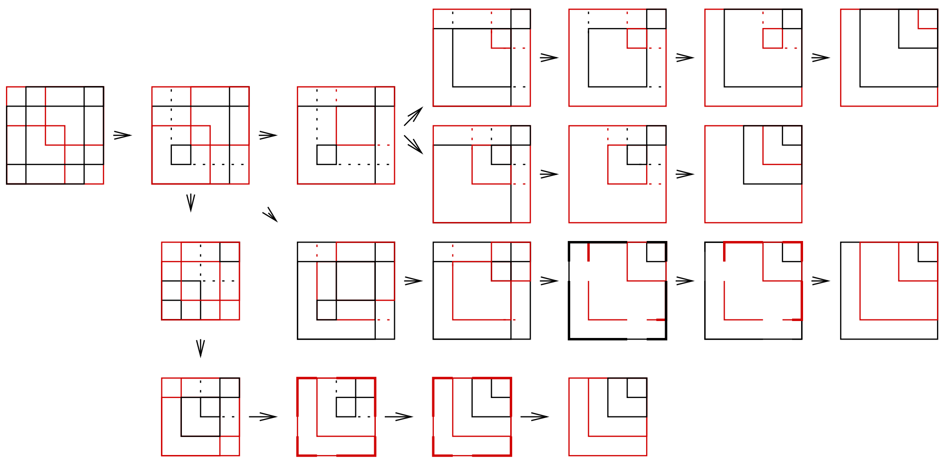

FIGURE 17. The calculation $\sigma^{2,1,2,1}_{1,0,0,0} \cdot \sigma^{2,1,2,1}_{1,0,0,0} = \sigma^{1,2,2,1}_{1,1,0,0} + \sigma^{2,1,2,1}_{1,1,1,0} + \sigma^{2,1,1,2}_{1,1,0,0} + \sigma^{2,2,1,1}_{1,1,1,1}$ in $F(2,4;5)$.

4.1.2. *Example 2: Fillers.* As a second example we compute $\sigma^{2,1,2,1}_{1,0,0,0} \cdot \sigma^{2,1,2,1}_{1,0,0,0}$ in $F(2,4;5)$. In this example, the MM, OS and S rules are satisfied for the initial tableau. We start by moving $A_{0,1}$. In this case, $A_{0,1}$ does not intersect the square of color C_1 to its northeast. The variety corresponding to Tableau 3 has strictly smaller dimension. We thus replace the tableau by Tableau 2 only. We continue by moving the next A square $A_{1,2}$. We replace this tableau by three new tableaux similar to the Tableaux 1, 2 and 3 in the previous example.

The most interesting case is Tableau 2 depicted to the southeast. Since both $A_{1,2}$ and $B_{1,2}$ are of color C_1, we do not delete either of them. The old $A_{1,2}$ square (now $D_{1,2}$) is the span of two squares $D_{0,1}$ (the side-length one square at the southwest corner of $D_{1,2}$) and $D'_{1,2}$ (the newly formed D square of color C_2). We will call squares like $D_{1,2}$ which are squares in C_1 spanned by the squares contained in them *spanned D squares*. We first nest the D squares by moving $D_{0,1}$. The new side-length one square $D_{0,1}$ at the southwest corner of $B_{1,2}$ is a *filler*. Note that this step also forces us to adjust the span of $B_{1,2}$. Fillers are squares of color C_2 left aligned with the lower left of a square of color C_1 that arise when we take the intersection of two squares of color C_1 or a square of color C_1 with one of color C_2. Fillers signal that the square S of color C_1 that are left aligned with them are the span of the filler and squares to the northeast of the filler contained in S. Fillers are the main new feature of two-step flag varieties. We have to modify the degeneration order to move fillers before moving other A squares.

We move the filler. When we move a filler F, we also move the parts of the squares that contain F and are upper right aligned with it. This may cause one or more of the squares containing F to become disconnected. In this example, when we move the filler, the squares $D_{1,2}$ and $A_{1,3}$ become disconnected. We will call such squares *chopped squares* and depict them by leaving blank the row and column corresponding to the unit square they do not contain. Geometrically chopped squares correspond to vector spaces which are the spans of non-consecutive basis elements. In the sequel we will have to take care that some of the squares may be chopped.

After we move the filler, we replace both the filler and $B_{0,1}$ with their new intersection $D_{0,1}$. Now all the D squares are nested and there are no fillers. We continue by moving the smallest A square $A_{1,3}$. Note that $A_{1,3}$ is now a chopped square. To say that we move $A_{1,3}$ means that we move its lower left most piece and leave the other pieces in place. The rest of the calculation is similar to the cases already discussed. We conclude that $\sigma^{2,1,2,1}_{1,0,0,0} \cdot \sigma^{2,1,2,1}_{1,0,0,0} = \sigma^{1,2,2,1}_{1,1,0,0} + \sigma^{2,1,2,1}_{1,1,1,0} + \sigma^{2,1,1,2}_{1,1,0,0} + \sigma^{2,2,1,1}_{1,1,1,1}$.

The attentive reader will have noticed the squares drawn in dotted lines. We will call these squares B' *squares*. B' squares are distinct, right justified squares that record the dimensions of intersection of (V_1, V_2) with the B-flag. Every B square is a B' square of the same color (since B and D squares are solid and B' squares are dotted, B' squares coinciding with B or D squares are not visible in the picture). Their purpose is to allow us to recognize fillers without labeling them. We can recognize fillers by the property: The number of B' squares of color C_2 between the lower left of the filler and the next B square $B_{i,j}$ of color C_2 to its northeast is less than the number of squares of color C_2 contained in the filler but not in $B_{i,j}$.

We indicate briefly what may go wrong in case we do not move fillers before moving other A squares. Consider the following example.

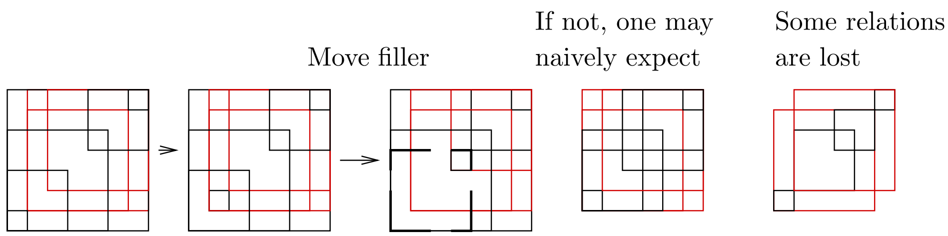

FIGURE 18. The reason to move fillers first.

After the first move the side-length one square is a filler. Suppose instead of moving it, we move the next A square. One potential limit occurs if both V_1 and V_2 lie in $\mathrm{B}_{2,4}$. We might naively try to depict this possibility by Tableau 3. The problem is that the variety associated to Tableau 3 has dimension greater than that of the original variety. The limiting pairs (V_1, V_2) have to satisfy further conditions. For example, this tableau does not record the non-trivial information that the subspace of V_1 contained in $\mathrm{A}_{1,4} \cap \mathrm{B}_{2,4}$ is also contained in the subspace of V_2 spanned by its intersection with the filler, $\mathrm{A}_{0,3} \cap \mathrm{B}_{1,3}$ and $\mathrm{A}_{1,4} \cap \mathrm{B}_{0,2}$. Of course, one can devise combinatorial objects that record these extra relations. However, we choose to eliminate fillers at the expense of creating chopped squares and having a slightly more complicated degeneration order. In return in the degenerations we never run into a situation where the apparent limit has dimension larger than the original variety.

Remark 4.2. R. Vakil gave a geometric interpretation of A. Knutson's conjectural Littlewood-Richardson rule for two-step flag varieties. In this interpretation the degeneration order does not take into account fillers. Consequently, the geometry becomes very complicated. Our treatment of fillers reflects the desire to keep the geometry always as simple as possible. We want all the limits to have a uniform description. Our order of degeneration will achieve this. V_2 will always be required to intersect k_2 vector spaces in given dimensions. V_1 will intersect k_1 subspaces of V_2 spanned by these intersections in given dimensions. The painted Mondrian tableaux record this information in a convenient manner.

4.1.3. *Example 3.* We give two more examples to describe a few remaining subtleties. In these examples we calculate $\sigma^{1,2,1}_{1,1,0} \cdot \sigma^{1,1,2}_{1,0,0}$ in $F(2,3;6)$ and $\sigma^{1,1,2}_{1,1,0} \cdot \sigma^{2,1,1}_{1,0,0}$ in $F(2,3;5)$.

After we apply the Outer Square rule, we replace $A_{1,1}$ with its intersection with $B_{2,2}$. We call A squares that no longer extend to the west and south edges of the tableau *floating squares*. When their turn comes, we do not move these squares, but delete the B square passing through their south and west edges and label them D squares.

We also need to take care that all our degenerations are codimension one degenerations. When the degenerations are not codimension one, then the limit may depend on the properties of the particular one-parameter family. The only time this issue arises is when the square we move forces other squares (lower left aligned with it) to move as well. As long as the complementary flag element in the B-flag does not impose any conditions on

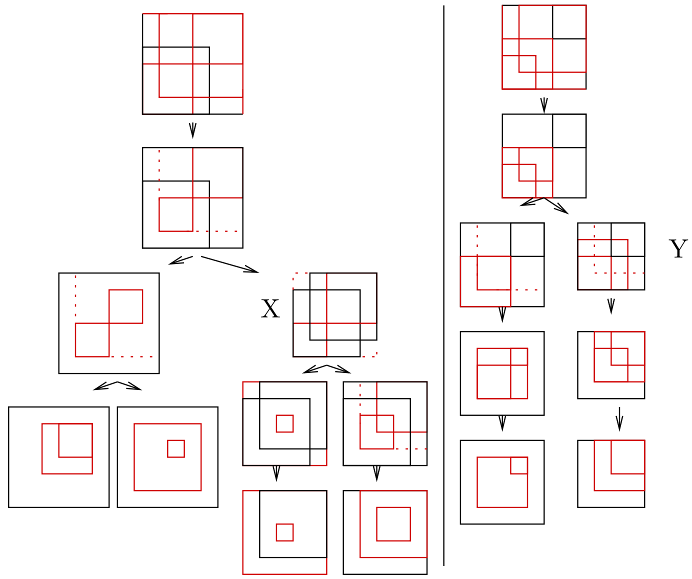

FIGURE 19. The calculations $\sigma^{1,2,1}_{1,1,0} \cdot \sigma^{1,1,2}_{1,0,0} = \sigma^{1,1,2}_{2,2,0} + \sigma^{1,1,2}_{3,1,0} + \sigma^{1,2,1}_{3,1,1} + \sigma^{1,1,2}_{2,1,1}$ in $F(2,3;6)$ and $\sigma^{1,1,2}_{1,1,0} \cdot \sigma^{2,1,1}_{1,0,0} = \sigma^{1,1,2}_{2,1,0} + \sigma^{1,1,2}_{1,1,1}$ in $F(2,3;5)$.

the vector spaces, this is a codimension one degeneration. When a square of color C_1 is lower left aligned with a square of color C_2, it is necessary to move the square of color C_1 first in case the complementary element in the B-flag imposes non-trivial conditions on the vector spaces. This is the case in the Tableau indicated by X in Figure 19. Hence, we first move the square $D_{1,1}$ before we move $A_{1,2}$. Whereas in the Tableau marked by Y in Figure 19, we move $A_{2,2}$.

The reader may find more examples in `http://www-math.mit.edu/~coskun/gallery.html`.

4.2. **Preliminary definitions.** We now describe the rule more precisely. After some terminology, we begin with characterizing the painted Mondrian tableaux that can occur during the game. We will then describe how to associate an irreducible subvariety of $F(k_1, k_2; n)$ to a painted Mondrian tableau. We will then describe how to simplify the tableaux.

