RIGIDITY OF SCHUBERT CLASSES IN ORTHOGONAL GRASSMANNIANS

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Abstract. A Schubert class σ in the cohomology of a homogeneous variety X is called *rigid* if the only projective subvarieties of X representing σ are Schubert varieties. A Schubert class σ is called *multi rigid* if the only projective subvarieties representing positive integral multiples of σ are unions of Schubert varieties. In this paper, we discuss the rigidity and multi rigidity of Schubert classes in orthogonal Grassmannians. For a large set of non-rigid classes, we provide explicit deformations of Schubert varieties using combinatorially defined varieties called *restriction varieties*. We characterize rigid and multi rigid Schubert classes of Grassmannian and quadric type. We also characterize all the rigid classes in $OG(2,n)$ if $n > 8$.

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1. Introduction

1.1. **The motivating questions.** Let G be an algebraic group and let P be a parabolic subgroup. A Schubert class σ in the cohomology of the homogeneous variety $X = G/P$ is called *rigid* if the only projective subvarieties of X representing σ are Schubert varieties. We will call a Schubert class σ *multi rigid* if the only projective subvarieties of X representing a positive, integral multiple $k\sigma$ are unions of k Schubert varieties. By definition, a multi rigid class is also rigid. However, the converse is often false. The class of a line in $\mathbb{P}^2$ is rigid but not multi rigid since, for example, twice the line class can be represented by a smooth conic.

In this paper, we will discuss the rigidity and multi rigidity of Schubert classes in orthogonal Grassmannians. The problem of determining rigid and multi rigid cohomology classes

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is a fundamental problem in differential geometry. In recent years, there has been significant progress in determining multi rigid Schubert classes in compact complex Hermitian symmetric (CCHS) spaces. In an approach initiated by Bryant [B] and Walters [W] and carried further by Hong [Ho1], [Ho2], Robles, The [RT] and others, one studies a differential system called the *Schur differential system* associated to each Schubert class (see [B] or [Ho1] for the definition and properties of the Schur differential system). Under suitable assumptions, using Lie algebra cohomology, one shows that the class is *Schur rigid*, that is the only integral varieties of this differential system are Schubert varieties, thereby concluding the multi rigidity of the corresponding Schubert classes. In fact, in this setting, multi rigidity is equivalent to Schur rigidity [RT]. For example, Hong has shown that if X is a CCHS space other than an odd-dimensional quadric, then the class of a smooth Schubert variety other than a non-maximal linear space or $\mathbb{P}^1 \subset C_n/P_n$ is multi rigid [Ho1]. She has also proved the multi rigidity of a large number of Schubert classes in the Grassmannian $G(k,n)$ [Ho2]. Robles and The have extended the analysis by determining the complete set of first-order obstructions to Schur rigidity. Consequently, they have extended Hong's results to many singular Schubert varieties in CCHS spaces and have sharpened Hong's results for Grassmannians [RT].

In [C1], using algebro-geometric techniques, we characterized the rigid Schubert classes in Grassmannians $G(k,n)$. The purpose of this paper is to extend the techniques introduced in [C1] to orthogonal Grassmannians. Since orthogonal Grassmannians are not in general CCHS spaces, this paper provides the first detailed study of the rigidity of Schubert classes in a non-CCHS homogeneous variety. The algebro-geometric techniques thus have two main advantages over the differential geometric techniques. First, they can prove the rigidity of a class even when the class is not multi rigid. Second, they apply to homogeneous varieties other than CCHS spaces. The main disadvantage is that studying multiples of Schubert classes gets successively more difficult as the multiple increases, making the technique less suitable for studying multi rigidity.

The study of rigidity is strongly motivated by the classical problem of determining whether a cohomology class can be represented by a smooth subvariety. There are several different versions of the problem:

(1) Given a cohomology class c, can it be represented by a smooth subvariety of X?
(2) Given a cohomology class c, can a multiple be represented by a positive linear combination of classes of smooth subvarieties of X?
(3) Given a cohomology class c, can a multiple be represented by an arbitrary linear combination of classes of smooth subvarieties of X?

For rigid, singular Schubert classes, the answer to the first question is negative. For multi rigid, singular Schubert classes, the answer to the second question is negative. These problems have been investigated by many authors for the Grassmannian $G(k,n)$ (see [HRT], [K1], [KL] and [C1]). We will say that a Schubert class is *smoothable* if it can be represented by a smooth subvariety. Our results on rigidity and multi rigidity imply that certain Schubert classes in orthogonal Grassmannians are not smoothable.

