PARAMETER SPACES OF SCHUBERT VARIETIES IN HYPERPLANE SECTIONS OF GRASSMANNIANS

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Abstract. Linear sections of Grassmannians provide important examples of varieties. The geometry of these linear sections is closely tied to the spaces of Schubert varieties contained in them. In this paper, we describe the spaces of Schubert varieties contained in hyperplane sections of $G(2,n)$. The group $PGL(n)$ acts with finitely many orbits on the dual of the Plücker space $P^*(\Lambda^2 V)$. The orbits are determined by the singular locus of $H \cap G(2,n)$. For $H$ in each orbit, we describe the spaces of Schubert varieties contained in $H \cap G(2,n)$. We also discuss some generalizations to $G(k,n)$.

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1. Introduction

Linear sections of Grassmannians provide examples that play an important role in many branches of algebraic geometry, including the classification of varieties, derived equivalences and mirror symmetry. For example, general codimension four linear sections of $G(2,5)$ are Del Pezzo surfaces of degree five (see [CH]) and general codimension seven linear sections of $G(2,7)$ are Calabi-Yau threefolds (see [BC], [R]). The geometry of a linear section $X$ of a Grassmannian is closely tied to the spaces of Schubert varieties contained in $X$, which provide crucial information about the cohomology and Hodge structure of $X$ (see [D] and Chapter 6 of [GH]). In this paper, we will describe the spaces of Schubert varieties contained in a hyperplane section of a Grassmannian.

Let $G(k,n)$ denote the Grassmannian parameterizing $k$-dimensional subspaces of a fixed $n$-dimensional vector space $V$. Let $\lambda$ denote a partition whose parts satisfy

$$n - k \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 0.$$  

When writing a partition, the parts that are equal to zero are often omitted. For many purposes, it is more convenient to group together the parts of $\lambda$ that are equal. We will write $\lambda$ also as $\lambda = (\mu_1, \ldots, \mu_t)$ and set $k_s = \sum_{j=1}^s i_j$, where $\mu_1 > \mu_2 > \cdots > \mu_t$ and

$$\mu_1 = \lambda_1 = \cdots = \lambda_{k_1}, \mu_2 = \lambda_{k_1+1} = \cdots = \lambda_{k_2}, \ldots, \mu_t = \lambda_{k_{t-1}+1} = \cdots = \lambda_k.$$
Given a partition \( \lambda \) and a flag \( F_\bullet : F_1 \subset F_2 \subset \cdots \subset F_n = V \), the Schubert variety \( \Sigma_\lambda(F_\bullet) \) is defined as
\[
\Sigma_\lambda(F_\bullet) = \{ [W] \in G(k,n) \mid \dim(W \cap F_{n-k+i-\lambda_i}) \geq i \}. 
\]
We will often abuse notation by dropping the reference to the flag. When we would like to emphasize the flag elements \( F_{n-k+i-\lambda_i} \) imposing rank conditions, we will write \( \Sigma_\lambda(F_{n-k+1-\lambda_1} \subset \cdots \subset F_{n-\lambda_n}) \). The cohomology class \( \sigma_\lambda \) of the Schubert variety depends only on the partition \( \lambda \) and not on the choice of flag. The Schubert classes \( \sigma_\lambda \), as \( \lambda \) varies over all allowed partitions, form a \( \mathbb{Z} \)-basis for the cohomology of \( G(k,n) \) \cite{GH} \S 1.5.

The Plücker map embeds the Grassmannian \( G(k,n) \) in \( \mathbb{P}(\wedge^k V) \). Let \( H \) be a hyperplane in \( \mathbb{P}(\wedge^k V) \). Let \( X(\lambda,H) = \{ \Sigma_\lambda(F_\bullet) \mid \Sigma_\lambda(F_\bullet) \subset G(k,n) \cap H \} \) denote the space of Schubert varieties with class \( \sigma_\lambda \) contained in \( G(k,n) \cap H \). In the next section, we will see that \( X(\lambda,H) \) is a closed algebraic subset of a suitable partial flag variety \( (X(\lambda,H) \) may be reducible). The purpose of this paper is to describe \( X(\lambda,H) \) in detail when \( k = 2 \) and \( H \) is arbitrary. We will also discuss some generalizations to larger \( k \).

There is a natural incidence correspondence
\[
I(\lambda) = \{ (\Sigma_\lambda(F_\bullet),H) \mid \Sigma_\lambda(F_\bullet) \subset H \}
\]
parameterizing pairs of a Schubert variety \( \Sigma_\lambda(F_\bullet) \) and a hyperplane \( H \) in the Plücker space containing \( \Sigma_\lambda(F_\bullet) \). Let \( \pi_2 \) denote the natural projection to \( \mathbb{P}^s(\wedge^k V) \). The first problem we address is characterizing the image of \( \pi_2 \). Before stating our theorems, we recall the case of \( G(2,4) \).

**Example 1.1** (Spaces of Schubert varieties in \( G(2,4) \)). The Plücker map embeds \( G(2,4) \) in \( \mathbb{P}^5 \) as a quadric hypersurface \( Q \). The image of a Schubert variety \( \Sigma_{2,1} \) is a line on \( Q \). Conversely, every line on \( Q \) is a Schubert variety with class \( \sigma_{2,1} \). Therefore, the Fano variety \( F_1(Q) \) parameterizing lines on \( Q \) is isomorphic to the flag variety \( F(1,3;4) \) \cite{HI} \S 6.

Let \( X = G(2,4) \cap H \) be a smooth hyperplane section of \( G(2,4) \). Then \( X \) is a smooth quadric threefold. The Fano variety \( F_1(X) \) parameterizing lines on \( X \) is the orthogonal Grassmannian \( OG(2,5) \), which is isomorphic to \( \mathbb{P}^3 \).

On the other hand, let \( Y = G(2,4) \cap \Sigma_1(V_2 \subset V_4) \) be a singular hyperplane section of \( G(2,4) \). Then \( Y \) is a cone over a smooth quadric surface whose vertex is the point corresponding to the two dimensional vector space \( V_2 \). The Fano variety \( F_1(Y) \) parameterizing lines on \( Y \) has two irreducible components \( Z_1 \) and \( Z_2 \). Both \( Z_1 \) and \( Z_2 \) are isomorphic to the blow-up of \( \mathbb{P}^3 \) along a line. The two components \( Z_1 \) and \( Z_2 \) intersect exactly along the exceptional divisors of the two blow-ups. The components \( Z_1 \) and \( Z_2 \) can be geometrically described as follows. Let \( l = \Sigma_{2,1}(F_1 \subset F_3) \) be a line on \( G(2,4) \). The line \( l \) is contained in \( Y \) if all the two-dimensional subspaces parameterized by \( l \) intersect \( V_2 \) defining \( \Sigma_1(V_2 \subset V_4) \) non-trivially. There are two possibilities. Either \( V_2 \subset F_3 \) and \( F_1 \) is an arbitrary one-dimensional subspace of \( F_3 \); or \( F_3 \) is arbitrary and \( F_1 = F_3 \cap V_2 \). These two possibilities correspond to the two components \( Z_1 \) and \( Z_2 \).

The image of a Schubert variety \( \Sigma_{1,1} \) or \( \Sigma_2 \) under the Plücker map is a plane on the quadric hypersurface \( Q \). Conversely, every plane on \( Q \) is a Schubert variety of the form \( \Sigma_{1,1} \) or \( \Sigma_2 \). These varieties are parameterized by \( \mathbb{P}^{3*} \) and \( \mathbb{P}^3 \), respectively. By the Lefschetz Hyperplane Theorem \cite{GH} \S 1.2, a smooth quadric threefold does not contain any planes. Therefore, the smooth hyperplane section \( X \) of \( G(2,4) \) does not contain any Schubert varieties \( \Sigma_{1,1} \) or \( \Sigma_2 \).
the other hand, \( Y \) is a cone over a quadric surface. Such a threefold has two one-dimensional families of planes both parameterized by \( \mathbb{P}^1 \). The two components are distinguished by the cohomology class of the planes they parameterize. Hence, the space of Schubert varieties of the type \( \Sigma_{1,1} \) or \( \Sigma_2 \) on \( Y \) are both parameterized by \( \mathbb{P}^1 \). Notice that in these two cases the incidence correspondences \( \mathcal{I}(1,1) \) and \( \mathcal{I}(2) \) both have dimension \( 5 = \dim(\mathbb{P}^*(\wedge^2 V)) \); however, the second projection is not surjective [H, Example 12.5].

In general, \( \mathbb{P}GL(n) \) acts with finitely many orbits on \( \mathbb{P}^*(\wedge^2 V) \) [D, §2]. The equation of a hyperplane \( H \) in the Plücker space \( \mathbb{P}(\wedge^2 V) \) can be expressed as \( \sum a_{ij} e_i \wedge e_j = 0 \). Therefore, \( H \) may be viewed as a skew-symmetric matrix \( Q_H \). The dimension of the kernel of \( Q_H \) is the invariant that determines the orbits of \( \mathbb{P}GL(n) \) on \( \mathbb{P}^*(\wedge^2 V) \) [D, §2]. The dense open orbit corresponds to hyperplanes \( H \) such that \( G(2, n) \cap H \) is smooth. The dual variety \( G(2, n)^* \) parameterizing hyperplanes \( H \) such that \( G(2, n) \cap H \) decomposes into finitely many orbits depending on the singular locus of \( H \cap G(2, n) \). For \( H \in G(2, n)^* \), the singular locus of \( G(2, n) \cap H \) is a Schubert variety of the form \( \Sigma_{2r,2r} \) for some \( 1 \leq r \leq \lfloor \frac{n-2}{2} \rfloor \) [D, §2]. Let \( S_r \) denote the locus in \( \mathbb{P}^*(\wedge^2 V) \) parameterizing hyperplanes \( H \) such that the singular locus of \( G(2, n) \cap H \) contains a Schubert variety of the form \( \Sigma_{2r,2r} \). By convention, we set \( S_{\lfloor \frac{n-2}{2} \rfloor} \) to be \( \mathbb{P}^*(\wedge^2 V) \). We thus have

\[
S_1 \subset S_2 \subset \cdots \subset S_{\lfloor \frac{n-2}{2} \rfloor}
\]

and the \( \mathbb{P}GL(n) \) orbits on \( \mathbb{P}^*(\wedge^2 V) \) are the locally closed subsets \( S_r/S_{r-1} \).