Painted Mondrian tableau. A *painted Mondrian tableau* for $F(k_1, k_2; n)$ is a diagram contained in a square of side-length at most n consisting of squares (possibly chopped) of colors C_1 or C_2 such that the anti-diagonal of each square is along the anti-diagonal of the outer square and the northeast and southwest corners of the squares each give at most k_2 distinct points on the anti-diagonal at most k_1 of which have color C_1.

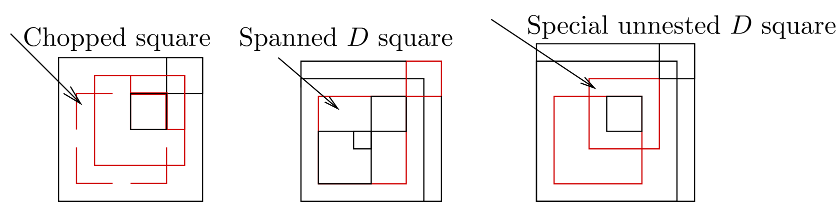

FIGURE 20. Examples of a chopped square, a spanned D square and a special unnested D square.

Chopped squares. In a Mondrian tableau a *chopped square* is a square that has gaps along the rows and columns corresponding to a collection of unit squares along the anti-diagonal (see Figure 20). The side-length of a chopped square is the number of unit squares contained in it excluding the gaps.

Since in a Mondrian tableau the unit squares along the anti-diagonal represent a basis for the ambient vector space, a chopped square represents a vector space that is the span of non-adjacent basis elements. We will say that a chopped square S_2 contains a square S_1, if the unit squares along the anti-diagonal contained in S_1 are also contained in S_2 (i.e., if there is an inclusion among the corresponding vector spaces).

D **squares.** An *unnested* D square is a D square that does not contain every D square of smaller second index. We will refer to a D square of color C_1 which is the span of an unnested D square and the largest nested D square contained in it as a *spanned* D square (see Figure 20). An unnested D square will be called a *special unnested D square* if there are not any squares of color C_2 between (the lower left corner of) this square and the largest D square to the lower left of it that does not contain it (see Figure 20).

Fillers. Some of the D squares will be designated as *fillers.* These will be D squares of color C_2 that arise as the intersections of A or D squares with B or unnested D squares of color C_1.

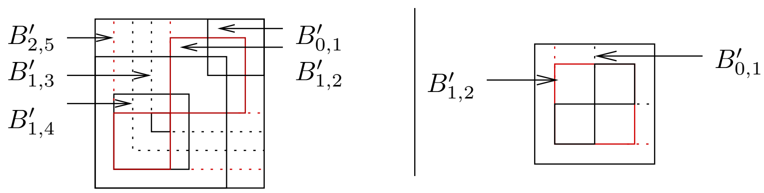

FIGURE 21. Drawing the B' squares.

B' **squares.** Given a Mondrian tableau for $F(k_1, k_2; n)$ we draw k_2 distinct, nested, right aligned squares in dotted lines such that exactly k_1 of them have color C_1 according to the following procedure. For every right aligned square, we draw as many squares of each color as the number of B and D squares entirely contained in that square omitting fillers and the largest nested square of color C_2 in a spanned D square. If some of the squares coincide, we order them $S_1, \dots, S_r$ listing the ones of color C_2 after all the ones of color C_1. We then shrink the side-length of S_i by $i-1$. We call these squares B' squares (see Figure 21.). We can associate a pair of indices to a B' square where the first index records the number of B' squares of color C_1 contained in that square and the second index records the number of B' squares in that square.

Geometrically the B' squares record the generic intersection of the vector spaces (V_1, V_2) parameterized by the variety corresponding to the Mondrian tableau with the B flag.

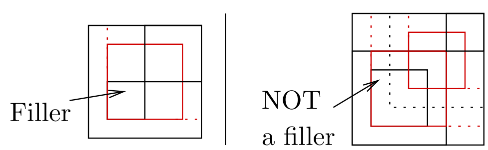

FIGURE 22. Recognizing fillers.

Recognizing fillers. The B' squares allow us to recognize fillers in painted Mondrian tableaux. A filler is a D square of color C_2 whose lower left most piece is left justified with either

- a B square of color C_1 that is not the second outer square, or
- an unnested D square of color C_1 that is not a spanned D square; and

satisfies the following property: The number of B' squares of color C_2 between the lower left of the filler and the next B square $B_{i,j}$ of color C_2 is less than the number of squares of color C_2 contained in the filler but not in $B_{i,j}$ (See Figure 22).

Restricting Mondrian tableau. We can restrict a Mondrian tableau to the B' square $B'_{i,j}$ passing through the lower left of a D square. The resulting tableau is a tableau for $F(i, j; |B'_{i,j}|)$. This tableau is obtained by taking all the D and B squares entirely contained in $B'_{i,j}$. Suppose there are k'_2 B squares k'_1 of which have color C_1. Suppose there are f fillers. Then we take k'_1 of the largest A squares of color C_1 and $k'_2 - k'_1 - f$ of the largest A squares of color C_2 and restrict them to $B'_{i,j}$. If there are any fillers left justified with the B' square to which we are restricting the tableau, we relabel that filler as an A square for the restricted tableau. If this filler abuts an unnested D square of color C_1 we declare that unnested D square an A square and the

B' square $B'_{i,j}$ a B square of color C_1. Geometrically, this tableau represents the intersection conditions on the subspaces of V_1 and V_2 contained in $\mathrm{B}'_{i,j}$.

The generalized MM and S rules. A priori, due to the presence of fillers, the MM and S rules may not be satisfied for the restricted tableau (even if they are satisfied for the larger tableau). When the squares are chopped, to say that the S rule is satisfied means that the lower left most piece of the A square $A_{h,l}$ touches or intersects the B' square with the complementary second index. We will say that the A and B squares satisfy the generalized MM and S rules if every restricted tableau satisfies the MM and S rules. More precisely, the restricted tableau satisfies the MM rule if of the k'_1 A squares of color C_1 and $k'_2 - k'_1 - f$ A squares of color C_2 that belong to the restricted tableau for $F(i, j; |B'_{i,j}|)$ the h-th one (counting in increasing order) intersects the $(k'_2 - f - h + 1)$-st B square among those that do not abut a filler (counting in increasing order) in a square of side-length at least one. Similarly, the restricted tableau satisfies the S rule if the h-th square among these A squares touches or intersects the $(k'_2 - f - h)$-th B square among those that do not abut a filler. From now on when we say the MM and S rules we will refer to the generalized MM and S rules. If the S rule is not satisfied, we remove the rows and columns from the restricted tableau by moving the A squares of the restricted tableau anti-diagonally up. If any of the pieces of the squares in the original tableau that are not contained in the restricted tableau become right justified with the squares we move, we slide those pieces as well (see Figure 23). In practice, we will only need to check the MM and S rules when a new filler forms and only for the whole tableau and the tableau restricted to the B' square passing through the lower left of the filler.

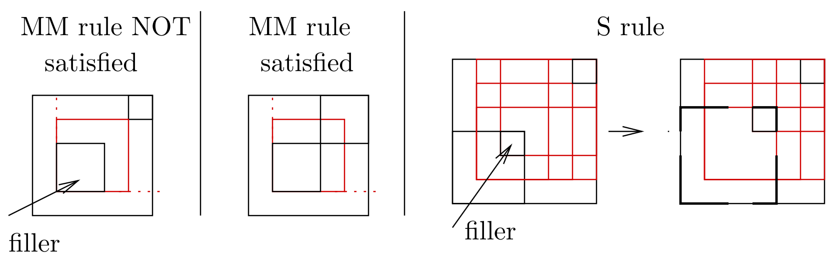

FIGURE 23. An illustration of the generalized MM and S rules.

We now describe the tableaux that arise during the game. Although at first glance this description may look long and complicated, each item describes a simple geometric requirement that pairs of vector spaces must satisfy. The reader who is only interested in using the algorithm may skip this description at the expense of not knowing what the most complicated possible painted Mondrian tableau that occurs during the game may look like.

Admissible Mondrian tableau. A painted Mondrian tableau is admissible for $F(k_1, k_2; n)$ if the squares that constitute the tableau are labeled as indexed A, B or D squares such that:

(1) The tableau has side length at most $m \leq n$ and satisfies the generalized MM and S rules.
(2) **The A squares:** The A squares are nested, distinct and possibly chopped. They contain the D squares. The lower left most pieces of the A squares are left justified with the second outer square. Suppose the smallest A square is $A_{i,j}$. Then the number of A squares is $k_2 - j + 1$. Among the squares that strictly contain $A_{i,j}$, $k_1 - i$ have color C_1. The first indices range between i and k_1 increasing each time an A square has color C_1. The second indices range between j and k_2.
(3) **The B squares:** The B squares are nested, distinct, right justified and not chopped. The number of B squares is the sum of the number of A squares and the number of fillers. The number of B squares of color C_2 to the upper right of a filler F is at least the number of fillers contained in F. If there are no fillers, then the number of A squares of color C_1 (respectively, C_2) equals the number of B squares of color C_1 (respectively, C_2). Otherwise, the number of B squares of color C_2 is equal to the sum of the number of fillers and the number of A squares of color C_2.
(4) **The D squares:** The number of D squares is $j - 1$ or j. If the number of D squares is j, then a D square of color C_1 is a spanned D square. Among the A and D squares k_1 of them have color C_1 and $k_2 - k_1$ or $k_2 - k_1 + 1$ of them have color C_2 depending on whether there exists a spanned D square.

(5) Some of the D squares may be unnested. There is at most one unnested D square of color C_2. There may be many unnested D squares of color C_1. If an unnested square D_{h_2,l_2} contains a D square of lower index D_{h_1,l_1}, then every D square of second index between l_1 and l_2 contains D_{h_1,l_1}. The unnested D square of color C_2 is either to the lower left of an unnested D square of color C_1 or is contained in it. Given two unnested D squares of color C_1 either one of them contains a filler that contains the other or one of them is a special unnested D square. An unnested D square has its lower left corner to the upper right of (the lower left corner of) all the D squares of lower index that it does not contain. If a D square D_{k_2,l_2} contains D_{k_1,l_1}, then $l_2 > l_1$ unless D_{k_2,l_2} is a spanned D square (in which case equality may hold).

(6) **Fillers:** Some of the D squares are designated as *fillers*. The lower left most pieces of fillers are left justified with a B square of color C_1 (other than the second outer square) or an unnested D square of color C_1 which is not a spanned D square. The fillers are nested. There exists a B square of color C_2 between the lower left hand corners of any two fillers. (In particular, a B or D square has at most one filler left justified with it.)

(7) **Chopped squares:** The A and D squares may be chopped. If an A or D square is chopped, then an A or D square of lower second index cannot be entirely contained to the lower left of the chop. If S_2 is a chopped square and S_1 is a square of lower second index that has pieces both to the lower left and to the upper right of a chop of S_2, then S_1 and S_2 coincide to the upper right of the chop. (In particular, if S_1 and S_2 are nested and if S_1 is not entirely contained to the upper right of a chop of S_2, then S_1 and S_2 coincide to the upper right of that chop.)

(8) **The relative position of B and D squares:** B squares may contain D squares. However, if a D square of color C_2 is strictly to the upper right of a B square $B_{i,j}$, then every D square not containing that D square is also to the upper right of $B_{i,j}$. If a D square is strictly to the upper right of a B square $B_{i,j}$ of color C_1, then every D square not containing that D square is also to the upper right of $B_{i,j}$.