We will now introduce the necessary notation and state our results more precisely.

1.2. **Notation.** We work over the field of complex numbers $\mathbb{C}$. In this paper, varieties are reduced but may be reducible. Let V be an n-dimensional vector space. Let $G(k,n)$ denote the Grassmannian parameterizing k-dimensional subspaces of V. Let Q be a non-degenerate, symmetric bilinear form on V. A k-dimensional subspace W of V is called *isotropic* if $w_1^T Q w_2 = 0$ for every $w_1, w_2 \in W$. If W is an isotropic subspace of V, then $2k \leq n$. If $n \neq 2k$, the space parameterizing k-dimensional isotropic subspaces of V is an irreducible homogeneous variety. When $n = 2k$, k-dimensional isotropic subspaces of V form two connected components. In this case, two linear spaces W, W' belong to the same connected component if and only if $\dim(W \cap W') = k$ modulo 2. Let $OG(k,n)$ denote the orthogonal Grassmannian that parameterizes k-dimensional isotropic subspaces of V. When $n = 2k$, we allow $OG(k, 2k)$ to have two isomorphic connected components. We warn the reader that in the literature it is customary to let $OG(k, 2k)$ denote only one of the connected components.

Geometrically, Q defines a smooth quadric hypersurface in $\mathbb{P}V$. W is isotropic if and only if $\mathbb{P}W$ lies on the quadric hypersurface defined by Q. Given an isotropic linear space W, we can define its orthogonal complement $W^\perp$ with respect to Q as the set of all vectors $v \in V$ such that $v^T Q w = 0$ for all $w \in W$. Geometrically, $\mathbb{P}W^\perp$ is the linear space everywhere tangent to the quadric defined by Q along $\mathbb{P}W$.

The cohomology groups of both $G(k,n)$ and $OG(k,n)$ are generated by the classes of Schubert varieties. To describe the Schubert varieties in $G(k,n)$, let $a_\bullet = 0 < a_1 < a_2 < \cdots < a_k \leq n$ be an increasing sequence of k positive integers. Let $F_\bullet = F_1 \subset F_2 \subset \cdots \subset F_n$ be a flag. The Schubert class $\sigma_{a_\bullet}$ is the cohomology class of the Schubert variety

$$\Sigma_{a_\bullet}(F_\bullet) = \{W \in G(k,n) \mid \dim(W \cap F_{a_i}) \geq i \text{ for } 1 \leq i \leq k\}.$$

To describe the Schubert classes in $OG(k,n)$, first assume that n is odd. We parameterize Schubert varieties in $OG(k,n)$ by pairs of sequences $(a_\bullet; b_\bullet)$ of total length k such that $0 < a_1 < a_2 < \cdots < a_s < \frac{n}{2}$ and $\frac{n}{2} - 1 > b_{s+1} > b_{s+2} > \cdots > b_k \geq 0$ with the property that $a_i \neq b_j + 1$ for any $1 \leq i \leq s$ and $s+1 \leq j \leq k$. Let $F_\bullet = F_1 \subset F_2 \subset \cdots \subset F_{\lfloor n/2 \rfloor}$ be an isotropic flag. The Schubert variety $\Sigma_{a_\bullet;b_\bullet}(F_\bullet)$ is the Zariski closure of the locus in $OG(k,n)$ defined by

$$\{W \in OG(k,n) \mid \dim(W \cap F_{a_i}) = i \text{ for } 1 \leq i \leq s, \dim(W \cap F_{b_j}^\perp) = j \text{ for } s < j \leq k\}.$$

Let $\sigma_{a_\bullet;b_\bullet}$ denote its cohomology class.

Next, assume that n is even. In this case, the notation has to distinguish between the two connected components of the space of half dimensional linear spaces. We denote half dimensional linear spaces in one connected component by $F_{n/2}$ and the other one by $F_{n/2-1}^\perp$. Technically, $F_{n/2-1}^\perp$ consists of the span of two half dimensional linear spaces in different connected components. However, this will significantly simplify notation. Schubert varieties are parameterized by pairs of sequences $0 < a_1 < a_2 < \cdots < a_s \leq \frac{n}{2}$ and $\frac{n}{2} - 1 \geq b_{s+1} > b_{s+2} > \cdots > b_k \geq 0$ such that $a_i \neq b_j + 1$ for any $1 \leq i \leq s$ and $s+1 \leq j \leq k$. The Schubert variety $\Sigma_{a_\bullet;b_\bullet}$ is the Zariski closure of the locus in $OG(k,n)$ defined by

$$\{W \in OG(k,n) \mid \dim(W \cap F_{a_i}) = i \text{ for } 1 \leq i \leq s, \dim(W \cap F_{b_j}^\perp) = j \text{ for } s < j \leq k\}.$$

Let $\sigma_{a_\bullet;b_\bullet}$ denote its cohomology class.