Our first theorem characterizes the image of \( \pi_2(\mathcal{I}(\lambda)) \) when \( k = 2 \).

**Theorem 1.2.** Let \( \lambda = (a, b) \) be a partition for \( G(2, n) \). The image of the map

\[
\pi_2 : \mathcal{I}(a, b) \to \mathbb{P}^2(\wedge^2 V)
\]

contains \( S_r \) if and only if \( \lceil \frac{a+b}{2} \rceil \geq r \). In particular, the map \( \pi_2 \) is surjective if and only if \( \lceil \frac{a+b}{2} \rceil \geq \frac{n-2}{2} \).

Theorem 1.2 implies that if \( H \in S_r/S_{r-1} \), then \( X((a, b), H) \) is not empty if and only if \( \lceil \frac{a+b}{2} \rceil \geq r \). This raises the question of describing \( X((a, b), H) \) in cases it is not empty. Our second theorem addresses this question.

Let \( Q \) be a skew-symmetric form on an \( n \)-dimensional vector space. If \( Q \) is non-degenerate, then \( n = 2r \) has to be even. A linear space \( W \) is called *isotropic* with respect to \( Q \) if the restriction of \( Q \) to \( W \) is identically zero. Given a vector space \( W \), let \( W^\perp \) denote the set of vectors \( v \in V \) such that \( v^T Q w = 0 \) for every \( w \in W \). If \( Q \) is non-degenerate, the variety parameterizing the \( k \)-dimensional isotropic subspaces of \( F_{2r} \) is called the isotropic Grassmannian \( SG(k, 2r) \). The isotropic Grassmannian \( SG(k, 2r) \) is a homogeneous variety for the symplectic group \( Sp(2r) \). An isotropic subspace of a non-degenerate skew-symmetric form has at most half the dimension, hence \( k \leq r \).

**Theorem 1.3.** Let \( H \) be a hyperplane in \( \mathbb{P}(\wedge^2 V) \) such that \( [H] \in \mathbb{P}^*(\wedge^2 V) \) is contained in the \( \mathbb{P}GL(n) \) orbit \( S_r/S_{r-1} \). Let \( F_{n-2r} \) be the kernel of the corresponding skew-symmetric form \( Q_H \).

Let \( (a, b) \) be a partition for \( G(2, n) \) such that \( \lceil \frac{a+b}{2} \rceil \geq r \). Let

\[
M = \max (0, n - 1 - a - \min(r, b)) \quad \text{and} \quad N = \min \left( n - a - 1, n - r - \frac{a + b + 1}{2} \right).
\]
Corollary 1.4. Let $k$ be odd. Then the irreducible components $Z_j$ of $X((a,b),H)$ are in one-to-one correspondence with integers $M \leq j \leq N$. The irreducible component $Z_j$ parameterizes pairs $(V_{n-a-1} \subset V_{n-b})$ in $F(n-a-1,n-b;n)$ such that $V_{n-a-1}$ is a $Q_H$-isotropic subspace with $\dim(V_{n-a-1} \cap F_{n-2r}) \geq j$ and $V_{n-b}$ is a linear space $V_{n-a-1} \subset V_{n-b} \subset V^\bot_{n-a-1}$ with $\dim(V_{n-b} \cap F_{n-2r}) \geq 2n-2r-a-b-1-j$. The dimension of $Z_j$ is given by

$$\dim(Z_j) = (a+1-b)(a+b+j-n+1) - j(4r+3a+3j-3n+4)$$

$$+ \frac{(n-a-1)(3a+j-n+4)}{2}.$$

Corollary 1.6. Let $H \in S_r/S_{r-1}$. Then $X((r,r),H)$ is isomorphic to the Lagrangian Grassmannian $SG(r,2r)$. In particular, $X((r,r),H)$ is irreducible of dimension $r(r+1)$.

Corollary 1.5. Let $H \in S_r/S_{r-1}$ and $a+b+1 = 2r$, then $X((a,b),H)$ is isomorphic to the isotropic Grassmannian $SG(b,2r)$. In particular, $X((a,b),H)$ is irreducible of dimension $b(2a-b+3)/2$.

Corollary 1.7. Let $a$ be odd. Then $\pi_2$ is a birational map from $I(a,0)$ to $S_{(a+1)/2}$. In particular, when $n$ is odd, a smooth hyperplane section of $G(2,n)$ contains a unique linear space of dimension $n-2$. Geometrically, this linear space corresponds to two-dimensional subspaces that contain the kernel of $Q_H$. Consequently, when $n$ is odd, the largest dimensional linear space on a general codimension two linear section of $G(2,n)$ has dimension $n-3$.

Corollary 1.8. Let $H \in S_1$ be the hyperplane defining the Schubert variety $\Sigma_1(F_{n-2} \subset F_n)$ and let $a > b > 0$. Then $X((a,b),H)$ is the union of the following two Schubert varieties in $F(n-a-1,n-b;n)$

1. ${\{V_{n-a-1} \subset V_{n-b} \mid V_{n-a-1} \subset F_{n-2}\}}$,
2. ${\{V_{n-a-1} \subset V_{n-b} \mid \dim(V_{n-b} \cap F_{n-2}) \geq n-b-1\}}$.

When $n-2 > k > 2$, $\mathbb{P}GL(n)$ no longer acts with finitely many orbits on $\mathbb{P}^*(\wedge^k V)$ (except when $k = 3$ and $n = 6,7$ or 8 [D §2.1]). It is, therefore, unrealistic to hope for as complete a classification of the spaces $X(\lambda,H)$. However, $X(\lambda,H)$ can be easily described for $H$ in certain orbits of $\mathbb{P}GL(n)$. We will give some examples below.
Proposition 1.9. Let λ be a partition for G(k, n) such that λ₁ < n − k and λₖ = 0. Then the image of the second projection π₂(ℐ(λ)) is contained in the dual variety G(k, n)*. In particular, π₂ is not surjective. On the other hand, let λ be a partition such that either λₖ₋₁ = n − k and λₖ > 0; or λ₁ = n − k and λₖ = n − k − 1. Then π₂ is surjective.

A corollary of the proof of the proposition is worth mentioning.

Corollary 1.10. Let λ be the partition λ₁ = · · · = λₖ₋₁ = n − k − 1 and λₖ = 0. Then π₂(ℐ(λ)) = G(k, n)*.

It is very rare to have an explicit, concrete resolution of singularities of a variety. Corollary 1.10 provides such a resolution for the dual of the Grassmannian in its Plücker embedding.

Corollary 1.11. Let n − 2 > k > 2. Let λ be the partition λ₁ = · · · = λₖ₋₁ = n − k − 1 and λₖ = 0. Let N = (n − k) − k(n − k) − 2. Then the incidence correspondence ℐ(λ) is a ℙ^N-bundle over G(k, n) and is smooth. The map π₂ is a birational map from ℐ(λ) onto G(k, n)* and gives a resolution of singularities of G(k, n)*.

Finally, we state the analogue of Corollary 1.8 for arbitrary k.

Proposition 1.12. Let H be the hyperplane in ℙ(∧^k V) defining the codimension one Schubert variety Σ₁(Fₙ−k ⊂ Fₙ−k+2 ⊂ · · · ⊂ Fₙ). Let λ = (µ₁, ..., µₖ) be a partition. Let δ denote the Kronecker delta function. Then X(λ, H) has t − δ₀,µ₁ irreducible components Z_µ with 1 ≤ j ≤ t − δ₀,µ₁. The component Z_µ is the Schubert variety in F(n−k+k−µ₁,...,n−µₖ−µ); parameterizing flags (V_{n−k+k−µ₈−µ₉−µ₅−µ₆} ⊂ ... ⊂ V_{n−µ₆}) such that dim(V_{n−k+k−µ₈−µ₉−µ₅−µ₆} ∩ F_{n−k}) ≥ n − k − µ₉ + 1.

The organization of the paper is as follows. In §2, we will recall basic facts about the geometry of Grassmannians, Schubert varieties and the dual variety to the Grassmannian in its Plücker embedding. In §3, we will prove Theorem 1.2, Proposition 1.9 and their corollaries. In §4, we will prove Theorem 1.3 and discuss its corollaries.

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2. Preliminaries about the geometry of Grassmannians

In this section, we recall some basic facts about the geometry of Grassmannians and Schubert varieties. For the reader’s convenience, we sketch the proofs of some classical facts about G(2, n)*. We refer the reader to [GH] and [H] for facts about Grassmannians and Schubert varieties, to [D] and [PVdV] for facts about the dual variety G(2, n)*, to [BL] and [C2] for facts about singularities of Schubert varieties.

In order to minimize confusion, we will denote the point in the Grassmannian G(k, n) corresponding to a k-dimensional subspace W by [W].

Parameter spaces of Schubert varieties. Although it is standard in the literature to define a Schubert variety by Equation (1), the Schubert variety does not determine the flag. In fact, the Schubert variety does not even determine the elements of the flag F_{n−k+i−λ₈} that impose the rank conditions defining the Schubert variety.