(9) **Side-lengths:** The side-length of an unnested D square D_{k_2,l_2} of color C_2 (resp., C_1) is at least s shorter than the side-length of its span with any other D square D_{k_1,l_1} (resp., of color C_1) if the number of D squares (resp., D squares of color C_1) contained in D_{k_1,l_1} but not contained in D_{k_2,l_2} is s. Similarly the side-length of any B square of color C_2 (resp., C_1) is at least s shorter than the side-length of the squares spanned by the B square and any collection of D squares (resp. D squares of color C_1), if the number of D squares (of color C_1) not contained in the B square, but contained in the span is s. When counting the D squares for the color C_2, we omit spanned D squares. The side-length of a B square $B_{*,j}$ is at least one larger than the sum of the side-length of $B_{*,j-1}$ and the number of D squares contained in $B_{*,j}$ but not in $B_{*,j-1}$ omitting any fillers and spanned D squares.

Remark 4.3. As long as we draw the B' squares in the diagram, we do not need to label the squares in an admissible diagram. We can recover the information of which D squares are fillers from the B' squares as described above. Once we know the number of fillers, we can recover the labels of the A, B and D squares as in the case of the ordinary Grassmannians. Hence, in the examples we never label the squares.

Remark 4.4. Note that the restriction of an admissible painted Mondrian tableau to a B' square passing through the lower left of a D square is also admissible.

The following definition will play a crucial role in telling which Mondrian tableaux have smaller than expected dimension.

Units. A *unit* U in a square S of a Mondrian tableau is a square such that all the squares between S and U are either contained in the Mondrian tableau or are B' squares and all the squares containing U (including the B' squares) have color at most the color of U (See Figure 24).

For example, a B square is a unit in a B' square if every right aligned square contained in the B' square and containing the B square is a B or B' square of the tableau and they all have color less than or equal to the color of the B square. A nested D square $D_{i,j}$ is a unit in the smallest A square if all the squares containing $D_{i,j}$ and contained in the A square are D squares of the tableau; and they and the smallest A square all have color less than or equal to the color of $D_{i,j}$. Geometrically, units represent vector spaces that do not impose additional constraints on the pair (V_1, V_2)

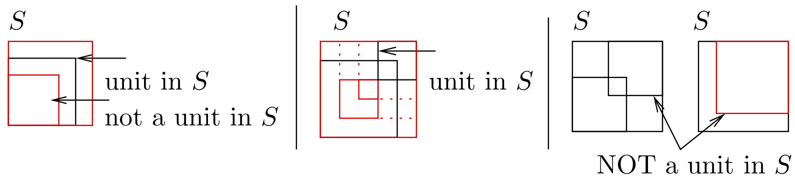

FIGURE 24. Some examples of units.

4.3. **The geometric interpretation of painted Mondrian tableaux.** To each admissible painted Mondrian tableau M, we associate an irreducible subvariety Σ_M of the flag variety $F(k_1, k_2; n)$. Suppose $A_{i,j}$ is the smallest A square. In an admissible Mondrian tableau each B square has two pairs of indices (i, j) and (i', j'). In the first pair (i, j), the index j denotes the number of B squares contained in that square and i denotes the number that are red among them. The second pair of indices (i', j') are the indices of the B' square that coincides with the B square. In other words, j' denotes the number of B' squares contained in that B square and i' denotes the number of those that are red. We will use the shorthand $B_{i,j}^{i',j'}$ to denote this B square.

AB^1**-squares.** Define the squares $AB_s^1(l_{k_1-s+1} - k_1 + s)$ to be the intersection of the s-th A square of color C_1 with the $(k_1 - s + 1)$-th B square $B_{k_1-s+1,*}^{l_{k_1-s+1},*}$ of color C_1 (See Figure 25). Geometrically, V_1 intersects the vector space represented by $AB_s^1(l_{k_1-s+1} - k_1 + s)$ in dimension $l_{k_1-s+1} - k_1 + s$.

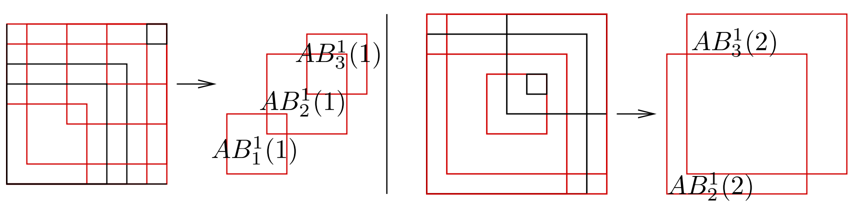

FIGURE 25. The definition of $AB_i^1(l)$ squares.

AB^2**-squares.** Similarly, define the squares $AB_t^2(h' - k_2 + t)$ as follows.

- If the smallest A square $A_{r,t}$ is of color C_2, take its intersection with the largest B square $B_{*,*}^{*,h'}$ that does not abut a filler and call the intersection $AB_t^2(h' - k_2 + t)$. Delete both the B and the A square.
- If the smallest A square $A_{r,t}$ is red and the $(k_1 - r + 1)$-th red B square is smaller than the largest remaining B square, take the intersection of $A_{r,t}$ with the largest B square $B_{*,*}^{*,h'}$ contained in the $(k_1 - r + 1)$-th red square B that does not abut a filler. Label the intersection $AB_t^2(h' - k_2 + t)$ and delete both squares. Otherwise, take the intersection of $A_{r,t}$ with the largest remaining B square $B_{*,*}^{*,h'}$ that does not abut a filler. Label the intersection $AB_t^2(h' - k_2 + t)$ and delete both squares.

Repeat the process until all the A and B squares are exhausted. This defines the AB_t^2 squares (see Figure 26).

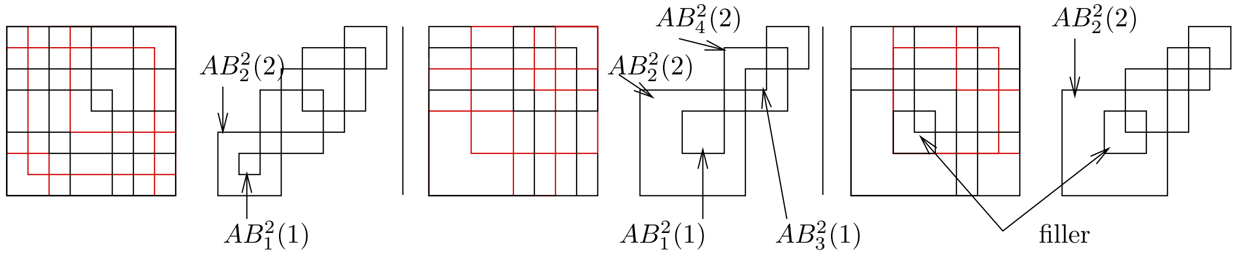

FIGURE 26. The definition of $AB_i^1(l)$ squares.

We can now define the subvariety Σ_M of $F(k_1, k_2; n)$ associated to the painted Mondrian tableau M. As always, we will denote the vector spaces represented by a square by the same symbols in Roman font. If the D squares are nested, then we consider the pairs of linear spaces (V_1, V_2) that satisfy the following conditions:

(1) $\dim(V_1 \cap \mathrm{D}_{h,l}) = h$, and $\dim(V_2 \cap \mathrm{D}_{h,l}) = l$, for $1 \leq k < i$ and $1 \leq j < k$.
(2) $\dim(V_1 \cap \mathrm{AB}^1_s(l_{k_1-s+1} - k_1 + s)) = l_{k_1-s+1} - k_1 + s$, and $\dim(V_2 \cap \mathrm{AB}^2_t(h' - k_2 + t)) = h' - k_2 + t$, where V_1 meets the vector spaces AB^1_s in the subspaces of V_2 contained in them.
(3) We require the intersections of V_2 with $\mathrm{AB}^2_{t_1}$ and $\mathrm{AB}^2_{t_2}$ or with AB^2_t and $\mathrm{D}_{h,l}$ to only intersect along vector spaces represented by squares common to both of the squares and otherwise to be independent. Similarly, we require the intersection of V_1 with $\mathrm{AB}^2_{s_1}$ and $\mathrm{AB}^2_{s_2}$ or with AB^2_s and the vector space represented by the h-th D square of color C_1 to meet only along vector spaces represented by squares of color C_1 common to both squares and otherwise to be independent.

The locus of pairs (V_1, V_2) satisfying these properties defines an irreducible quasi-projective subvariety of $F(k_1, k_2; n)$. We will denote its closure by Σ_M, the variety associated to the Mondrian tableau (see Figure 27).

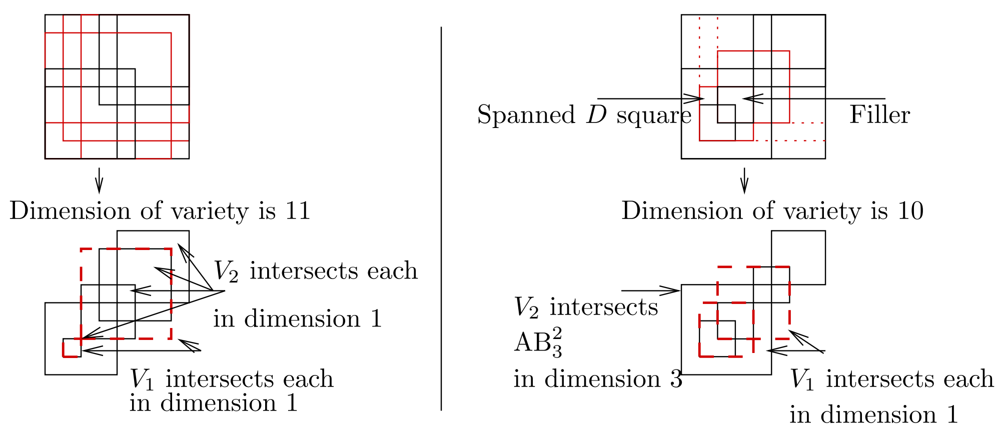

Figure 27. Two examples of varieties associated to painted Mondrian tableaux.

If the D squares are not nested, (1) in the definition needs to be accordingly modified. We require V_2 to meet each of the vector spaces represented by the D squares in dimension equal to the number of D squares contained in that square, except when a D square is a spanned D square. In the latter case, we require V_2 to meet the vector space represented by the spanned D square in one fewer dimensions. We require V_1 to meet each of the vector subspaces of V_2 contained in a vector space represented by a D square in dimension equal to the number of D squares of color C_1 contained in that square. Furthermore, we demand the intersections of V_2 with any of the two $\mathrm{AB}^2_{t_1}$ and $\mathrm{AB}^2_{t_2}$ or AB^2_t and $\mathrm{D}_{h,l}$ to only intersect along vector spaces represented by squares common to both of the squares and otherwise to be independent. Similarly, we require the intersection of V_1 with the linear spaces $\mathrm{AB}^2_{s_1}$ and $\mathrm{AB}^2_{s_2}$ or with AB^2_s and the vector space represented by the h-th D square of color C_1 to meet only along vector spaces represented by squares of color C_1 common to both squares and otherwise to be independent. Otherwise, we keep the definition as above. The closure of the locus of pairs (V_1, V_2) satisfying these properties defines an irreducible subvariety of $F(k_1, k_2; n)$. We will denote this subvariety as Σ_M, the variety associated to the Mondrian tableau.