Definition 1.1. We call a Schubert class $\sigma_{a_\bullet}$ in $OG(k,n)$ of *Grassmannian type* if in the sequence defining it $s = k$ and $a_k < \frac{n}{2}$. We call a Schubert class $\sigma_{b_\bullet}$ in $OG(k,n)$ of *quadric type* if in the sequence defining it $s = 0$ and $\frac{n}{2} - 1 > b_1$.

Remark 1.2. A Schubert class in $OG(k,n)$ is of Grassmannian type if every flag element in its definition is isotropic. In addition, if n is even, the largest dimensional flag element is not maximal isotropic. A Schubert class in $OG(k,n)$ is of quadric type, if none of the flag elements in its definition are isotropic. Under the intersection pairing, the dual of a class of Grassmannian type is a class of quadric type and vice versa.

A word is in order about our notation. It is customary to parameterize Schubert classes in $G(k,n)$ by partitions $\lambda : n-k \geq \lambda_1 \geq \cdots \geq \lambda_k \geq 0$. The relation between the two notations is given by $a_i = n-k+i-\lambda_i$. Similarly, for the maximal isotropic Grassmannians $OG(k, 2k+1)$ (respectively, $OG(k,2k)$), it is customary to denote the Schubert classes by strictly decreasing partitions $\lambda_1 > \cdots > \lambda_s$ of length at most k (respectively, $k-1$). Note that in these cases the sequence $b_\bullet$ is uniquely determined by the sequence $a_\bullet$ by the requirement that $a_i \neq b_j + 1$. The translation in this case is given by $a_i = k - \lambda_i + 1$ (respectively, $a_i = k - \lambda_i$, noting that when $a_s = \frac{n}{2}$, $\lambda_s = 0$ is omitted). For the non-maximal orthogonal Grassmannians, there are several different notations in use. The translation between our notation and other conventions is straightforward. The advantage of our notation is that it is well adapted for maps between different Grassmannians. For example, under the natural inclusions $G(k,n) \hookrightarrow G(k,n+1)$ (respectively, $OG(k,n) \hookrightarrow G(k,n)$) the classes of Schubert varieties (respectively, those of Grassmannian type) are denoted by the same sequences.

Given an increasing sequence of positive integers $a_\bullet$, we can associate a non-decreasing sequence of non-negative integers $\alpha_\bullet$ by setting $\alpha_i = a_i - i$. The associated sequence plays an important role in the geometry of $G(k,n)$. For instance, the dimension of the Schubert variety $\Sigma_{a_\bullet}(F_\bullet)$ in $G(k,n)$ is $\sum_{i=1}^k \alpha_i$. Similarly, given the sequences $(a_\bullet; b_\bullet)$, it is convenient to record the sequences $(\alpha_\bullet, \beta_\bullet)$ by setting $\alpha_i = a_i - i$ and $\beta_j = n - b_j - j$. The concatenation of the sequences $\alpha_\bullet$ and $\beta_\bullet$ is a sequence of non-decreasing, non-negative integers. Furthermore, given such a sequence $c_\bullet$, it is helpful to group together the parts that are equal. We will write the sequence as $(\tilde{c}_1^{i_1}, \dots, \tilde{c}_t^{i_t})$, where the sequence has t distinct parts $\tilde{c}_1 < \cdots < \tilde{c}_t$ that are strictly increasing and the j-th part $\tilde{c}_j$ is repeated i_j times, i.e.,

$$c_1 = \cdots = c_{i_1} = \tilde{c}_1, c_{i_1+1} = \cdots = c_{i_1+i_2} = \tilde{c}_2, \dots, c_{i_1+\cdots+i_{t-1}+1} = \cdots = c_{i_1+\cdots+i_t} = \tilde{c}_t.$$

We will use the different ways of denoting a sequence interchangeably.

1.3. **Results and examples.** We will now give several illuminating examples and summarize our results. The classical theorem of Bertini asserts that the general member of a linear system on a smooth variety is smooth away from the base points of the linear system [H, III.10.9]. Bertini's Theorem allows in many instances to deform Schubert varieties as the following example shows.