For example, Σ₁₁(F₂ ⊂ F₃) and Σ₁₁(F₁′ ⊂ F₃) define the same Schubert variety in G(2, 4) for any two F₂ and F₁′, two-dimensional subspaces contained in F₃. Once a two-dimensional
In the Plücker embedding, the linear subspaces of the image of the Grassmannian under this embedding is the space of totally decomposable wedges. The Plücker embedding of the Grassmannian is much harder to understand.

For simplicity, set $k_\lambda = \sum_{j=1}^s i_j$. In particular, $k_t = k$. The Schubert variety $\Sigma_\lambda(F_\bullet)$ can equivalently be defined as

$$\Sigma_\lambda(F_\bullet) = \{ [W] \in G(k,n) \mid \dim(W \cap F_{n-k+k_j-m_j}) \geq k_j \text{ for } 1 \leq j \leq t \}. \quad (2)$$

Once $W$ intersects $F_{n-k+k_s-m_s}$ in a $k_s$-dimensional subspace, it intersects $F_{n-k+k_s-m_s-j}$ in a subspace of dimension at least $k_s - j$. Consequently, the rank conditions in Equation (2) imply all the rank conditions in Equation (1). Conversely, it is easy to see that the Schubert variety determines the linear spaces $F_{n-k+k_s-m_s}$ for $1 \leq s \leq t$. Consequently, we can use the partial flag variety $F(n-k+k_1-\mu_1, \ldots, n-\mu_t; n)$ as a parameter space for Schubert varieties in $G(k,n)$ with cohomology class $\sigma_\lambda$. The space $X(\lambda, H)$ is then naturally a closed algebraic subset of $F(n-k+k_1-\mu_1, \ldots, n-\mu_t; n)$.

We have a natural incidence correspondence $\mathcal{I}(\lambda)$

$$\mathcal{I}(\lambda) = \{ ([W] \in G(k,n) \mid \dim(W \cap F_{n-k+k_s-m_s}) \geq k_s) \}$$

consisting of pairs of a Schubert variety $\Sigma_\lambda(F_\bullet)$ and a hyperplane containing it. The first projection $\pi_1$ realizes $\mathcal{I}(\lambda)$ as a projective bundle over the partial flag variety $F(n-k+k_1-\mu_1, \ldots, n-\mu_t; n)$. The fibers are isomorphic to $\mathbb{P}H^0(I_{\Sigma_\lambda}(1))$, where $I_{\Sigma_\lambda}$ denotes the ideal sheaf of $\Sigma_\lambda$, and are all projective spaces of the same dimension. Consequently, $\mathcal{I}(\lambda)$ is irreducible and smooth [3, Theorem I.6.8]. Note, however, that the second projection $\pi_2$ is rarely flat and much harder to understand.

The Plücker embedding of the Grassmannian. The Grassmannian $G(k,n)$ is a smooth, projective variety of dimension $k(n-k)$. The Plücker map embeds $G(k,n)$ into $\mathbb{P}(\Lambda^k V)$. The image of the Grassmannian under this embedding is the space of totally decomposable wedges. In the Plücker embedding, the linear subspaces of $G(k,n)$ have a concrete description. A line on $G(k,n)$ corresponds to a family of $k$-dimensional subspaces of $V$ that contain a fixed $(k-1)$-dimensional subspace and are contained in a fixed $(k+1)$-dimensional subspace. More generally, a linear space of dimension $s$ on $G(k,n)$ corresponds to either a family of $k$-dimensional subspaces that contain a fixed $(k-1)$-dimensional space and are contained in a fixed $(k+s)$-dimensional subspace; or a family of $k$-dimensional subspaces that are contained in a fixed $(k+1)$-dimensional subspace and contain a fixed $(k-s)$-dimensional subspace [4, §6]. Of course, the first possibility can only exist when $k+s \leq n$ and the second possibility can only exist when $s \leq k$.

The tangent space $T_{[W]}G(k,n)$ is naturally isomorphic to $\text{Hom}(W,V/W)$ [4, §16]. We can also explicitly describe the projectivized tangent space to $G(k,n)$ in the Plücker embedding. Choose a basis $e_1, \ldots, e_n$ for $V$ so that $W$ is given as the span of the vectors $e_1, \ldots, e_k$. Then under the Plücker embedding, the image of $[W]$ is $e_1 \wedge e_2 \wedge \cdots \wedge e_k$. Let $i_1 < \cdots < i_k$ be a set of indices such that the cardinality of the set $\{i_1, \ldots, i_k\} - \{1, 2, \ldots, k\}$ is at most one. Since we
can replace any of the elements $1 \leq i \leq k$ by one of the elements $k < j \leq n$, there are $k(n-k)+1$ such sets. The projectivized tangent space to $G(k,n)$ at $W$ is spanned by the $k(n-k)+1$ points $e_1 \wedge \cdots \wedge e_k$ in $\mathbb{P}(\Lambda^k V)$ defined by setting all the Plücker coordinates but $p_{i_1,\ldots,i_k}$ equal to zero. To prove this description of the tangent space, observe that the line spanned by $e_1 \wedge \cdots \wedge e_k$ and $e_1 \wedge \cdots \wedge e_k$ is contained in the Grassmannian $G(k,n)$. Since the tangent space at $[W]$ contains every line in $G(k,n)$ passing through $[W]$, we conclude that the projectivized tangent space at $[W]$ contains the span of the points $e_1 \wedge \cdots \wedge e_k$. Since both these projective spaces have dimension $k(n-k)$, we conclude that they are equal.

Let $\Sigma_{\lambda}$ be a Schubert variety in $G(k,n)$. Then $\Sigma_{\lambda}$ is cut out on $G(k,n)$ by hyperplanes. These hyperplanes can be explicitly written as follows. Suppose we choose our flag so that $F_i$ is the span of the vectors $e_1,\ldots,e_i$. Then the Schubert variety $\Sigma_{\lambda}(F_*)$ is cut out by the Plücker coordinates $p_{i_1,\ldots,i_k} = 0$, where at least for one $j$, $i_j > n-k+j-\lambda_j$ [IP]. Specializing to the case $k = 2$, we obtain the following lemma.

**Lemma 2.1.** The dimension of the vector space of linear spaces containing a Schubert variety $\Sigma_{a,b}$ in $G(2,n)$ is given by

$$h^0(I_{\Sigma_{a,b}}(1)) = \binom{n}{2} - \binom{n-b}{2} + \binom{a-b+1}{2}.$$

Applying the Theorem on the Dimension of Fibers [S Theorem I.6.7] to the first projection $\pi_1: \mathcal{I}(a,a) \to F(n-a-1,n-b;n)$, we obtain the following corollary.

**Corollary 2.2.** If $a = b$, then the first projection $\pi_1: \mathcal{I}(a,a) \to F(n-a-1,n-b;n) = G(n-a,n)$ exhibits $\mathcal{I}(a,a)$ as a projective space bundle over $G(n-a,n)$ with fibers of dimension

$$\binom{n}{2} - \binom{n-a}{2} - 1.$$

In particular, $\mathcal{I}(a,a)$ is smooth, irreducible and

$$\dim(\mathcal{I}(a,a)) = \frac{a(4n-3a-1)}{2} - 1.$$ 

If $a > b$, then the first projection $\pi_1: \mathcal{I}(a,b) \to F(n-a-1,n-b;n)$ exhibits $\mathcal{I}(a,b)$ as a projective space bundle over $F(n-a-1,n-b;n)$ with fibers of dimension

$$\binom{n}{2} - \binom{n-b}{2} + \binom{a-b+1}{2} - 1.$$

In particular, $\mathcal{I}(a,b)$ is smooth, irreducible and

$$\dim(\mathcal{I}(a,b)) = n(a+b+1) - \frac{a^2 + 3a}{2} - b^2 - 2.$$ 

**Singularities of Schubert varieties.** Given a partition $\lambda$, a singular partition $\lambda^s$ associated to $\lambda$ is obtained by adding a hook to the partition $\lambda$. More explicitly, if $\lambda = (\mu_1^1,\ldots,\mu_t^t)$, then $\lambda^s$ is any of the partitions

$$(\mu_1^1,\ldots,\mu_{a-2}^i, (\mu_{u-1}+1)^{i_{u-1}+1}, \mu_u^{i_u+1}, \mu_{u+1}^{i_{u+1}}, \ldots, \mu_t^i)$$

provided that they are admissible for $G(k,n)$, where it is understood that if $\mu_{u-1} + 1 = \mu_{u-2}$ those parts have to be grouped together. For example, if $(5,3,2,2,1)$ is a partition for $G(5,11)$,
then the singular partitions are \((6, 2, 2, 1), (5, 4, 4, 1)\) and \((5, 3, 3, 3, 3)\). The singular locus of the Schubert variety \(\Sigma_l(F)\) is the union of \(\Sigma_{\lambda^s}(F)\) as \(\lambda^s\) varies over all allowable singular partitions associated to \(\lambda\). In particular, \(\Sigma_{a,b}\) in \(G(2, n)\) is smooth if and only if \(a = n - 2\) or \(a = b\). Otherwise, the singular locus of \(\Sigma_{a,b}(F_{n-1-a} \subset F_{n-b})\) is \(\Sigma_{a+1,a+1}(F_{n-2-a} \subset F_{n-1-a})\) [BL].

Lemma 2.3. Let \(H\) be a hyperplane in \(\mathbb{P}(\wedge^k V)\). Let \(V_1\) be a linear space with \(\dim(V_1) \geq k\) such that \(H \cap G(k, n)\) is singular at every \([W] \in G(k, n)\) such that \(W \subset V_1\). Then for any linear space \(U\) such that \(\dim(U \cap V_1) \geq k - 1\), \([U] \in G(k, n) \cap H\).