To see that the variety associated to a Mondrian tableau is irreducible we can construct it as an open set in a tower of projective and Grassmannian bundles over an irreducible subvariety of $G(k_2, n)$. We consider the projection π_2 of $F(k_1, k_2; n)$ to $G(k_2, n)$ that sends the pair (V_1, V_2) to V_2. The image of the projection is given by the variety associated to the Mondrian tableau given by the D squares (omitting any spanned D squares) and the AB^2_t squares. Note that this is a generalized Mondrian tableau in $G(k_2, n)$ except that the squares are allowed to be chopped as indicated in the definition of an admissible painted Mondrian tableaux. It is easy to see that allowing the squares to be chopped does not change the dimension estimates in Lemma 3.7. By the argument given in the proof of Theorem 3.6, this variety is irreducible. In the open set in which we defined

Σ_M π_2 has equi-dimensional and irreducible fibers. The fiber over a point in the image of π_2 corresponds to choices of a k_1 dimensional subspace so that in each square of color C_1 in the tableau we select a subspace of V_2 that has dimension equal to the number of squares of color C_1 contained in that square. It follows that Σ_M is irreducible. This description also allows us to compute the dimension of the variety Σ_M.

Let $|S|$ the side-length of the square S. Let $\sum_l' |D_{h,l}|$ denote the sum of the side-lengths of the D squares omitting any spanned D squares. Let R_1 be the number of containment relations among the D and AB_t^2 squares omitting any spanned D squares. When we count the number of containment relations between squares, we always include a square itself among the squares contained in it. By Lemma 3.8 the dimension of the projection of Σ_M to $G(k_2, n)$ is given by the following:

$$\sum_l{}' |D_{h,l}| + \sum_t |AB_t^2| - R_1.$$

The fiber dimension of the projection is also easy to determine. For a D or AB_s^2 square S of color C_1, let $||S||$ denote the number of D or AB_t^2 squares contained in that square, except when a D square is a spanned D square. In the latter case we let $||S||$ to be one less than the number of D squares contained in it. Let R_2 denote the number of containment relations that the AB_s^1 and the D squares of color C_1 satisfy. Finally, we will write $\sum_{C_1}$ to denote that a sum is taken over squares of color C_1. The fiber dimension of the projection is

$$\sum_{C_1} ||D_{h,l}|| + \sum_s ||AB_s^1|| - R_2.$$

We have thus determined the dimension of Σ_M.

Lemma 4.5. *The dimension of the variety associated to a painted Mondrian tableau is given by*

$$\sum_l{}' |D_{h,l}| + \sum_t |AB_t^2| + \sum_{C_1} ||D_{h,l}|| + \sum_s ||AB_s^1|| - R_1 - R_2.$$

4.4. **The rule.** We now describe how to simplify an admissible painted Mondrian tableau which does not consist of k_2 nested squares k_1 of which have color C_1.

The moves. If the order of degeneration dictates that we move a square S, we slide the left most piece of S anti-diagonally up by one unit. If the square is not chopped, this means that we slide all of S up by one unit. If S is chopped, we move its lower left most piece and keep the other pieces fixed. If there are any D squares contained in S that abut its lower left, we slide the lower left most pieces of these D squares up by one unit. The D squares that do not abut the lower left of S remain fixed. If the piece of S we move is right justified with a piece of an A or D square that contains S, we also move that the portion of the A or D square that coincides with the lower left most piece of S. See Figure 28.

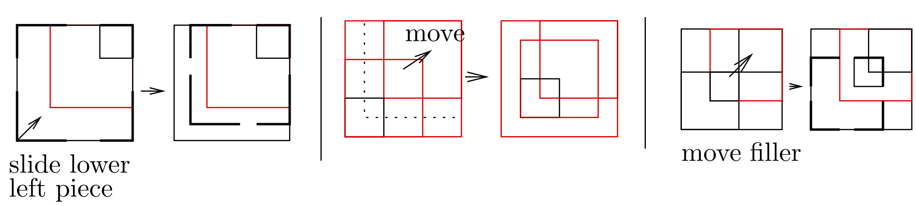

FIGURE 28. The moves.

The order of degeneration. We now specify the order in which we slide the squares. We follow the following flow chart.

Step 1. If there are any fillers or unnested D squares, we proceed to Step 2. If there are not any unnested D squares or fillers, we consider the smallest A square.

- If the smallest A square $A_{i,j}$ has color C_1, we check whether the first outer square is lower left justified with the second outer square.

  - If not, we take the intersection of $A_{i,j}$ and the largest B square of color C_1 before deleting them. We label the intersection $D_{i,j}$. We apply the OS rule.
  - If yes, we move the smallest A square anti-diagonally up by one unit.
- If the smallest A square $A_{i,j}$ has color C_2, we check whether there are any squares of color C_1 lower left justified with it. If not, we move $A_{i,j}$ anti-diagonally up by one unit. If yes, we first run Step 5 before moving $A_{i,j}$ anti-diagonally up by one unit.

Step 2. If there are fillers and/or unnested D squares, we check whether there are any spanned D squares or special unnested D squares. If not, we proceed to Step 3. If there is a spanned D square $D_{i,j}$, we move the largest D square contained in $D_{i,j}$ abutting the lower left of $D_{i,j}$ up by one unit. If there are not any spanned D squares and there is a special unnested D square, then we move the largest D square to the lower left of the lower left most special unnested D square not containing it.
Step 3. If there are no spanned or special unnested D squares, but there are fillers and/or unnested D squares, we check whether there are fillers. If not, we proceed to Step 4. If there are fillers, we check whether the smallest filler is contained in every D square not contained in it and whether the D squares contained in the D square are nested. If yes, we move the smallest filler up by one unit. If not, we proceed to Step 4.
Step 4. If there is a filler and a D square not contained in the smallest filler F does not contain F, we move the largest D square to the lower left of F that does not contain it. In case this square has color C_2 we first run Step 5. If all the D squares not contained in the smallest filler F contain it, we nest the D squares contained in F starting with the lower left most D square not containing the others. In case this square has color C_2 we first run Step 5. Finally, if there are no fillers, we nest the D squares by moving the lower left most D square not containing the lower left most unnested D square. In case this square has color C_2 we first run Step 5.
Step 5. Before we slide a square S of color C_2, we check whether there are any D squares of color C_1 contained in S that abut its lower left. If not, we slide S. If there are such D squares, we consider the largest one S' among them. If the number of B' squares of color C_1 between the B' square passing through the lower left of S' and the next B square or unnested D square S'' of color C_1 is greater than the number of squares of color C_1 contained in S' but not in S'', we still move S. Otherwise, we first move S' before we move S.

We now describe the outcomes of the moves. We will divide the discussion into two cases depending on whether the square we move has color C_1 or C_2.

If the square S we move has color C_1, we replace the tableau M by the following three tableaux. In case the variety associated to one or more of the three tableaux have smaller dimension than the variety associated to M, we discard that tableau. The instances when one or more of the tableaux need to be discarded is determined by the combinatorics of M. We will make this explicit below.

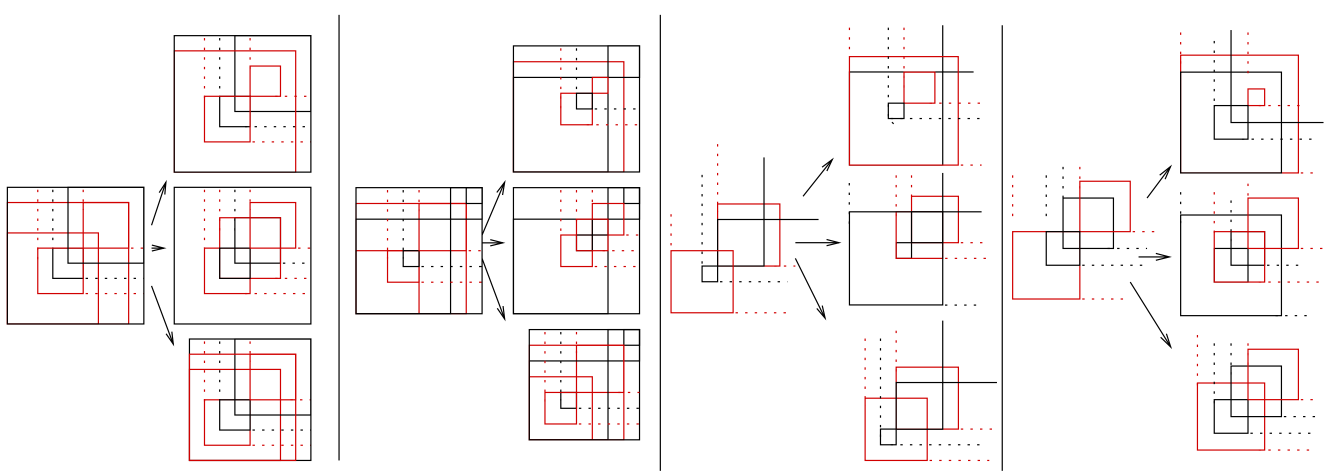

FIGURE 29. Typical admissible painted Mondrian tableaux and the three tableaux that we replace them with.

Tableau 1: If the move increases the side-length of the intersection of S with the first square S' of color C_1 to its northeast, we draw the new intersection and the old span of S and S' in C_1. We delete S and S'. If there is a filler contained in S', we extend the north and east sides of the filler to the old span. This square is still a filler. If there are any D squares (necessarily of color C_2) contained between S and S' but not in S, we extend the north and east sides of it to its old span with S. If S' is a B square, this D square becomes a filler. We also extend the south and west sides of it to its old span with S'. We keep every other square as in M. We apply the Span rule.

Tableau 2: We draw the new intersection of S with the first square S' to its northeast in C_2. If S' is of color C_2, we delete it. If S' is of color C_1, we do not delete it. In this case, the new intersection becomes a filler. We shrink S so that it is the span of the largest square of color C_2 contained in S, but not in S' and the new intersection. S becomes a spanned D square. We apply the MM and S rules. If the generalized MM rule is not satisfied, we discard this tableau. If S does not contain any squares of color C_2 not contained in S' and S' has color C_2, we draw the new intersection in color C_1 instead. If S does not contain any squares of color C_2 not contained in S' and S' has color C_1; or if there are no squares of color C_2 to the northeast of S not contained in S, we discard this tableau. We keep all the other squares as in M.

Tableau 3: We place S in its new position. If the second outer square passes through the south and west corners of S before the move, we restrict the tableau to the northeast of the new position of S. We keep all the other squares as in M. If S starts containing a square of color C_2 as a result of the move and the largest D square of color C_2 is not a unit in S, we still draw this tableau. In this case we shrink the square of color C_2 until it is one unit smaller than (and left justified with) the smallest unit of color C_1 in S. In case S is the A square $A_{i,j}$, we relabel it $A_{i,j+1}$. If there are no units of color C_1 in S, we label the new square of color C_2 by $A_{i-1,j}$. If there are units of color C_1, we slide them up by one unit and draw in C_2 the intersection of the largest B square in color C_2 with largest unit of color C_1 in S. The latter square becomes a spanned D square.

If the square S we move has color C_2, we replace the tableau M with the following three tableaux. Again we discard a tableau in case the variety corresponding to it has smaller dimension.