Example 1.3. The orthogonal Grassmannian $OG(1,n)$ is a quadric hypersurface Q in $\mathbb{P}^{n-1}$. Let Q_d^r denote a quadric of corank r obtained by restricting Q to $\mathbb{P}\Lambda_d$, where Λ_d is a linear space of dimension d. Let L_j denote an isotropic subspace of dimension j. The Schubert varieties in $OG(1,n)$ consist of:

- Linear spaces $\mathbb{P}L_j$ for $0 < j \leq \frac{n-1}{2}$ and the linear spaces $\mathbb{P}L_k$ and $\mathbb{P}L_k'$ if $n = 2k$, where the linear spaces L_k and L_k' belong to different connected components.
- The quadrics Q_d^{n-d} for $n \geq d > \frac{n}{2} + 1$.

The linear spaces are smooth and their classes are rigid. When $d < n$, the quadrics Q_d^{n-d} are singular. Nevertheless, the cohomology class of Q_d^{n-d} is the same as the cohomology class of any quadric Q_d^r with $r \leq n-d$ contained in Q. Therefore, the latter classes are not rigid. In particular, since Q_d^0 is a smooth quadric, every Schubert class in $OG(1,n)$ can be represented by a smooth subvariety of $OG(1,n)$. The classes of the linear spaces $\mathbb{P}L_j$, for $1 < j \leq \frac{n-1}{2}$, are rigid but not multi rigid. For example, twice the class of $\mathbb{P}L_j$ can be represented by a smooth quadric of the same dimension. If $2k = n$, the classes of the Schubert varieties $\mathbb{P}L_k$ and $\mathbb{P}L_k'$ are multi rigid [Ho1].

Example 1.3 demonstrates that by deforming the quadrics to less singular quadrics, we can obtain deformations of Schubert varieties. It also shows that, while the isotropic spaces are rigid, their multiples may be deformed to quadrics. We will systematically use these two facts to prove the failure of rigidity or multi rigidity for many Schubert classes.

Our first two theorems characterize the rigidity and multi rigidity of classes of Grassmannian type and quadric type.

Theorem 1.4. *Let $\sigma_{a_\bullet}$ be a Schubert class of Grassmannian type in the cohomology of $OG(k,n)$. Let $\alpha_\bullet$ be the sequence associated to $a_\bullet$ by setting $\alpha_i = a_i - i$. Express $\alpha_\bullet$ as $(\tilde{\alpha}_1^{i_1}, \dots, \tilde{\alpha}_t^{i_t})$ by grouping the equal parts. Then:*

(1) *The class $\sigma_{a_\bullet}$ is rigid if and only if there does not exist an index $1 \leq u < t$ such that $i_u = 1$ and $0 < \tilde{\alpha}_u = \tilde{\alpha}_{u+1} - 1$.*
(2) *The class $\sigma_{a_\bullet}$ is multi rigid if and only if $i_u \geq 2$ for every $2 \leq u \leq t$, $i_1 \geq 2$ unless $\tilde{\alpha}_1 = 0$, and $\tilde{\alpha}_u \leq \tilde{\alpha}_{u+1} - 2$ for every $1 \leq u < t$.*
(3) *The class $\sigma_{a_\bullet}$ is not smoothable if there exists an index $1 \leq u < t$ such that $0 < \tilde{\alpha}_u < \tilde{\alpha}_{u+1} - 1$, or an index $1 \leq u < t$ such that $\tilde{\alpha}_u > 0$ and $i_u > 1$.*

Theorem 1.5. *Let $\sigma_{b_\bullet}$ be a Schubert class of quadric type in the cohomology of $OG(k,n)$. Let $\beta_\bullet$ be the sequence derived from $b_\bullet$ by setting $\beta_j = n - b_j - j$ and group equal terms to express it as $(\tilde{\beta}_1^{i_1}, \dots, \tilde{\beta}_t^{i_t})$. Then:*

(1) *The class $\sigma_{b_\bullet}$ is not rigid unless $t = 1$ and $\tilde{\beta}_1 = n - k$.*
(2) *The class $\sigma_{b_\bullet}$ is not smoothable if there exists an index $1 \leq u < t$ such that $\lfloor \frac{n+1}{2} \rfloor < \tilde{\beta}_u < \tilde{\beta}_{u+1} - 1$ or an index $1 \leq u < t$ such that $\lfloor \frac{n+1}{2} \rfloor < \tilde{\beta}_u$ and $i_u > 1$.*