Proof. First, observe that if a line \(l\) on \(G(k, n)\) intersects the singular locus of \(H \cap G(k, n)\), then by Bezout’s Theorem [Ha, I.7.7], \(l\) is contained in \(H \cap G(k, n)\). Hence, for any \(k\)-dimensional subspace \(U\) that intersects \(V_1\) in a subspace of dimension \(k - 1\), we have \([U] \in H \cap G(k, n)\). This is immediate by assumption if \(U \subset V_1\). We may assume that \(U \not\subset V_1\). Let \(F_{k-1} = U \cap V_1\) and let \(W\) be a \(k\)-dimensional subspace of \(V_1\) containing \(F_{k-1}\). Then the \(k\)-dimensional subspaces contained in \(\text{Span}(U, W)\) and containing \(F_{k-1}\) are parameterized by a line \(l\) in \(G(k, n)\). The line \(l\) contains \([W]\) which is a singular point of \(H \cap G(k, n)\) by assumption. Hence \(l \subset H \cap G(k, n)\). Since \([U]\) is also a point on \(l\), we conclude that \([U] \in H \cap G(k, n)\). This concludes the proof of the lemma.

Lemma 2.4. Let \(H\) be a hyperplane in \(\mathbb{P}(\wedge^2 V)\). Let \(V_1, V_2\) be two linear subspaces of \(V\) such that \(\dim(V_i) \geq 2\). Assume that \(H \cap G(2, n)\) is singular along every two-dimensional subspace contained in \(V_i, 1 \leq i \leq 2\). Then \(H \cap G(2, n)\) contains every two-dimensional subspace that intersects \(\text{Span}(V_1, V_2)\) non-trivially and is singular along every two-dimensional subspace that is contained in \(\text{Span}(V_1, V_2)\).

Proof. Let \(W\) be a two-dimensional subspace that intersects \(\text{Span}(V_1, V_2)\) in a one-dimensional subspace \(F_1\). Then there exists a two-dimensional subspace \(W'\) such that \(F_1 \subset W', W' \cap V_1 \neq 0\) and \(W' \cap V_2 \neq 0\). To construct \(W'\), take the span of two one-dimensional subspaces \(G_1 \subset V_1 \cap \text{Span}(F_1, V_2)\) and \(G_1' \subset V_2 \cap \text{Span}(F_1, G_1)\). Let \(F_3 = \text{Span}(W, W')\). The two-dimensional subspaces contained in \(F_3\) are parameterized by a plane \(P\) in \(G(2, n)\). There are two special lines \(l_1\) and \(l_1'\) on the plane \(P\), parameterizing two-dimensional subspaces containing \(G_1\), respectively, \(G_1'\) and contained in \(F_3\). Since each of these two-dimensional spaces intersects \(V_1\) or \(V_2\) non-trivially, by Lemma 2.3, \(l\) and \(l'\) are contained in \(H \cap G(2, n)\). By Bezout’s Theorem, we conclude that \(P \subset H \cap G(2, n)\). Therefore, \([W] \in H \cap G(2, n)\). Since \(H \cap G(2, n)\) is projective and contains the dense open subset of the Schubert variety of \([W]\) such that \(\dim(W \cap \text{Span}(V_1, V_2)) = 1\), we conclude that \(H \cap G(2, n)\) contains every \([W]\) such that \(W \cap \text{Span}(V_1, V_2) \neq 0\). This proves the first part of the lemma.

Next, we prove that a hyperplane section of \(G(2, n)\) that contains a Schubert variety of the form \(\Sigma_{a,0}(F_{n-1-a} \subset F_n)\) is singular along a Schubert variety of the form \(\Sigma_{a+1,a+1}(F_{n-2-a} \subset F_{n-1-a})\). This will conclude the proof of the second part of the lemma. Let \(v \wedge w\) represent the Plücker point of a two-dimensional subspace contained in \(F_{n-1-a}\). Choose coordinates for \(V\) so that \(F_{n-1-a}\) is spanned by \(e_1, \ldots, e_{n-1-a}\) with \(e_1 = v\) and \(e_2 = w\). Then a hyperplane containing \(\Sigma_{a,0}\) is a linear combination of the Plücker coordinates \(p_{i,j}\) with \(n - 1 + a < i < j \leq n\). The tangent space to the \(G(2, n)\) in its Plücker embedding at the point \(e_1 \wedge e_2\) is given by the span of the points \(e_1 \wedge e_i\) and \(e_2 \wedge e_j\) with \(2 \leq i \leq n\) and \(3 \leq j \leq n\). All the Plücker coordinates containing \(\Sigma_{a,0}\) vanish at all these points spanning the tangent space to the Grassmannian. Hence, all these hyperplanes contain the tangent space at all the points of \(\Sigma_{a+1,a+1}\). We conclude that the linear
section $H \cap G(2, n)$ is singular along $v \wedge w$. By homogeneity, it follows that $H \cap G(2, n)$ is singular along $\Sigma_{a+1, a+1}$. This concludes the proof of the lemma. \hfill \square

We chose to give this proof because similar arguments can be used for $G(k, n)$. For $G(2, n)$, one can prove the previous lemma using the correspondence between hyperplanes and skew-symmetric forms. By assumption, $V_1$ and $V_2$ are in the kernel of the skew-symmetric form $Q_H$. Therefore, the span of $V_1$ and $V_2$ is also in the kernel. The lemma then follows by observing that $H \cap G(2, n)$ is singular along $[W]$, where $W$ is in the kernel of $Q_H$.

It follows from Lemma 2.4 that the singular locus of a hyperplane section $H \cap G(2, n)$ is either empty or a Schubert variety of the form $\Sigma_{a,a}$ parameterizing two-dimensional subspaces contained in a vector space $F_{n-a}$. Simply let $F_{n-a}$ be the span of all the two dimensional subspaces $W$ such that $[W]$ is a singular point of $G(2, n) \cap H$. Furthermore, $a$ has to be even. To see this we use the correspondence between the hyperplane $H$ and the skew-symmetric form $Q_H$. The codimension of the kernel of a skew-symmetric form is even since the restriction of the skew-symmetric form to a complementary linear space is non-degenerate. Hence, $a$ has to be even. Conversely, every $\Sigma_{2r,2r}$ occurs as the singular locus of some hyperplane section of $G(2, n)$. This can be seen by explicitly writing the skew-symmetric form $e_1 \wedge e_2 + e_3 \wedge e_4 + \cdots + e_{2r-1} \wedge e_{2r}$, whose kernels has codimension $2r$. Finally, Darboux's Theorem [MS, §2] guarantees that the hyperplanes corresponding to the skew-symmetric forms with the same dimensional kernel form one orbit under $\text{PGL}(n)$. This recalls the proof of the following beautiful statement from Ron Donagi's paper [D] alluded to in the Introduction.

**Proposition 2.5.** ([D, §2]) The group $\text{PGL}(n)$ acts with finitely many orbits on $\mathbb{P}^s(\bigwedge^2 V)$. The orbits are indexed by an integer $1 \leq r \leq \lceil \frac{n-1}{2} \rceil$. The orbit corresponding to $r < \lceil \frac{n-1}{2} \rceil$ consists of hyperplanes $H$ such that the singular locus of $H \cap G(2, n)$ is a Schubert variety of the form $\Sigma_{2r,2r}$. The open orbit corresponding to $r = \lceil \frac{n-1}{2} \rceil$ is the complement of the dual variety $G(2, n)^*$ parameterizing hyperplanes $H$ such that $H \cap G(2, n)$ is smooth.

Let $r \leq \frac{n-2}{2}$. Since a hyperplane $[H] \in S_r/S_{r-1}$ is singular along $\Sigma_{2r,2r}$ parameterizing linear spaces contained in $F_{n-2r}$, by Lemma 2.3, $H \cap G(2, n)$ contains the Schubert variety $\Sigma_{2r-1,0}$ parameterizing linear spaces intersecting $F_{n-2r}$. Conversely, we saw in the proof of Lemma 2.4 that a hyperplane containing $\Sigma_{2r-1,0}(F_{n-2r} \subset F_n)$ is singular along the Schubert variety $\Sigma_{2r,2r}$ parameterizing linear spaces that are contained in $F_{n-2r}$. We conclude that $H$ contains a unique $\Sigma_{2r-1,0}$. In particular, the map $\pi_2 : \mathcal{I}(2r-1,0) \to S_r$ is birational and a resolution of singularities of $S_r$. Furthermore, the Theorem on the Dimension of Fibers and Lemma 2.2 then imply the following corollary.

**Corollary 2.6.** ([D, §2]) The codimension of $S_r$ in $\mathbb{P}^s(\bigwedge^k V)$ is $\binom{n-2r}{2r}$.

In particular, we have the following well-known corollary.

**Corollary 2.7.** ([D, §2] or [PYdV]) When $n$ is even, then the dual $G(2, n)^*$ is a hypersurface. When $n$ is odd $G(2, n)^*$ has codimension three.

Finally, recall that if $n - 2 > k > 2$, then the dual of $G(k, n)$ in its Plücker embedding is a hypersurface and at a general point $[H] \in G(k, n)^*$ the singular locus of $H \cap G(k, n)$ consists of one singular point. For the convenience of the reader, we briefly sketch an elementary proof. Since $G(k, n)$ is isomorphic to $G(n-k, n)$, we may further assume that $2k \leq n$. First, observe that the projective tangent spaces $\mathbb{P}T_{[W_1]} \cap \mathbb{P}T_{[W_2]} = \emptyset$ if $\dim(W_1 \cap W_2) < k - 2$, $\mathbb{P}T_{[W_1]} \cap \mathbb{P}T_{[W_2]} = \mathbb{P}^3$ if $\dim(W_1 \cap W_2) = k-2$ and $\mathbb{P}T_{[W_1]} \cap \mathbb{P}T_{[W_2]} = \mathbb{P}^{n-1}$ if $\dim(W_1 \cap W_2) = k-1$. 
Let $U = G(k, n) \times G(k, n) - \Delta$ be the complement of the diagonal $\Delta$ in $G(k, n) \times G(k, n)$. Consider the incidence correspondence

$$J = \{([W_1], [W_2], H) \mid \mathbb{P}T_{[W_1]} \cap \mathbb{P}T_{[W_2]} \subset H\}$$

consisting of a point $([W_1], [W_2])$ in $U$ and a hyperplane $H$ containing the projective tangent spaces to $G(k, n)$ at both points. Let $\pi_1$ and $\pi_2$ denote the projection to $U$ and $\mathbb{P}^*(\wedge^k V)$, respectively.