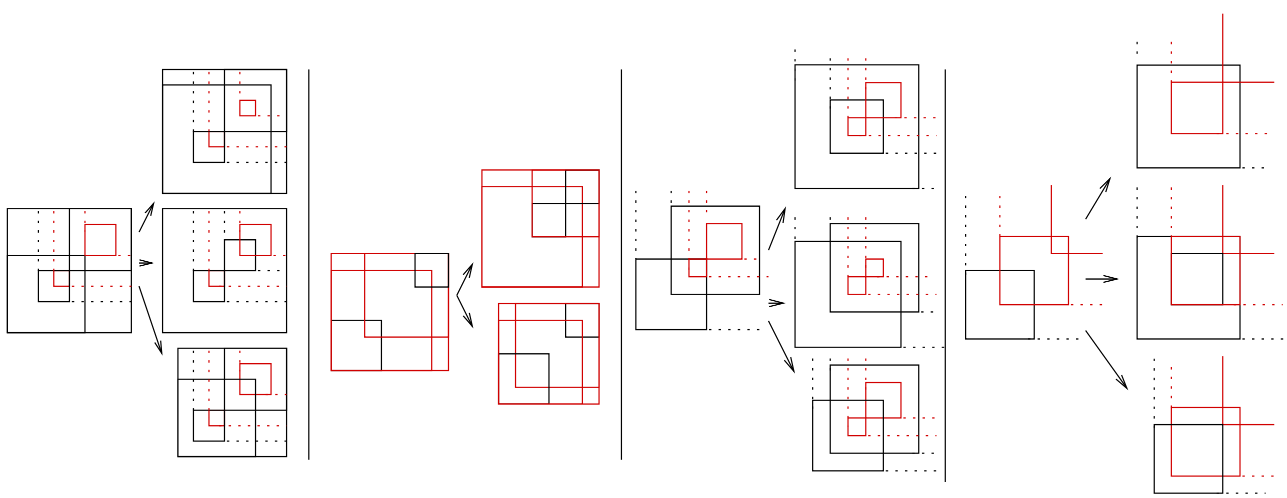

FIGURE 30. Typical admissible painted Mondrian tableaux and the tableaux that we replace them with.

Tableau 1: If the move increases the side-length of the intersection of S with a square S' of color C_1 to its northeast and there are no squares between S and S' in the tableau restricted to the lower left of S, we delete S and S'. We draw the new intersection in C_1 and the old span in C_2. We keep all the other squares as in M. We apply the MM and S rules.

Tableau 2: We draw the intersection of S with the first square S' to the northeast of S in C_2. We delete S. If S' has color C_2, we delete S'. If S' has color C_1, we do not delete S'. The new intersection is then a filler. We draw the old span in C_2 unless the old span is a B or D square of color C_1. In the latter case we keep the old span in C_1. If every square to the northeast of S and not contained in S is red, we omit this tableau. We apply the MM and S rules. We keep all the other squares as in M.

Tableau 3: We place S in its new position. We shrink the side-lengths of all the squares that contain S and have the same lower-left corner as S. If a B square of color C_2 is lower left justified with S and S becomes left justified with a B or B' square of color C_1, we shrink the B square so that it is a unit but does not overlap with any B or B' square of color C_1. We keep all other squares as in M.

We label the squares in these tableaux as follows. With this labeling each of the tableaux are again admissible painted Mondrian tableaux. When we take the intersection between an A square $A_{i,j}$ and a B square, we label the new intersection $D_{i,j}$ and the old span as a B square. In case $A_{i,j}$ becomes a spanned D square, we label it as $D_{i,j}$ and the new intersection as $D'_{i,j}$. When an intersection occurs between two D squares, we label the old span by the index of the unnested square and the new intersection by the indices of the square we move. When we take the intersection between a D square and a B square, we label the old span as a B square and the new intersection by a D square with the same indices as the D square. When drawing Tableau 3 if we need to shrink a B square of color C_2 so that it does not overlap with a B or B' square of color C_1, we adjust the indices of the B squares so that they reflect the number of B squares of each color contained in them. Similarly, when we need to add a unit to S in drawing Tableau 3, we add one to the second indices of all the D squares containing this unit.

We now specify in detail the instances where we have to discard one or more of the tableaux because the corresponding variety has smaller dimension. See Figures 31 and 32 for examples.

When we move a square S of color C_1, then

- We omit Tableau 1 in case the length of the intersection of S with any square of color C_1 does not increase.
- We omit Tableau 1 when S is the largest nested D square contained in a spanned D square.
- We omit Tableau 2 if S' has color C_1 and if either all the squares contained in S but not in S' have color C_1 or if all the squares to the northeast of S' (including S') but not in S have color C_1.
- We omit Tableaux 2 and 3 if either the next square in the $A-D$ flag containing S is of color C_1 and of side-length one larger than S, or if the first square of color C_1 to the northeast of S is a unit in its span with S. In these cases Tableau 1 is the only possibility. (Informally, if S and S' of color C_1 are as large as they can be, Tableau 1 is the only possibility.)
- We omit Tableau 3 if either the next square in the restricted $A-D$ flag containing S has side-length one larger and the largest square of color C_2 contained in S is a unit, or if the next square of color C_2 to the northeast of S is a unit in its span with S. (Informally, if S and the largest square of color C_2 contained in S are as large as they can be, or if the next square of color C_2 to the northeast of S is as large as it can be, then we omit Tableau 3.)

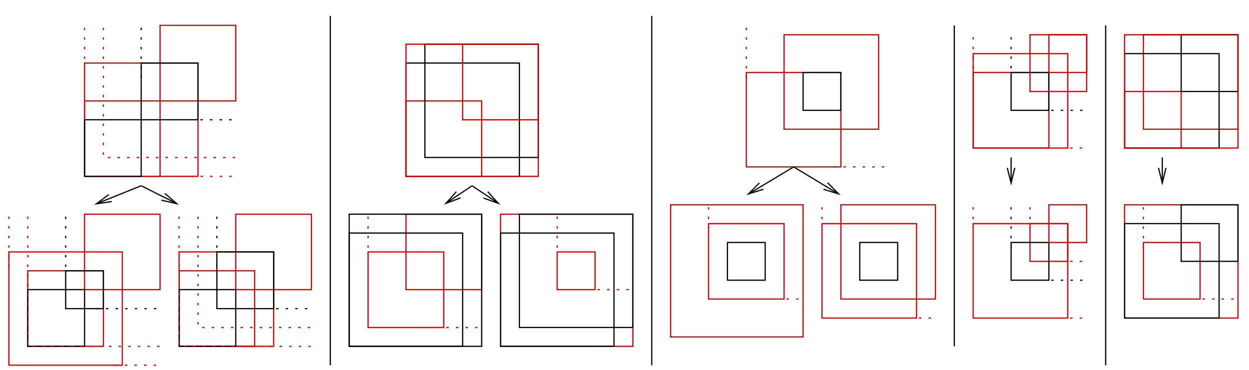

FIGURE 31. Examples where some of the tableaux are discarded because they correspond to varieties of smaller dimension.

When we move a square S of color C_2, then

- We omit Tableau 1, if the move does not increase the length of the intersection of S with a square of color C_1 to its northeast, or if there is a square of color C_2 in the restricted tableau between S and the first square of color C_1 to its northeast, or if in the restricted tableau S is lower left justified with a square of color C_1 containing it but not containing the square of color C_1 to its northeast.
- We omit Tableaux 2 and 3, if after the move a square S' of color C_1 to the northeast of S is contained in S.
- We omit Tableau 3, if the next square in the $A-D$ flag has side-length one larger than S, or if the next square of color C_2 to the northeast of S is a unit in its span with S.

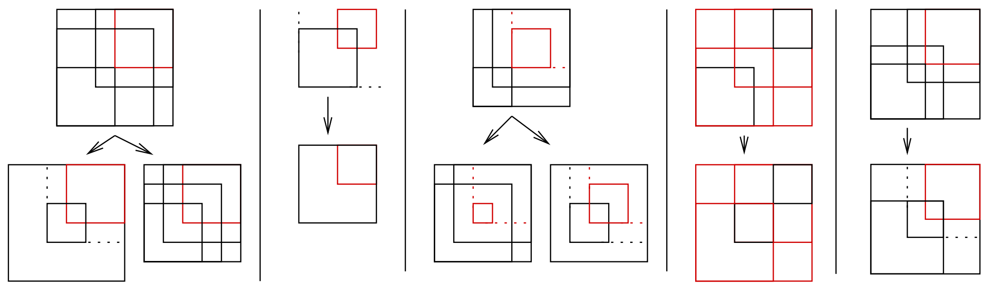

FIGURE 32. Examples where some of the tableaux are discarded because they correspond to varieties of smaller dimension.

When to stop. After applying the MM, OS and S rules, the initial painted Mondrian tableau associated to the product of two Schubert cycles is admissible. It is straightforward, if slightly tedious, to check that given an admissible painted Mondrian tableau, the tableaux that result from the moves are again admissible. Therefore, we can apply the rules to every tableau that arises from the initial tableau. At the end of each cycle of moving an A square, nesting the resulting D squares and eliminating the fillers, we obtain a tableau with one fewer A and B squares and one more nested D square. After a finite number of repetitions, all the squares (ignoring the B' squares) will be nested. We stop applying the algorithm to a tableau at the first instance when all the squares (ignoring the B' squares) in the tableau are nested. When all the squares in all the tableaux are nested, we obtain a collection of tableaux with k_2 nested squares exactly k_1 of which have color C_1. Such a tableau corresponds to a Schubert cycle in $F(k_1, k_2; n)$.

The main theorem of this section is that the coefficient of a Schubert cycle σ_ν in the product of two Schubert cycles $\sigma_\lambda \cdot \sigma_\mu$ in $F(k_1, k_2; n)$ is equal to the number of times the painted Mondrian tableau associated to σ_ν results in the game starting with the tableaux of σ_λ and σ_μ in an $n \times n$ square.

Theorem 4.6. *Let σ_λ and σ_μ denote two Schubert cycles in the flag variety $F(k_1, k_2; n)$. Let their product be $\sigma_\lambda \cdot \sigma_\mu = \sum_\nu c_{\lambda\mu}^\nu \sigma_\nu$. The coefficient $c_{\lambda\mu}^\nu$ is equal to the number of times the painted Mondrian tableau of σ_ν occurs in a game of Mondrian tableau played by starting with the Mondrian tableaux of σ_λ and σ_μ in an $n \times n$ square.*

Proof. The proof is similar to the proof of Theorem 3.6. We interpret the moves on painted Mondrian tableaux as codimension one degenerations of the A and B flags. First, a dimension count allows us to identify the set theoretic limits. Then an easy local calculation shows that the limits occur with multiplicity one.

Lemma 4.5 allows us to compute the dimension of the variety associated to a painted Mondrian tableau. An inspection of the possible outcomes after each move allows us to conclude that the varieties corresponding to Tableaux 1,2 and 3 have the same dimension as the original variety. We need to show that there are no other loci in the limit of the degeneration of this or larger dimension. More generally, the same dimension calculation as in Lemma 4.5 holds for a variety corresponding to a painted Mondrian tableau that is built by taking a generalized Mondrian tableau in $G(k_2, n)$ (with possibly chopped squares) and choosing k_1 squares to be squares spanned by a collection of these k_2 squares satisfying the following two properties:

(1) None of the squares of color C_1 is equal to the span of the squares of color C_1 contained in it.

(2) Let S_1 and S_2 be any two squares of color C_1. Assume that there are s squares of color C_1 contained in their span and not contained in S_1. Then the dimension of the subspace of V_2 contained in S_1 is at least s less than the dimension of the subspace of V_2 contained in their span.

The dimension of such a variety is equal to the sum of the dimension of the variety corresponding to the generalized Mondrian tableau in $G(k_2, n)$ and the fiber dimension of the projection to $G(k_2, n)$. The fiber dimension is given by the total number of squares of the tableau for $G(k_2, n)$ contained in each square of color C_1 minus one for every containment relation between squares of color C_1. Tableaux built this way correspond to irreducible subvarieties of $F(k_1, k_2; n)$. Every component of the limit cycle is contained in such a variety.

Now we run through the possibilities and see how the moves transform an admissible painted Mondrian tableau to three or fewer admissible painted Mondrian tableaux. The most important case is when the D squares are nested and there are no fillers. The other cases reduce to this case for the tableau restricted to the B' square passing through their lower left.

The D squares are nested and there are no fillers. Let $A_{i,j}$ be the smallest A square. First, suppose $A_{i,j}$ has color C_1. If $A_{i,j}$ is a floating square (i.e., the largest B square of color C_1 is not left justified with the second outer square), Step 1 instructs us to delete $A_{i,j}$ and the largest B square of color C_1; draw their intersection in C_1 and label it $D_{i,j}$; and apply the OS rule. In this case there are no degenerations, we simply rename the AB_i^1 square $D_{i,j}$. It is clear that the resulting tableau is an admissible painted Mondrian tableau. The number of A and B squares of color C_1 both decrease by one. There is at most one unnested D square (of color C_1). There are no fillers; the relative positions of the chopped squares remains the same and the condition about side-lengths is obviously satisfied. In this case, there is nothing to prove.

Next suppose we move the smallest A square $A_{i,j}$. When we move its lower left most piece, we do not cause any new squares to be chopped. If $A_{i,j}$ has color C_1, then in Tableau 1 there is one fewer A and B squares of color C_1 and one new possibly unnested D square $D_{i,j}$ of color C_1. There are no fillers. Some B squares of color C_2 may contain $D_{i,j}$, but the only B square of color C_1 that contains it is the second outer square. Hence, Tableau 1 is an admissible painted Mondrian tableau. In Tableau 2, there are two newly formed D squares. The old $A_{i,j}$ is a spanned D square and it is the span of the largest D square of color C_1 contained in $A_{i,j}$ and the new intersection with the largest B square (which is not the second outer square). Hence there are two new D squares, one of color C_1 and one of C_2. If the B square is of color C_2, then the number of A squares of color C_1 decreases by one and the number of B squares of color C_2 decreases by one. If the B square is of color C_1, then the new D square of color C_2 is a filler. The second outer square is an extra B square of color C_2. Applying the S rule may create new chopped squares. However, since no new chopped squares are created until the filler becomes right justified with an A square containing it, the conditions about chopped squares are satisfied. The conditions about the side-lengths hold since for this possibility to occur a B square of color C_1 cannot be a unit. Hence, Tableau 2 is an admissible painted Mondrian tableau. In Tableau 3 the number of A, B and D squares remain constant and there are no fillers. Since this tableau does not occur if any of the B squares are a unit, the condition about side-lengths hold. Hence, Tableau 3 is admissible. If the move forces $A_{i,j}$ to contain the next A square of color C_2, the Tableau 3 described above is admissible. If there are no units of color C_1 contained in $A_{i,j}$, the number of A, B and D squares remains constant. There are no fillers or unnested D squares. If there is a unit of color C_1, then the number of A squares decreases by one. Either the number of B squares also decreases or there is a new filler. Hence, Tableau 3 is admissible in these cases as well. A very similar analysis applies if in Step 5, we move the largest left justified D square of color C_1.

Now suppose the smallest A square $A_{i,j}$ we move has color C_2. In Tableau 1, the number of A and B squares of color C_1 decrease by one. There is a new square $D_{i,j}$ of color C_1. In this case there are no fillers. Hence, Tableau 1 is an admissible tableau. In Tableau 2, if the largest B square (which is not the second outer square) has color C_2, then the number of A and B squares of color C_2 decrease by one and $D_{i,j}$ is a new square of color C_2. In this case there are no fillers. Hence, the tableau is admissible. If the B square has color C_1, then the number of A squares of color C_2 decreases by one. The new intersection $D_{i,j}$ is a filler. The number of B squares does not change. Applying the S rule may create new chopped squares. However, since this does not happen until the filler is right justified with a square containing it, the conditions about chopped squares are satisfied. Hence, this tableau is admissible. Finally in Tableau 3, the number of A, B and D squares remains the same. The conditions about side-lengths are satisfied because this tableau does not occur if any of the B squares are units. Hence, Tableau 3 is also admissible.

Now we argue that the varieties corresponding to these tableaux are the only possible limits. Using the same reduction as in the proof of Theorem 3.6 we can assume that $A_{i,j}$ is the only square that moves during the degeneration. Otherwise we first move the D squares abutting the lower left of $A_{i,j}$ up by one unit starting with the smallest one. When we move such a D square, the argument we will now give applies to show that there is a unique outcome given by simply sliding that D square up by one unit in the original tableau. We can factor the move to many moves where we first move the D squares that abut the lower left of $A_{i,j}$ in increasing order and then move $A_{i,j}$. The outcome of carrying out the degeneration in a few steps is the same as moving $A_{i,j}$ and dragging any of the D squares that abut its lower left with it.

Recall that in order to obtain the image of the projection π_2 of an admissible painted tableau to $G(k_2, n)$ we take all the D squares and draw them in C_2 omitting any spanned D squares and take all the squares AB_t^2. When we carry out a degeneration, any of the linear spaces (V_1, V_2) contained in the limit have to continue to satisfy the rank conditions with respect to the vector spaces represented by the A, B and D squares that they satisfied prior to the degeneration. The argument we gave in the proof of Theorem 3.6 determines the possible limiting positions of the linear spaces V_2 and estimates the dimension of each subloci corresponding to a generalized Mondrian tableau. To obtain a characterization of the equi-dimensional loci we have to bound the fiber dimension of the projection π_2 over the various loci of V_2. The fiber dimension of the projection can increase if the total number of squares contained in the squares of color C_1 increases. As long as the fiber dimension is constant, the projection to $G(k_2, n)$ must be an equi-dimensional variety. We have determined these varieties in §3.

Let (V_1, V_2) be a general point on a component of the limit cycle when we move $A_{i,j}$. If the subspace of V_1 contained in $\mathrm{A}_{i,j}$ remains independent from the subspace of V_1 contained in the vector spaces represented by the B squares except along the vector spaces represented by the D squares common to both, then V_1 must be contained in the new span of $\mathrm{A}_{i,j}$ and the vector space represented by the largest interior B square of color C_1. If, in addition, the subspace of V_2 contained in $\mathrm{A}_{i,j}$ also remains independent from the subspace of V_2 contained in the vector spaces represented by the B squares except along vector spaces represented by the D squares common to both, then V_2 must be contained in the new span of $\mathrm{A}_{i,j}$ and the vector space represented by the largest interior B square. Since the rank conditions are closed conditions, V_1 and V_2 must continue to satisfy the intersection conditions required by the D squares. A priori the subspaces of V_1 and V_2 might meet the AB_s^1 and AB_t^2 in more specialized ways.

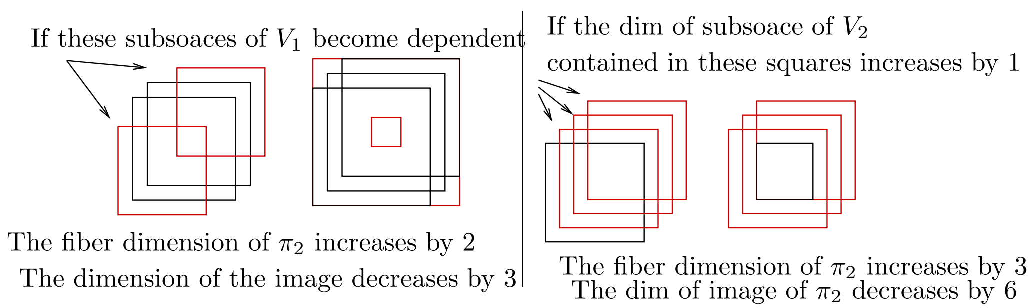

FIGURE 33. The dimension decreases if the subspaces contained in the AB squares become more dependent.

However, it is easy to see that the dimension is maximized only when the subspaces of V_1 and V_2 contained in AB_s^1 and AB_t^2 remain independent. By the discussion in §3 the dimension of the projection decreases unless the subspaces of V_2 contained in AB_t^2 remain independent. However, the fiber dimension may increase if the dimension of V_2 contained in AB_s^1 increases. This happens when V_2 meets AB_s^1 in larger dimension.

If V_2 meets the vector space represented by AB_t^2 along the intersection of the vector spaces represented by j of the AB_s^1 squares of color C_1 that do not contain AB_t^2, then the dimension changes as follows. Suppose AB_t^2 contains j_0 of the AB_s^1 squares. Then the number of containment relations among squares in the projection increases by $j - j_0$. On the other hand, the side-length of the squares decreases by at least $j_0 + 1$. The fiber dimension increases by j. Hence such loci cannot form a component of the limit.

If V_1 meets the intersection of the vector spaces represented by two AB_s^1 squares, then we draw the intersection in C_1. We leave the squares that contain either of the two squares unchanged. We replace the squares between the two AB_s^1 squares by the span of the consecutive squares. The change in dimension in this case can be computed as follows. The dimension of the subspace of V_2 contained in the vector spaces represented by the squares of color C_1 increases by the number of squares of color C_2 contained between the two squares. The number of containment relations between squares of color C_1 increases by one more than the number of C_1 squares contained between the two squares. The image of the projection π_2 drops by the number of squares between the two squares. Hence such loci cannot form a component of the limit (see Figure 33).

We conclude that any component where the subspaces of V_1 and V_2 contained in $\mathrm{A}_{i,j}$ remain independent from the subspaces contained in the vector spaces represented by the B squares except along subspaces represented by the D squares common to both, must be contained in the locus corresponding to Tableau 3. Since the locus corresponding to Tableau 3 is irreducible and of the correct dimension, any such component must be supported on this locus.

If $A_{*,j+1}$ is a square one unit larger or the largest interior B square is as large as it can be given the second outer square, then V_1 or V_2 must meet the intersection of $\mathrm{A}_{i,j}$ and the vector space represented by the largest interior B square. However, it is still possible that V_2 lies in the new span of $\mathrm{A}_{i,j}$ and the vector space represented by the largest interior B square. We will shortly see that when a square of color C_2 specializes to lie in a square of color C_1, the corresponding dimension of the image of the projection π_2 decreases by the number of squares that are contained in the square of color C_1 that are units for V_2. On the other hand, the fiber dimension of the projection increases by the number of squares of color C_1 among these squares that are units for C_2. It follows that if $A_{i,j}$ is of color C_2, the second outer square is of color C_2 and the largest interior B square of color C_1 is not a unit, then V_2 may lie in the new span. Reciprocally, if $A_{i,j}$ is of color C_1, $A_{i,j+1}$ is one larger and of color C_2 and the largest D square of color C_2 is not a unit in $A_{i,j}$, V_2 may lie in the new span. The cases where Tableau 3 is discarded follow from this dimension computation.

We can now assume that at a general point of a component of the limit cycle, the subspace of V_1 or V_2 contained in $\mathrm{A}_{i,j}$ meets the subspace of V_1 or V_2 contained in the vector space represented by the largest interior B square. We would like to prove that at a general point of any component of the limit cycle either V_1 intersects $\mathrm{A}_{i,j}$ along the vector space represented by the largest B square of color C_1 in dimension one larger and the subspaces contained in the $\mathrm{D}_{h,l}$ and AB_s^1 remain otherwise independent; or V_2 intersects $\mathrm{A}_{i,j}$ along vector space represented by the largest B square in dimension one more than the number of D squares common to both and the other subspaces remain independent. There will be exceptions when all the squares in $A_{i,j}$ not contained in the largest B square are of color C_1 or all the B squares are of color C_1.