Let $U_1$ be the locus in $U$ parameterizing $\{([W_1], [W_2]) \mid \dim(W_1 \cap W_2) < k-2\}$. Then the fibers of $\pi_1$ over $U_1$ are projective spaces of dimension $\binom{n}{k} - 2k(n-k) - 3$. Since $U_1$ has dimension $2k(n-k)$, the Theorem on the Dimension of Fibers implies that $\pi_2(\pi_1^{-1}(U_1))$ has codimension at least two in $\mathbb{P}^*(\wedge^k V)$.

Let $U_2$ be the locus in $U$ parameterizing $\{([W_1], [W_2]) \mid \dim(W_1 \cap W_2) = k-2\}$. Then the fibers of $\pi_1$ over $U_2$ are projective spaces of dimension $\binom{n}{k} - 2k(n-k) + 1$. Since $U_2$ has dimension $k(n-k) + 2n - 4$, the Theorem on the Dimension of Fibers implies that $\pi_1^{-1}(U_2)$ has dimension $\binom{n}{k} - k(n-k) + 2n - 3$. Hence, $\pi_2(\pi_1^{-1}(U_2))$ has codimension at least two in $\mathbb{P}^*(\wedge^k V)$ if $k \geq 4$ or if $k = 3$ and $n \geq 9$. If $k = 3$ and $n = 6, 7$ or $8$, we observe that the fiber dimension of $\pi_2$ on $\pi_1^{-1}(U_2)$ is at least $6, 4$ and $2$, respectively. Let $W_1 = \text{Span}(e_1, e_2, e_3)$ and let $W_2 = \text{Span}(e_1, e_4, e_5)$. A hyperplane $H$ containing $\mathbb{P}T_{[W_1]}$ and $\mathbb{P}T_{[W_2]}$ can be expressed as $\sum_{i=0}^{n}(a_ip_{2i} + b_ip_{3i} + c_ip_{25i} + d_ip_{35i}) = 0$ in Plücker coordinates. Consider two-dimensional subspaces $Y$ in $\text{Span}(e_2, e_3, e_4, e_5)$ that satisfy $a_ie_2 \wedge e_4 + \cdots + d_ie_3 \wedge e_5 = 0$ for $6 \leq i \leq n$. Then $H$ contains the tangent space to the three-dimensional subspace $\text{Span}(e_1, Y)$. The claim about the fiber dimension of $\pi_2$ follows. Hence, $\pi_2(\pi_1^{-1}(U_2))$ has codimension at least two in $\mathbb{P}^*(\wedge^k V)$ in these cases as well.

Let $U_3$ be the locus in $U$ parameterizing $\{([W_1], [W_2]) \mid \dim(W_1 \cap W_2) = k-1\}$. Then the fibers of $\pi_1$ over $U_3$ are projective spaces of dimension $\binom{n}{k} - 2k(n-k) + n - 3$. The locus $U_2$ consists of pairs of points $([W_1], [W_2])$ such that the line spanned by them is contained in $G(k, n)$. Hence, $\dim(U_2) = 2k + (k+1)(n-k-1)$. Note that if a hyperplane $H$ is tangent to $G(k, n)$ at both $W_1$ and $W_2$, then it is tangent at all points along the line spanned by $[W_1]$ and $[W_2]$. Consequently, the fibers of $\pi_2$ over $\pi_1^{-1}(U_2)$ have dimension at least two. By the Theorem on the Dimension of Fibers, the codimension of $\pi_2(\pi_1^{-1}(U_2))$ will be less than two if $2k + (k+1)(n-k-1) - 2k(n-k) + n - 2 > 0$. Rewriting this inequality, $0 > (k-2)n - k^2 + 3$. Using $n \geq 2k$, we immediately see that this inequality cannot be satisfied if $k \geq 4$. When $k = 3$, the inequality becomes $6 > n$. Hence, we conclude that the inequality is not satisfied for $k \geq 3$ and $n \geq 2k$. It follows that if $n - 2 > k > 2$, $G(k, n)^*$ is a hypersurface and a general tangent hyperplane is tangent at a unique point. We have proved the following well-known fact for which we could not find a convenient reference.

**Proposition 2.8.** If $2 < k < n-2$, then $G(k, n)^*$ in $\mathbb{P}^*(\wedge^k V)$ is a hypersurface. Furthermore, a general hyperplane parameterized by $G(k, n)^*$ is tangent to $G(k, n)$ at one point.

### 3. The proof of Theorem 1.2

In this section, we prove Theorem 1.2 and discuss its generalizations to $G(k, n)$.

**Proof of Theorem 1.2.** Let $\Sigma_{a,b}(F_{n-1-a} \subset F_{n-b})$ be a Schubert variety with class $\sigma_{a,b}$ in $G(2, n)$. Suppose that $H$ is a hyperplane in $\mathbb{P}(\wedge^2 V)$ containing $\Sigma_{a,b}(F_{n-1-a} \subset F_{n-b})$. Notice that $\Sigma_{a,b}(F_{n-1-a} \subset F_{n-b}) \subset G(2, F_{n-b})$. There are two possibilities. Either $G(2, F_{n-b}) \subset H$; or
$H \cap G(2, F_{n-b})$ is a hyperplane section of $G(2, F_{n-b})$ that contains $\Sigma_{a,b}(F_{n-1-a} \subset F_{n-b})$. We will now analyze each of these possibilities.

First, assume that $H \cap G(2, F_{n-b})$ is a hyperplane section of $G(2, F_{n-b})$. A linear embedding $V' \hookrightarrow V$, induces an embedding $G(2, V') \hookrightarrow G(2, V)$. The following lemma analyzes the relation between the singular loci of $H \cap G(2, V)$ and $H \cap G(2, V')$.

**Lemma 3.1.** Let $G(2, n) \hookrightarrow G(2, n+1)$ be the embedding induced by the embedding of $V_n \hookrightarrow V_{n+1}$. Let $H \cap G(2, n)$ be a linear section of $G(2, n)$ in $\mathbb{P}(\wedge^2 V_n)$ with singular locus $\Sigma_{2k,2k}$. Let $H'$ be a general hyperplane in $\mathbb{P}(\wedge^2 V_{n+1})$ such that $H' \cap G(2, n+1)$ restricts to $H \cap G(2, n)$. Then the singular locus of $H' \cap G(2, n+1)$ is $\Sigma_{2(k+1),2(k+1)}$.

**Proof.** Pick a basis $e_1, \ldots, e_{n+1}$ of $V_{n+1}$ such that $V_n$ is spanned by the first $n$ vectors and the singular locus of $H \cap G(2, n)$ parameterizes two-dimensional subspaces contained in the span $F_{n-2k}$ of the first $n-2k$ vectors. Then $H$ is defined by a linear equation $L(p_{i,j}) = 0$, where $L$ is a linear combination of the Plücker coordinates $p_{i,j}$ for $i < j$ and $n - 2k < j \leq n$. A hyperplane in $\mathbb{P}(\wedge^2 V_{n+1})$ that contains $H$ may be expressed as $L(p_{i,j}) + \sum_{i=1}^n a_i p_{i,n+1} = 0$.

By Bertini’s Theorem [Ha, III.10.9], the singular locus of $H' \cap G(2, n+1)$ for a general hyperplane containing $H$ is contained in $H \cap G(2, n)$. Let $W$ be the $(n-2k-1)$-dimensional linear space cut out on $F_{n-2k}$ by the linear equation $\sum_{i=1}^n a_i x_i = 0$. Then $H' \cap G(2, n+1)$ contains the tangent space to $G(2, n+1)$ at any two-dimensional space contained in $W$. At a point, $u \wedge v$ with $u, v \in W$, the tangent space is spanned by replacing at most one of $u$ or $v$ by elements of a basis. All the Plücker coordinates defining $H'$ clearly vanish at all these points. Hence, $H' \cap G(2, n+1)$ is singular along two-dimensional subspaces contained in $W$. We conclude that the singular locus of $H' \cap G(2, n+1)$ contains a $\Sigma_{2(k+1),2(k+1)}$ of two-dimensional subspaces contained in $W$. Conversely, for a two-dimensional space not contained in that hyperplane, there exists a vector $v$ such that $\sum a_i v_i \neq 0$. Hence, the point $v \wedge e_{n+1}$ is not contained in $H'$, but it is contained in the tangent space to a line $w \wedge v$. Hence, the singular locus does not contain all of $\Sigma_{k,k}$. The lemma follows. □

We are now ready to prove the theorem in the case $H$ does not contain $G(2, F_{n-b})$. There are two cases that we need to analyze separately. First, assume that $a = n-2$. Since the Grassmannian contains linear spaces of the form $\Sigma_{n-2,0}$. Any hyperplane section contains linear spaces $\Sigma_{n-2,1}$ of one smaller dimension. Hence, $\pi_2$ is surjective for $\lambda = (n-2, i)$ for $i > 0$. We now have to analyze the case $\lambda = (n-2, 0)$. In this case, the flag variety $F(1, n; n)$ is isomorphic to $\mathbb{P}^{n-1}$. Hence, $\dim(I) = \binom{n}{2} - 1$. If $n$ is even, then the general singular hyperplane section $X$ of $G(2, n)$ is singular along a point $[\Lambda] \in G(2, n)$. Furthermore, in this case the dual variety $G(2, n)^*$ is a hypersurface, hence has dimension $\binom{n}{2} - 2$. By Lemma [2.4] if $F_1 \subset \Lambda$, then every two-dimensional subspace containing $F_1$ is contained in $X$. Since the space of one-dimensional subspaces of $\Lambda$ is isomorphic to $\mathbb{P}^1$, the general fiber of $\pi_2$ over $G(2, n)^*$ has dimension greater than or equal to one. By the Theorem on the Dimension of Fibers, $\dim(\pi_2^{-1}(G(2, n)^*)) \geq \binom{n}{2} - 1$. However, since $\pi_2^{-1}(G(2, n)^*) \subset I(n-2, 0)$, $\dim(\pi_2^{-1}(G(2, n)^*)) \leq \binom{n}{2} - 1$. We conclude that $\pi_2^{-1}(G(2, n)^*) = I(n-2, 0)$ and consequently, $\pi_2$ is not surjective.

If $n$ is odd, then the skew-symmetric form $Q_H$ corresponding to any hyperplane $H$ in $\mathbb{P}(\wedge^2 V)$ must have non-trivial kernel. Let $v$ be a vector in the kernel of $Q_H$. Then any two-dimensional subspace $W$ such that $v \in W$ is isotropic with respect to $Q_H$. Consequently, $H$ contains the Schubert variety $\Sigma_{n-2,0}$ parameterizing the two-dimensional subspaces containing $v$. For a general hyperplane $H$, the kernel of $Q_H$ is one-dimensional and $H$ contains a unique Schubert variety of the form $\Sigma_{n-2,0}$.
Now we can discuss the case $\Sigma_{a,0}$ with $a < n - 2$. If $a$ is odd, then the singular locus of a general hyperplane contains $\Sigma_{a+1,a+1}$. Conversely, a linear section whose singular locus is $\Sigma_{a+1,a+1}$ contains a Schubert variety of the form $\Sigma_{a,0}$. We conclude that $\pi_2(\mathcal{I}(a,0)) = S_{(a+1)/2}$. If $a$ is even, then the singular locus of a hyperplane section containing $\Sigma_{a,0}$ contains $\Sigma_{a+1,a+1}$. However, since the singular loci have to be of the form $\Sigma_{2k,2k}$, it follows that the singular locus has to contain a Schubert variety of the form $\Sigma_{a,0}$. Conversely, a hyperplane section whose singular locus has the form $\Sigma_{a,a}$ contains a Schubert variety of the form $\Sigma_{a,0}$. We conclude that the image of $\pi_2$ is $S_{a/2}$.

Returning to the original argument, if $b > 0$, then $\Sigma_{a,b}$ is a Schubert variety with class $\sigma_{a-b,0}$ in $G(2, n - b)$. Hence, any hyperplane section of $G(2, n - b)$ containing $\sigma_{a-b,0}$ is singular along a Schubert variety of the form $\Sigma_{a-b+1,a-b+1}$ if $a - b$ is odd or $\Sigma_{a-b,a-b}$ if $a - b$ is even. Using Lemma 3.1 $b$-times, we conclude that if $a - b$ is even, then the general hyperplane containing $\Sigma_{a,b}$ is smooth if $a + b > n - 3$ or singular along a Schubert variety of the form $\Sigma_{a+b+1,a+b+1}$ when $a + b \leq n - 3$. Similarly, when $a - b$ is odd, then a hyperplane section of $G(2, n - b)$ containing $\Sigma_{a-b,0}$ is singular along $\Sigma_{a-b,a-b}$. Using Lemma 3.1 $b$-times, we conclude that a general hyperplane containing $\Sigma_{a,b}$ is smooth when $a + b > n - 2$ or singular along $\Sigma_{a+b,a+b}$ when $a + b \leq n - 2$.

Finally, we analyze the cases when the hyperplane contains $G(2, n - b)$ or when $a = b$. The first observation is that the only hyperplanes containing a Schubert variety of the form $\Sigma_{1,1}(F_{n-2} \subset F_{n-1})$ are Schubert varieties $\Sigma_{1}(G_{n-2} \subset G_{n})$. The flag variety $F(n - 1; n) \cong (\mathbb{P}^{n-1})^*$, hence has dimension $n - 1$. The fiber dimension of $\pi_1$ over a point in $F(n - 1; n)$ is $n - 2$. Hence the dimension of $\mathcal{I}(1,1)$ is $2n - 3$. The locus of Schubert varieties in $\mathbb{P}^*(\bigwedge^2 V)$ has dimension $2(n-2)$. If $F_{n-1}$ contains $G_{n-2}$, then $\Sigma_{1,1}(F_{n-2} \subset F_{n-1}) \subset \Sigma_{1}(G_{n-2} \subset G_{n})$. Hence, the fiber of $\pi_2$ over a hyperplane corresponding to a Schubert variety has dimension at least one. We conclude that $\dim(\pi_2^{-1}(S_1)) = 2n - 3 = \dim(\mathcal{I}(1,1))$. Hence, $\pi_2(\mathcal{I}(1,1)) = S_1$ and every hyperplane containing a Schubert variety $\Sigma_{1,1}$ is a Schubert variety $\Sigma_{1}$. Applying Lemma 3.1 $(b-1)$-times, we conclude that a general hyperplane section containing $\Sigma_{b,b}$ is smooth if $2b > n - 2$ or singular along a Schubert variety of the form $\Sigma_{2b,2b}$ if $2b \leq n - 2$. This also concludes the discussion of the case $a \neq b$. Let $H$ and $H'$ be two hyperplanes containing $\Sigma_{a,b}$. If $G(2,F_{n-b}) \subset H$ and $G(2,F_{n-b}) \not\subset H'$, then we have just proved that the dimension of the singular locus of $G(2,n) \cap H$ is greater than or equal to the dimension of the singular locus of $H' \cap G(2,n)$. This concludes the proof of the theorem. □

Since the proof of Proposition 1.9 uses similar techniques, we include it in this section.

**Proof of Proposition 1.9.** Let $\lambda$ be a partition of the form $\lambda_1 = \lambda_{k-1} = n - k$ and $\lambda_k > 0$, then the Plücker image of $\Sigma_{\lambda}$ is a linear space. Since the Grassmannian contains linear spaces with cohomology class $\sigma_{\mu}$, where $\mu = ((n-k)^{k-1}, 0)$, every hyperplane section contains linear spaces with cohomology class $\sigma_{\lambda}$. The same argument applies for a partition $\lambda$ with $\lambda_1 = n - k$ and $\lambda_k \geq n - k - 1$ by considering linear spaces with cohomology class $\sigma_{\nu}$, where $\nu = ((n-k-1)^{k})$. This proves the second part of the proposition.

To prove the first part of the proposition, we will show that if $\lambda$ is a partition such that $\lambda_1 < n - k$ and $\lambda_k = 0$, then any hyperplane $H$ containing $\Sigma_{\lambda}$ is singular. Fix a basis $e_1, \ldots, e_n$ of $V$. Let $F_\bullet$ be the flag where the flag element $F_i$ is the span of the basis vectors $e_1, \ldots, e_i$. Let $H$ be a hyperplane containing $\Sigma_{\lambda}(F_\bullet)$. Then the equation defining $H$ must be a linear combination of the Plücker coordinates defining $\Sigma_{\lambda}(F_\bullet)$. Recall that the Plücker coordinates vanishing on $\Sigma_{\lambda}(F_\bullet)$ are $p_{i_1,\ldots,i_k}$ with $i_1 < \cdots < i_k$ such that $i_j > n - k + j - \lambda_j$ for at least one
j. Since by assumption \( \lambda_k = 0 \) and we cannot have \( i_k > n \), there must exist \( j < k \) such that \( i_j > n - k + j - \lambda_j \). In particular, \( i_{k-1} > n - k + j - \lambda_j + k - j - 1 = n - \lambda_j > k \).

It follows that \( H \cap G(k, n) \) is singular at the point \( p = e_1 \wedge \cdots \wedge e_k \). The tangent space to \( G(k, n) \) at \( p \) is spanned by Plücker coordinates \( p_{i_1,\ldots,i_k} \) where the set \( \{i_1, \ldots, i_k\} \) differs from \( \{1, \ldots, k\} \) in at most one element. On the other hand, the Plücker coordinates occurring in the equation of \( H \) have indices that differ from \( \{1, \ldots, k\} \) in at least two elements. Hence, \( H \) vanishes at all the points spanning the tangent space to \( G(k, n) \) at \( p \). We conclude that \( H \cap G(k, n) \) is singular at \( p \). This concludes the proof of the proposition.

**Proofs of Corollaries 1.10 and 1.11.** When \( \lambda \) is the partition \( \lambda_1 = \cdots = \lambda_{k-1} = n - k - 1 \) and \( \lambda_k = 0 \), then, by Proposition 1.9 for any hyperplane \( H \) containing \( \Sigma_\lambda \) the hyperplane section \( H \cap G(k, n) \) is singular at a point. Conversely, if \( H \cap G(k, n) \) is singular at a point \( p = e_1 \wedge \cdots \wedge e_k \), then by Lemma 2.3 the Schubert variety \( \Sigma_\lambda \) parameterizing \( k \)-dimensional subspaces that intersect \( \text{Span}(e_1, \ldots, e_k) \) in a subspace of dimension at least \( k - 1 \) is contained in \( H \). In this case, we conclude that the image of \( \pi_2(\mathcal{I}(\lambda)) \) is precisely the dual variety.