A priori there may be components where V_1 and V_2 meet the vector spaces represented by the D and B squares in more specialized ways. We would like to show that such loci have strictly smaller dimension, hence cannot form components of the limit cycle. Given the hypothetical limit of the degeneration we can construct a generalized painted Mondrian tableau that contains this locus. If V_1 and V_2 in the limit intersect the D and B squares along their intersections, we draw these squares of color C_1 for the intersections of V_1 and of color C_2 for the intersections of V_2. We then complete the tableau by drawing squares so that the number of squares of color C_1 in each square of color C_1 is equal to the number before the degeneration and the number of squares in each square is equal to the number of squares before the degeneration. This tableau must contain the hypothetical limit because the rank conditions are closed conditions and hence must be satisfied by the limit cycles. Once we make this construction we are reduced to comparing dimensions of generalized painted Mondrian tableaux.

Let (V_1, V_2) be a general point of a component of the limit cycle. Suppose the subspaces of V_1 or V_2 contained in the vector spaces represented by some of the D squares meet AB_s^1 and AB_t^2 in dimensions larger than the number of squares common to the squares.

If V_1 meets the intersection of the vector spaces represented by two squares of color C_1, then we draw the intersection in C_1. We leave any squares that contain either of the two squares or are contained in the intersection unchanged. We also leave any squares that do not contain the intersection unchanged. We redraw the squares in between the two squares so that they are the spans of the consecutive squares. Since the limit variety has to satisfy the rank conditions with respect to all the A, B and D squares, if V_1 meets the intersection of a vector space represented by a D square of color C_1 with AB_s^1, the locus has to be contained in a variety constructed in this way. If this intersection happens between two squares of color C_1 that we did not move, then the change in the dimension of the tableau is given as follows:

The number of AB_t^2 squares in some of the newly formed squares of color C_1 may increase. This increase in the fiber dimension of the projection π_2 is at most equal to the number of AB_t^1 squares of color C_2 between the two squares of color C_1 not contained in either. On the other hand, the dimension of the image of the projection drops by one more than the number of squares between these squares not contained in either of them of any color. The increase in the fiber dimension is at least one less than the decrease in the dimension of the image of π_2.

If V_1 intersects the intersection of vector spaces represented by a square of color C_1 and a square of color C_2, then we draw the intersection in C_1. We keep the AB_t^2 and D squares that contain either of the two squares or do not contain the intersection unchanged. For the squares between the two squares we take the spans of the consecutive squares.

If V_1 meets the intersection of a square of color C_1 and a square of color C_2 that we did not move and no other changes happen, then the fiber dimension of the projection π_2 drops by an amount equal to the number of squares of color C_2 contained in the square of color C_1 and not in the square of color C_2. On the other hand, the dimension of the image drops by one more than the number of squares between these two squares not contained in either or not containing either.

If V_2 meets the intersection of vector spaces represented by a square of color C_1 and a square of color C_2, then there are two possibilities. Either the dimension of the subspace of V_2 contained in the vector space represented by the square of color C_1 remains constant or it increases by one. If the dimension of the subspace of V_2 contained in the vector space represented by the squares of color C_1 remain constant, then the fiber dimension of the projection π_2 remains constant, but the dimension of the image of π_2 decreases by the discussion in §3. We can assume that we are in the case when the dimension of the subspace of V_2 contained in the vector space represented by the square of color C_1 increases. In this case we delete the square of color C_2 and draw the new intersection in C_2. We leave the other squares unchanged.

If V_2 meets the intersection of vector spaces represented by a square of color C_1 and a square of color C_2 that we did not move, then the fiber dimension of the projection increases by the number of squares of color C_1 that contain the intersection, but not the square of color C_2. On the other hand, the dimension of the image decreases by one more than the number of squares that do not contain the square of color C_2 and are in between the square of color C_1 and the square of color C_2. The increase in the fiber dimension is at least one less than the drop in the dimension of the image of the projection.

Finally, V_1 or V_2 may meet the intersection of vector spaces represented by two squares of color C_2 that we did not move. In case V_1 meets the intersection it is easy to see that the resulting variety has smaller dimension. In case V_2 meets the intersection, then the dimension of the fiber increases by the number of squares of color C_2 not containing either of them. The image of the projection decreases by one more than the number of squares between the two squares. Again the gain in the fiber dimension is less than the drop in the dimension of the image.

Now suppose that when we move $A_{i,j}$ the vector spaces V_1 or V_2 meet $\mathrm{A}_{i,j}$ along its intersection with one of vector spaces represented by the B squares. The above calculation of the fiber dimensions of π_2 do not change. However, the dimension of the image of π_2 increases by one since the sum of the side-lengths of the squares increases by one.

Since we can build a generalized Mondrian tableau that contains any component of the limit cycle by repeatedly taking the intersections of various D and AB squares, our dimension count determines the possible limits of the degenerations.

First, taking the intersections of any two squares that we do not move strictly decreases the dimension. Taking the intersection of $A_{i,j}$ with any of the squares either keeps the dimension equal or strictly decreases it. We can only obtain an equi-dimensional locus if the only intersection occurs between $A_{i,j}$ and a B square and the subspaces of V_1 and V_2 contained in the remaining D and AB squares remain as independent as possible.

Suppose $A_{i,j}$ has color C_1. If V_1 meets the intersection of $\mathrm{A}_{i,j}$ with a vector space represented by a B square $B_{h.l}$ of color C_1, then the dimension of the fiber of the projection of π_2 increases by the number of B squares

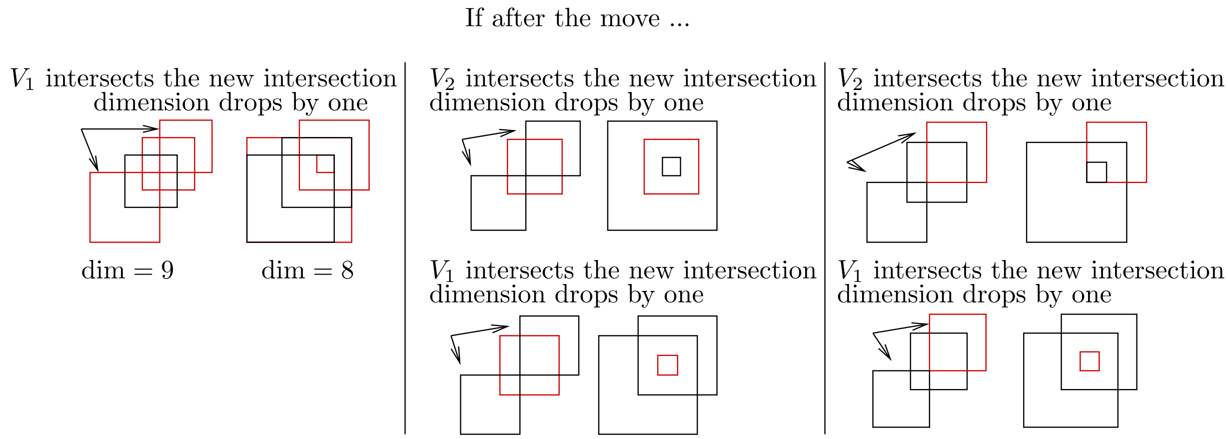

FIGURE 34. Examples of the dimension counts. The dimension drops if subspaces of V_1 and V_2 other than those described in the game become dependent.

of color C_2 containing $B_{h,l}$. On the other hand, the dimension of the image of π_2 decreases by the number of B squares containing $B_{h,l}$. We can only have an equality if $B_{h,l}$ is the largest B square of color C_1.

If V_1 meets the intersection of $\mathrm{A}_{i,j}$ with a vector space represented by a B square $B_{h,l}$ of color C_2, then $B_{h,l}$ must be the largest interior B square and there cannot be any squares of color C_2 contained in $A_{i,j}$ and not $B_{h,l}$. Otherwise, the fiber dimension decreases. Note that if the largest interior B square of color C_1 is a unit or $A_{i+1,j+1}$ is a square of color C_1 one unit larger than $A_{i,j}$, then this possibility must occur.

If V_2 meets the intersection of $\mathrm{A}_{i,j}$ and $\mathrm{B}_{h,l}$, then the B square has to be the largest interior B square. There are two possibilities, either the dimension of V_2 contained in $\mathrm{A}_{i,j}$ remains constant. This case corresponds to Tableau 2. The dimension of V_2 contained in $\mathrm{A}_{i,j}$ may increase by one. If the largest D square of color C_2 is a unit, then the gain in fiber dimension is less than the drop in the dimension of the image of π_2. Similarly if $A_{*,j+1}$ is a square at least two units larger, then the image of the projection drops by more than the gain in the fiber dimension. We conclude that the latter case occurs when $A_{i,j+1}$ is a square of color C_2 and one unit larger than $A_{i,j}$ provided the largest D square of color C_2 in $A_{i,j}$ is not a unit.

If $A_{i,j}$ is of color C_2, then V_2 may meet the intersection of $\mathrm{A}_{i,j}$ and the vector space represented by the largest interior B square. If the largest interior B square is of color C_1, then there are two possibilities. These cases are reciprocal to the cases above. Either V_2 meets the vector space represented by the largest interior B square in one larger dimension or in dimension equal to its previous dimension. The latter case corresponds to Tableau 2. In the first case if we do not have Tableau 3, the second outer square cannot be of color C_1 and the largest square of color C_2 contained in the largest interior B square cannot be a unit and finally the move must force $A_{i,j}$ to be contained in the largest interior B square.

Finally, V_1 may meet the intersection of $\mathrm{A}_{i,j}$ and the vector space represented by the largest interior B square of color C_1. If $A_{i,j+1}$ is not of color C_1, then the fiber dimension of π_2 decreases by at least one and the image of π_2 can have at most the same dimension. If $A_{i+1,j+1}$ is a floating square, then this forms a neighbor of $A_{i,j}$ in the projection. By the proof of Theorem 3.6 that case has the same dimension. If $A_{i+1,j+1}$ is not a floating square, even if the fiber dimension of the projection π_2 remains constant, the dimension of the image of π_2 decreases. We recover the description of Tableau 1. See schematic representations in Figure 34.

The case when we move a filler. We now assume that every D square not contained in the smallest filler F contains it and the D squares contained in it are nested. In this case the algorithm instructs us to move F. When we move F (which is always a square of color C_2), the only possibilities are recorded by Tableaux 2 and 3. The filler can only interact with a B square to its northeast. If this square is of color C_1, the new intersection is again a filler. Note that any filler that contained F continues to contain F and since F was the smallest filler, all the fillers are still nested. This case does not change any of the numbers of A, B or D squares. Hence it is clear that the tableau is admissible. If F interacts with a B square of color C_2, then the intersection is no longer a filler. In this case we have one fewer B square of color C_2 and one fewer filler. The number of A and

D squares remain the same. It is again easy to see that the tableau is admissible. In Tableau 3 the number of A, B, and D squares remains unchanged. No new fillers are created. It is clear that Tableau 3 is admissible.

The new twist in this case is that moving the filler might increase the dimension of intersection of some A and D squares that contain the filler with the next B square. We can by the usual reduction assume that the D squares that are contained in the filler do not move. Suppose the index of the B square that passes through the lower left of the filler is $B_{h,l}^{h',l'}$. If at a general point of a component of the limit the linear subspaces of V_1 and V_2 contained in A_{*,k_2-l} remain independent from the linear subspaces of V_1 and V_2 contained in $\mathrm{B}_{h,l}$ except along the vector spaces represented by D squares common to both, then we can take the projection of the variety to the flag variety $F(h', l', |B_{h,l}^{h',l'}|)$ by sending (V_1, V_2) to their subspaces contained in $\mathrm{B}_{h,l}$. By assumption the fiber dimension of this map remains equal to the fiber dimension of the map prior to the degeneration. Note that since $B_{h,l}$ is of color C_1, there cannot be any floating squares in tableau restricted to $F(h', l', |B_{h,l}^{h',l'}|)$. We can think of moving the filler as moving the smallest A square in the restricted tableau for $F(h', l', |B_{h,l}^{h',l'}|)$. Since there are no floating squares only Tableaux 2 and 3 occur.