Note that \( h^0(\mathcal{I}_\lambda(1)) = \binom{n}{k} - k(n-k) - 1 = N \). Hence, the incidence correspondence \( \mathcal{I}(\lambda) \) is a projective space bundle over \( G(k, n) \) with fibers of dimension \( N - 1 \). In particular, \( \dim(\mathcal{I}(\lambda)) = \binom{n}{k} - 2 \). When \( n-2 > k > 2 \), the dual variety \( G(k, n)^* \) is a hypersurface and the general tangent hyperplane to \( G(k, n) \) is tangent at a unique point. Therefore, \( \pi_2 \) is a birational map. Hence, \( \pi_2 : \mathcal{I}(\lambda) \to G(k, n)^* \) gives a resolution of singularities of \( G(k, n)^* \). This concludes the proofs of Corollary 1.10 and Corollary 1.11.

4. **The Proof of Theorem 1.3**

In this section, we prove Theorem 1.3 and discuss some generalizations to \( G(k, n) \).

**Proof of Theorem 1.3.** Let \( H \) be a hyperplane in \( \mathbb{P}(\wedge^2 V) \) such that \( [H] \in S_r/S_{r-1} \). Then \( H \cap G(2, n) \) is singular along a Schubert variety \( \Sigma_{2r,2r} \) parameterizing two-dimensional subspaces of \( V \) contained in a linear subspace \( F_{n-2r} \). First, suppose that \( a \neq b \). Let \( (V_{n-a-1} \subset V_{n-b}) \) be the partial flag defining a Schubert variety \( \Sigma_{a,b} \subset H \cap G(2, n) \). Suppose that \( \dim(V_{n-a-1} \cap F_{n-2r}) = j \). Then clearly

\[
0 \leq j \leq \min(n-a-1, n-2r).
\]

Consider the restriction of \( H \) to \( G(2, V_{n-b}) \). Either \( H \) identically vanishes on \( G(2, V_{n-b}) \); or \( H \) defines a hyperplane section of \( G(2, V_{n-b}) \).

If \( H \) identically vanishes on \( G(2, V_{n-b}) \), then both \( V_{n-a-1} \) and \( V_{n-b} \) are \( Q_H \)-isotropic. Hence, trivially \( V_{n-a-1} \subset V_{n-b} \subset V_{a-b-1} \). Take a linear space \( S_{2r} \) of dimension \( 2r \) complementary to \( F_{n-2r} \). Then the restriction of \( Q_H \) to \( S_{2r} \) is non-degenerate. Since \( \text{Span}(V_{n-a-1}, F_{n-2r}) \cap S_{2r} \) is isotropic with respect to the restriction of \( Q_H \) to \( S_{2r} \), its dimension \( n-a-1-j \) must be less than or equal to \( r \). Equivalently, \( n-a-1-r \leq j \). Similarly, since \( V_{n-b} \) is isotropic, \( n-b \leq n-r \).

In particular, \( b \geq r \). Hence, the inequality \( n-a-1-r \leq j \) holds.

Next, suppose that \( H \) defines a hyperplane section of \( G(2, V_{n-b}) \). By our assumption that \( \Sigma_{a,b}(V_{n-a-1} \subset V_{n-b}) \subset H \cap G(2, n) \), we must have that \( [W] \in H \cap G(2, n) \) for every two-dimensional subspace \( W \) that intersects \( V_{n-a-1} \) non-trivially and is contained in \( V_{n-b} \). In particular, \( [W] \) is contained in \( H \cap G(2, n) \) for every two-dimensional subspace \( W \) contained in \( V_{n-a-1} \). We conclude that the skew-symmetric form \( Q_H \) vanishes identically on \( V_{n-a-1} \). Hence, \( V_{n-a-1} \) is \( Q_H \)-isotropic. Hence, \( \text{Span}(V_{n-a-1}, F_{n-2r}) \) is also \( Q_H \)-isotropic. The dimension of this vector space, which by assumption is \( n-a-1+n-2r-j \), has to be less than or equal to \( n-r \). We conclude that \( n-a-1-r \leq j \).
Finally, since the restriction of $Q_H$ to $V_{a-b}$ must contain $V_{n-a-1}$ in its kernel, we must have that $V_{a-b} \subset V_{n-a-1}$. By assumption, the dimension of $V_{n-a-1}$ is $n - 1 - a - j$. Hence, $n - a - 1 - j \leq b$. Combining all these inequalities, yields the inequality

$$ \max(0, n - a - 1 - \min(b, r)) \leq j \leq \min(n - a - 1, n - 2r). $$

Note that by assumption $2r \leq a + b + 1$, so for $j$ satisfying the assumptions of the theorem, these inequalities hold.

Conversely, suppose $j$ satisfies the inequalities

$$ \max(0, n - a - 1 - \min(b, r)) \leq j \leq \min(n - a - 1, n - 2r). $$

Then every Schubert variety $\Sigma_{a,b}(V_{n-a-1} \subset V_{n-b})$ is contained in $H \cap G(2, n)$ provided $V_{n-a-1}$ is $Q_H$ isotropic and $V_{a-b} \subset V_{n-a-1}$. This is clear since the kernel of $Q_H$ restricted to $V_{n-a-1}$ contains $V_{n-a-1}$. Hence, every two-dimension space intersecting $V_{n-a-1}$ non-trivially is $Q_H$ isotropic.

Furthermore, there exists flags $(V_{n-a-1} \subset V_{n-b})$ such that $\dim(V_{n-a-1} \cap F_{n-2r}) = j$. To construct such a flag, let $S_{2r}$ be a linear space complementary to $F_{n-2r}$. Pick a $Q_H$ isotropic subspace $W$ of dimension $n - a - 1 - j$ in $S_{2r}$. This is possible since $n - a - 1 - j \leq r$. Pick a $j$-dimensional subspace $W'$ of $F_{n-2r}$. Let $V_{n-a-1} = \text{Span}(W, W')$. Then $V_{n-a-1}$ is isotropic and has dimension $n - a - 1$. Next, consider $V_{n-a-1}^\perp$, which has dimension $a + 1 + j$. Since by assumption $n - a - 1 - b \leq j$, $n - b \leq a + 1 + j$. Therefore, there exists $n - b$ dimensional subspaces of $V_{n-a-1}^\perp$ containing $V_{n-a-1}$.

Let $Z_j$ denote the locus of two-step flags $(V_{n-a-1} \subset V_{n-b})$ in $F(n - a - 1, n - b; n)$ such that $V_{n-a-1}$ is $Q_H$ isotropic, $\dim(V_{n-a-1} \cap F_{n-2r}) \geq j$ and $V_{a-b} \subset V_{n-a-1}$ and $\dim(V_{n-b} \cap F_{n-2r}) \geq 2n - 2r - a - b - 1 - j$. It is clear from the construction in the previous paragraph that $Z_j$ is irreducible. Recall the following definitions from the statement of the theorem

$$ M = \max(0, n - 1 - a - \min(r, b)) \quad \text{and} \quad N = \min\left(n - a - 1, n - r - \frac{a + b + 1}{2}\right). $$

We have shown that

$$ X((a, b), H) = \bigcup_{j=M}^{\min(n-a-1,n-2r)} Z_j $$

and in this range each $Z_j$ is non-empty. Finally, there remains to check that $Z_j$ is an irreducible component of $X((a, b), H)$ if $M \leq j \leq N$ and $X((a, b), H) = \bigcup_{j=M}^{N} Z_j$.

The dimension $\dim(V_{n-a-1} \cap F_{n-2r})$ is an upper-semi-continuous function. Consequently, if $j_1 > j_2$, then linear spaces intersecting $F_{n-2r}$ in a $j_1$-dimensional subspace cannot specialize to linear spaces intersecting $F_{n-2r}$ in a $j_2$-dimensional subspace. Therefore, $Z_{j_2}$ cannot be contained in $Z_{j_1}$. On the other hand, $\dim(V_{n-b} \cap F_{n-2r})$ is also an upper-semi-continuous function. By construction, for a general point $(V_{n-a-1}, V_{n-b})$ in $Z_{j_2}$, $\dim(V_{n-b} \cap F_{n-2r}) = \max(j, 2n - 2r - a - b - 1 - j)$ since $V_{n-b}$ is an arbitrary linear space containing $V_{n-a-1}$ and contained in the $(a + j + 1)$-dimensional space $V_{n-a-1}^\perp$. Suppose $n - r - \frac{a + b + 1}{2} \geq j_1 > j_2$, then the dimension of $V_{n-b} \cap F_{n-2r}$ for a general point in $Z_{j_1}$, respectively, $Z_{j_2}$ is given by $2n - 2r - a - b - 1 - j_1 < 2n - 2r - a - b - 1 - j_2$. Hence, $Z_{j_1}$ cannot be contained in $Z_{j_2}$. We conclude that for $M \leq j \leq N$, $Z_j$ form irreducible components of $X((a, b), H)$.