To complete the argument in the case when we move F, we need to show that the loci where the linear subspaces contained in the vector spaces represented by A or D squares that contains F intersect the linear subspace contained in $\mathrm{B}_{*,l-1}$ have strictly smaller dimension. Here we give the argument for V_1. The case of V_2 is easier. Suppose the linear subspace of V_1 contained in A_{*,k_2-l} has an intersection with the linear subspace of V_1 contained in $\mathrm{B}_{h,l}$ of dimension larger than the number of D squares of color C_1 common to both. By the usual argument, the dimension is maximized when V_1 meets the intersection of the next B square of color C_1 along one of the squares of color C_1 whose intersection with that square increased as a result of the move. We can draw this intersection and delete the two squares of color C_1. Since the linear spaces V_1 and V_2 must still satisfy the rank conditions with respect to the A and D squares.

When we compare this tableau to Tableau 2, we see that this tableau differs from Tableau 2 in that a D or A square of color C_1 and the filler have been swapped. Hence the dimension of such a tableau is at least one smaller. We conclude that in any limit the subspace of V_1 contained in A_{*,k_2-l} does not intersect the subspaces of V_1 contained in any of the B squares in dimension larger than the number of D squares common to both (see Figure 35).

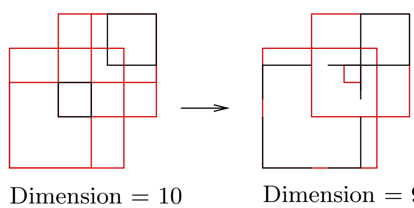

FIGURE 35. The dimension decreases if V_1 meets an A or D square containing a filler with its intersection with a B square.

Nesting the D squares. First, suppose there is a spanned D square. When we move the largest nested D square abutting the lower left of the spanned D square, the only tableaux that occur are Tableaux 2 and 3. Tableau 3 is clearly admissible. If S has color C_1, in Tableau 2 the new intersection is still a filler. The new S is (possibly) a spanned square. The old spanned square is now a special unnested D square. If S has color C_2, the new intersection is still a filler. There are no longer any spanned D squares. The old spanned D square is (possibly) a special unnested D square.

Suppose there are no spanned D squares, but there are special unnested D squares. When we move the squares to the lower left of the lower left most special unnested D square, the only tableaux that occur are Tableaux 1 and 3. It is easy to check that these are admissible tableaux. In neither of these cases any fillers are formed. In each case the number of A, B and D squares remains the same.

Suppose there are no spanned or special unnested D squares, but there is a D square which does not contain the smallest filler. The next smallest filler (if it exists) contains both the square S we move and the smallest filler. If S has color C_1, in Tableau 1 the new filler is contained in every D square not contained in it. The number of A, B and D squares remain constant. It is clear that Tableaux 2 and 3 are admissible.

Suppose there are no spanned or special D squares and all the squares not contained in the smallest filler contain it. In this case the analysis is easier. If any fillers form, then they will be automatically contained in all the existing fillers. The conditions about the numbers of A, B and D squares are easy to check as in the previous cases. Hence all the tableaux are admissible. Finally the analysis in the case when there are no special or unnested D squares or any fillers is easy.

We observe that when nesting the D squares, the dimension counts reduce to the previous discussion. We can restrict the painted Mondrian tableau to the B' square passing through the lower left of the D square we are moving. To obtain the restricted painted Mondrian tableau we take the sub-tableau consisting of the D, AB_s^1 and AB_t^2 squares contained in the relevant B' square. We relabel the D square we move as an A square and keep the squares contained in it as D squares. We also relabel any D square contained in the restricted tableau but not contained in the D square we are moving as AB squares. Once we do this the same dimension count given above applies to this tableau. The dimension count given in the case of moving the fillers shows that if the vector subspace of V_1 or V_2 contained in the vector space represented by a D square not contained in the restricted tableau intersects the vector space represented by the next B or unnested D square, we get a strictly smaller dimensional variety. We can, therefore, confine our attention to the restricted tableau. We leave the details of specifying the possibilities in this case to the reader.

Now that we have identified the limits we need to know the multiplicities with which they occur. To conclude the proof of the theorem it suffices to show that all the multiplicities are one. We will prove this theorem by reducing it to Monk's formula.

It suffices to carry out these calculations in the case the square we move is $A_{i,j}$. The cases when we move a D square reduces to this case by restricting to the tableau contained in the B' square passing through the lower left of the D square. Suppose the square we move has color C_1 and V_1 meets the intersection of $\mathrm{A}_{i,j}$ with the vector space represented by the next B square of color C_1. We can consider the projection to the Grassmannian $G(k_1, n)$ by π_1. At a general point of the relevant loci this morphism is smooth since it the fibers are open sets in towers of flag variety bundles. Hence this case follows from Theorem 3.6

If V_2 meets the intersection of $\mathrm{A}_{i,j}$ with the vector space represented by the largest B square or both V_1 and V_2 lie in the new span of $\mathrm{A}_{i,j}$ and the vector space represented by the largest interior B square, then we can consider the projection to $G(k_2, n)$. Again the multiplicity follows by pulling back the multiplicity where we know it to be one by Theorem 3.6.

Now suppose $A_{i,j}$ has color C_2. Suppose V_2 meets the intersection of $\mathrm{A}_{i,j}$ and the vector space represented by the largest B square. We can take the projection to $G(k_2, n)$. This is smooth at a general point of the relevant locus. The problem reduces to a question in the Grassmannian. That the multiplicity is 1 follows as in the proof of Theorem 3.6 by Pieri's formula for lines.

If V_1 meets the intersection of $A_{i,j}$ with $A_{i+1,j+1}$ which is a floating square, then we can reduce the local picture to one in $F(1, 2; n)$. In that case the multiplicity is one. We pull it back to deduce it for that possibility. It is clear that Possibility 3 occurs with multiplicity one.

The multiplicity calculations for when we move the filler are identical. There is a projection from $F(k_1, k_2; n)$ to the smaller flag variety contained in the B square that passes through the lower left of the filler. This is obtained by sending the pair (V_1, V_2) to their intersections in the B square. In this case the degeneration reduces to the case when there are no fillers and we move an $A_{i,j}$ square of color C_2. The discussion of that case proves that the multiplicity is one in this case as well. This completes the proof. □

5. A quantum Littlewood-Richardson rule for Grassmannians

In this section we obtain a quantum Littlewood-Richardson rule for Grassmannians $G(k, n)$ as a corollary to Theorem 4.6.

Given a Mondrian tableau for σ_λ in $G(k, n)$ and an integer $d \leq k$, we can associate to it a painted Mondrian tableau in $F(k-d, k+d; n)$ as follows: The Mondrian tableau associated to the Schubert variety σ_λ consists of k nested squares. We take the largest $k-d$ squares (those of index $d+1, \ldots, k$) and color them in C_1. We color the remaining squares in C_2. Finally, we add d squares of color C_2 at the largest available places in the flag defining the Mondrian tableau of σ_λ (see Figure 36 for two examples). We call the resulting painted Mondrian tableau the quantum Mondrian tableau of degree d associated to σ_λ. This tableau is none other than the painted Mondrian tableau associated to the special Schubert variety $X_\lambda^{(d)}$ of $F(k-d, k+d; n)$ defined in §2.

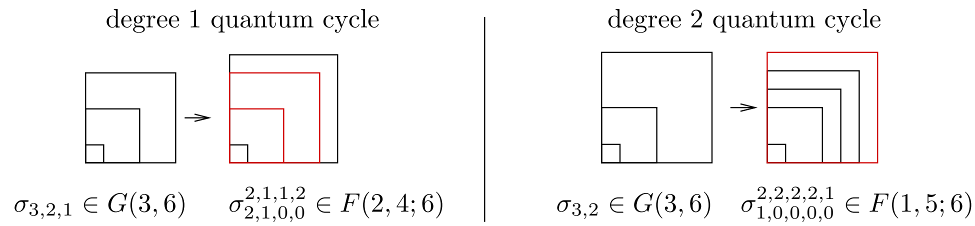

FIGURE 36. The quantum Mondrian tableaux associated to two Schubert varieties.

Let $\sigma_\lambda, \sigma_\mu$ and σ_ν be three Schubert cycles in $G(k,n)$ that satisfy the equality

$$|\lambda| + |\mu| + |\nu| = k(n-k) + dn.$$

Apply the algorithm described in the previous section to the quantum Mondrian tableau of degree d associated to σ_λ and σ_μ to express their intersections as a sum of Schubert cycles in $F(k-d, k+d; n)$. Then apply the algorithm to the quantum Mondrian tableau of degree d associated to σ_ν and each of the summands of the previous product. Theorem 4.6 and Lemma 2.6 imply that the Gromov-Witten invariant $I_d(\sigma_\lambda, \sigma_\mu, \sigma_\nu)$ is equal to the number of times we obtain the class of a point as a result of these multiplications. We have obtained the following theorem.

Theorem 5.1. *The three-pointed Gromov-Witten invariant $I_d(\sigma_\lambda, \sigma_\mu, \sigma_\nu)$ is equal to the number of times the point class occurs as a result of applying the Littlewood-Richardson rule for the two-step flag varieties to the quantum Mondrian tableau of degree d associated to σ_ν and each outcome of the product of the quantum Mondrian tableaux of degree d associated to σ_λ and σ_μ.*

We illustrate the use of Theorem 5.1 by computing the Gromov-Witten invariant

$$I_{G(3,6),d=1}(\sigma_{3,2,1}, \sigma_{3,2,1}, \sigma_{2,1}) = 2.$$

Figure 37 demonstrates the computation. The quantum cycle of $d = 1$ associated to $\sigma_{3,2,1}$ (respectively, $\sigma_{2,1}$) is $\sigma_{2,1,0,0}^{2,1,1,2}$ (respectively, $\sigma_{1,0,0,0}^{2,1,2,1}$). In order to calculate the Gromov-Witten invariant we have to find how many times $\sigma_{2,2,2,1}^{1,2,1,2}$ (the dual of $\sigma_{1,0,0,0}^{2,1,2,1}$) occurs in the square of the class $\sigma_{2,1,0,0}^{2,1,1,2}$. An easy calculation with painted Mondrian tableaux shows that the answer is 2.

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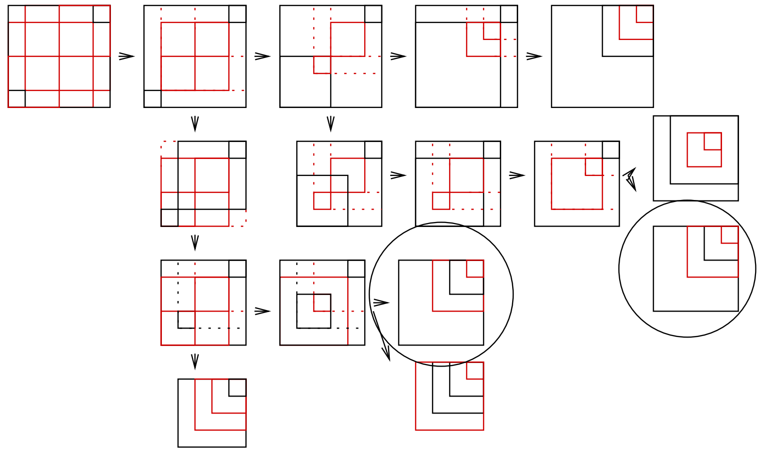

FIGURE 37. Computing the Gromov-Witten invariant $I_{G(3,6),d=1}(\sigma_{3,2,1}, \sigma_{3,2,1}, \sigma_{2,1}) = 2$.

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