There remains to show that when $2j > 2n - 2r - a - b - 1$, then $Z_j$ is contained in $Z_{j-1}$. Let $(V_{n-a-1} \subset V_{n-b})$ be a point of $Z_j$ such that $\dim(V_{n-a-1} \cap F_{n-2r}) = \dim(V_{n-b} \cap F_{n-2r}) = j$. Let $E$ be a codimension one linear space in $V$ containing the vector space $\text{Span}(V_{n-b}, F_{n-2r})$. By
assumption,
\[
\dim(\text{Span}(V_{n-b}, F_{n-2r})) = 2n - 2r - b - j < a + 1 + j \leq n.
\]
Hence, we can always find a codimension one linear space \( E \) containing \( \text{Span}(V_{n-b}, F_{n-2r}) \). Since a non-degenerate skew-symmetric form can only exist in an even dimensional vector space, the dimension of the kernel of \( Q_H \) restricted to \( E \) has to have dimension greater than or equal to \( n - 2r + 1 \). Denote this kernel by \( K_E \). Let \( V_{a+1-b} \) be a general subspace in \( V_{n-b} \) complementary to \( V_{n-a-1} \). Pick a pencil of linear spaces \( V_{n-a-1}(t) \) such that \( V_{n-a-1}(0) = V_{n-a-1}, V_{n-a-1}(t) \subset K_E \) and \( V_{n-a-1}(t) \not\subset F_{n-2r} \) for \( t \neq 0 \). Consider the pencil of flags \( (V_{n-a-1}(t) \subset \text{Span}(V_{n-a-1}(t), V_{a+1-b})) \). First, notice that when \( t = 0 \), this is simply \( (V_{n-a-1}) \subset V_{n-b} \). Hence, except for finitely many \( t \), these flags are contained in \( F(n - a - 1, n - b; n) \). By construction, \( \dim(V_{n-a-1}(t) \cap F_{n-2r}) = j - 1 \). Since \( V_{n-a-1}(t) \subset K_E, \text{Span}(V_{n-a-1}(t), V_{a+1-b}) \subset V_{n-a-1}(t) \perp \). Hence, the general member of this family is contained in \( Z_{j-1} \). We conclude that \( Z_j \subset Z_{j-1} \).

The computation of the dimension of \( Z_j \) is standard. We have to choose a \( Q_H \) isotropic subspace \( V_{n-a-1} \) that intersects the kernel of \( Q_H \) in a subspace of dimension \( j \). The reader can easily check that the dimension of the space of such isotropic subspaces is
\[
\frac{(n - a - 1)(3a + j - n + 4)}{2} - j\left(\frac{4r + 3a + 3j - 3n + 4}{2}\right).
\]
Then we need to choose an \((a - b)\)-dimensional subspace in the \((a + j + 1)\)-dimensional subspace \( V_{n-a-1} \) containing \( V_{n-a-1} \). The dimension of the space of such linear spaces \( V_{n-b} \) is
\[
(a + 1 - b)(a + b + j - n + 1).
\]
This immediately yields the dimension formula for \( Z_j \).

Next, suppose that \( a = b \). In this case, the Schubert variety is determined by one flag element \( V_{n-a} \). Since \( \Sigma_{a,a} \subset H \cap G(2, n), V_{n-a} \) is \( Q_H \) isotropic. Conversely, if \( V_{n-a} \) is \( Q_H \)-isotropic, then \([W] \in H \cap G(2, n)\) for every two dimensional subspace \( W \subset V_{n-a} \). We conclude that \( X((a, a), H) \) is the space of \( Q_H \)-isotropic linear spaces of dimension \( n - a \). It is standard that this space is irreducible and has the claimed dimension.

\[\square\]

**Example 4.1.** Let \( H \) be a hyperplane in \( \mathbb{P}({\wedge}^2 V_8) \) such that \([H] \in S_2\). Let \( H \cap G(2, 8) \) be the corresponding hyperplane section of \( G(2, 8) \). Then the space \( X((5, 4), H) \) parameterizing Schubert varieties of the form \( \Sigma_{5,4}(V_2 \subset V_4) \) has two irreducible components \( Z_0 \) and \( Z_1 \). The singular locus of \( H \cap G(2, 8) \) consists of two-dimensional subspaces contained in a four-dimensional vector space \( F_4 \). The component \( Z_0 \) parameterizes pairs \((V_2 \subset V_4)\) such that \([V_2] \in H \cap G(2, 8)\) and \( V_4 \subset V_2^\perp \) and \( \dim(V_2 \cap V_4) \geq 2 \). The component \( Z_1 \) parameterizes \( V_2 \) such that \( \dim(V_2 \cap F_4) \geq 1 \) and \( V_4 \subset V_2^\perp \). Note that \( Z_1 \) contains the pairs where \( V_2 \subset F_4 \) and \( V_4 \subset V_2^\perp \).

The corollaries are obtained by specializing the numbers \( a \) and \( b \).

**Proof of Corollary 4.1.** When \( a = b = r \), we are in Case (2) of Theorem 1.3 \( X((a, a), H) \) parameterizes \((n-a)\)-dimensional isotropic subspaces of \( Q_H \). These are maximal dimensional isotropic subspaces, hence they all contain the kernel \( F_{n-2r} \) of \( Q_H \). Passing to the quotient \( V/F_{n-2r} \), we see that \( X((a, a), H) \) parameterizes \( r \)-dimensional isotropic subspaces of a \( 2r \)-dimensional vector space under a non-degenerate skew-symmetric form. We conclude that \( X((a, a), H) \) is isomorphic to \( SG(r, 2r) \). This variety is irreducible of dimension \( \binom{r + 1}{2} \). \[\square\]
Proof of Corollary 1.7. When \( a + b + 1 = 2r \), we are in Case (1) of Theorem 1.3. The integers \( a \) and \( b \) must satisfy the inequalities \( b < r \leq a \). Hence \( n - a - b - 1 = n - 2r \leq j \leq n - r - \frac{a + b + 1}{2} = n - 2r \). We conclude that \( j = n - 2r \) and that \( X((a, 2r - a - 1), H) \) is irreducible. The linear space \( V_{n-a-1} \) must contain the kernel of \( Q_H \), which by assumption has dimension \( n - 2r = j \). Furthermore, \( \dim(V_{n-a-1}^\perp) = n - 2r + a + 1 = n - b \). Hence, \( V_{n-b} = V_{n-a-1}^\perp \). Therefore, \( X((a, 2r - a - 1), H) \) can be identified with \( SG(b, 2r) \). \( \square \)

Proof of Corollary 1.6. When \( b = 0 \), we are in Case (1) of Theorem 1.3. In this case, \( n - a - 1 \leq j \leq n - a - 1 \). Hence, there is only one component and \( V_{n-a-1} \) is contained in \( F_{n-2r} \). Therefore, in this case, \( X((a, 0), H) \) parameterizes linear spaces \( V_{n-a-1} \) contained in \( F_{n-2r} \). This is the Grassmannian \( G(n - a - 1, n - 2r) \), which has dimension \( (n - a - 1)(a + 1 - 2r) \). \( \square \)

Proof of Corollary 1.7. If \( a \) is odd and \( \lambda = (a, 0) \), then we are in Case (1) of Theorem 1.3 and \( j = n - a - 1 \). Hence, \( V_{n-a-1} = F_{n-2r} \) and \( V_{n-b} = V \). Hence, \( \pi_2 \) is a birational map from \( I(\lambda) \) to \( S_{a+1} \). In particular, when \( n \) is odd a smooth hyperplane section contains a unique linear space of the form \( \Sigma_{n-2,0} \). The rest of the corollary is obvious. \( \square \)

Finally, we prove Proposition 1.12 which clearly specializes to Corollary 1.8 when \( k = 2 \).

Proof of Proposition 1.12. Let \( H = \Sigma_1(F_{n-k} \subset F_{n-k+2} \subset \cdots \subset F_n) \). A Schubert variety \( \Sigma_\lambda \) is contained in \( H \) if and only if every \( k \)-dimensional subspace parameterized by \( \Sigma_\lambda \) intersects \( F_{n-k} \) non-trivially. Let \( V_{n-k+k_1-\mu_1} \subset V_{n-k+k_2-\mu_2} \subset \cdots \subset V_{n-\mu_t} \) be the linear spaces defining \( \Sigma_\lambda \). Let \( W \) be any \( k \)-dimensional subspace such that \( [W] \in \Sigma_\lambda \). If for some \( j \), \( \dim(V_{n-k+k_1-\mu_1} \cap F_{n-k}) \geq n - k - \mu_j + 1 \), then we can estimate \( \dim(W \cap F_{n-k} \cap V_{n-k+k_1-\mu_1}) \) as follows. \( \dim(W \cap V_{n-k+k_1-\mu_1}) \geq k_j \) since \( [W] \in \Sigma_\lambda \). Hence, \( \dim(W \cap F_{n-k} \cap V_{n-k+k_1-\mu_1}) \geq k_j + n - k - \mu_j + 1 - (n - k + k_j - \mu_j) = 1 \). We conclude that \( [W] \in H \cap G(k, n) \), hence \( \Sigma_\lambda \subset H \cap G(k, n) \). Note that if \( \mu_t = 0 \), then the condition \( \dim(V_{n-\mu_t} \cap F_{n-k}) \geq n - k + 1 \) is impossible to satisfy. Therefore, that case has to be treated separately.

Conversely, suppose that \( \dim(V_{n-k+k_1-\mu_1} \cap F_{n-k}) = n - k - \mu_j \) for every \( 1 \leq j \leq t \). Then there exists a \( k_1 \)-dimensional subspace in \( V_{n-k+k_1-\mu_1} \) that does not intersect \( F_{n-k} \). This can be extended to a \( k_2 \)-dimensional subspace in \( V_{n-k+k_2-\mu_2} \) that does not intersect \( F_{n-k} \). Continuing this way, we construct a \( k \)-dimensional subspace \( W \) such that \( [W] \in \Sigma_\lambda \), but \( [W] \notin H \cap G(k, n) \).

Let \( S_j \) be the Schubert variety in the flag variety \( F(n - k + k_1 - \mu_1, \ldots, n - \mu_t, n) \) defined by

\[ S_j = \{(V_{n-k+k_1-\mu_1} \subset \cdots \subset V_{n-\mu_t} \mid \dim(V_{n-k+k_j-\mu_j} \cap F_{n-k}) \geq n - k - \mu_j + 1)\} \]

We have shown that \( X(\lambda, H) = \bigcup_{j=1}^{r_0} S_j \). Since the Schubert varieties \( S_j \nsubseteq S_i \) for \( i \neq j \), we conclude that the \( t - \delta_0,t \) Schubert varieties \( S_j \) form the irreducible components of \( X(\lambda, H) \).

This concludes the proof of the proposition. \( \square \)

References


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