Example 1.6. The orthogonal Grassmannian $OG(2,5)$ is isomorphic to $\mathbb{P}^3$. The codimension two Schubert varieties $\Sigma_{1;1}$ in $OG(2,5)$ are lines; however, not all lines in $OG(2,5)$ are Schubert varieties. A Schubert variety $\Sigma_{1;1}$ is determined by specifying a point p on $Q \subset \mathbb{P}^4$. Projectively, the Schubert variety parameterizes lines in Q that contain the point p. In particular, the space of Schubert varieties with class $\sigma_{1;1}$ is a quadric threefold. Let $Q' \subset Q$ be a codimension one smooth quadric and let $l \subset Q'$ be a line. Then the space of lines that are contained in Q' and intersect l is also a line in $OG(2,5)$. We will later see that this is an example of a restriction variety. The lines parameterized by the Schubert variety $\Sigma_{1,1}$ sweep out the singular quadric surface $T_pQ \cap Q$, whereas the lines parameterized by the restriction variety sweep out the smooth quadric surface Q'. Since $OG(2,5)$ is isomorphic to $\mathbb{P}^3$, the space of lines in $OG(2,5)$ is isomorphic to the Grassmannian $G(2,4)$ parametrizing lines in $\mathbb{P}^3$. The Grassmannian $G(2,4)$ admits a map to $(\mathbb{P}^4)^*$ sending a point $q \in G(2,4)$ to the hyperplane in $\mathbb{P}^4$ spanned by the linear spaces parameterized by the line corresponding to q. This is a two-to-one map branched over the locus of Schubert varieties. This is one of the first examples where restriction varieties provide an explicit deformation of Schubert varieties. This example also shows that a Schubert class may be represented by a variety that is isomorphic, even projectively equivalent (under $GL(n)$ but not $SO(n)$), to a Schubert variety but is not a Schubert variety.

We now turn our attention to more general cohomology classes. The next two theorems give criteria that guarantee that the class is not rigid or multi rigid.

Theorem 1.7. *Let $\sigma_{a_\bullet;b_\bullet}$ be a Schubert class in $OG(k,n)$. Let $(\alpha_\bullet;\beta_\bullet)$ be the sequences defined by setting $\alpha_j = a_j - j$ and $\beta_i = n - b_i - i$. Express these sequences by grouping the equal terms $(\tilde{\alpha}_1^{j_1}, \dots, \tilde{\alpha}_t^{j_t}; \tilde{\beta}_1^{i_1}, \dots, \tilde{\beta}_v^{i_v})$. Assume that one of the following conditions holds for $(a_\bullet; b_\bullet)$:*

(1) *$\tilde{\beta}_1 < n - k$ and $b_{s+i_1} \neq a_j$ for any $1 \leq j \leq s$.*

(2) *There exists an index* $1 \leq u < t$ *such that* $j_u = 1$, $0 < \tilde{\alpha}_u = \tilde{\alpha}_{u+1} - 1$ *and* $a_{j_1+\cdots+j_u} \neq b_i$ *for any* $s < i \leq k$.

(3) $\#\{j | a_j \leq b_{s+1}\} = s + b_{s+1} - \frac{n-3}{2}$ *and there exists an index* $1 \leq h \leq s$ *such that* $a_h = b_{s+1}$.

Then $\sigma_{a_\bullet, b_\bullet}$ *is not rigid.*

Theorem 1.8. *Let* $\sigma_{a_\bullet;b_\bullet}$ *be a Schubert class in* $OG(k,n)$. *Let* $(\alpha_\bullet; \beta_\bullet)$ *be the sequences defined by setting* $\alpha_j = a_j - j$ *and* $\beta_i = n - b_i - i$. *Express these sequences by grouping the equal terms* $(\tilde{\alpha}_1^{j_1}, \ldots, \tilde{\alpha}_t^{j_t}; \tilde{\beta}_1^{i_1}, \ldots, \tilde{\beta}_v^{i_v})$. *Assume that either one of the conditions in Theorem 1.7 or one of the following conditions holds for* $(a_\bullet; b_\bullet)$:

(1) *There exists an index* $1 \leq u \leq t$ *such that* $a_{j_1+\cdots+j_u} \neq b_i$ *for any* $s+1 \leq i \leq k$ *and either* $j_u = 1$ *with* $0 < \tilde{\alpha}_u < \frac{n}{2}$ *or* $\tilde{\alpha}_u = \tilde{\alpha}_{u+1} - 1$.

(2) $a_{s-1} + 1 < a_s < \frac{n}{2}$ *and either* $a_s > b_{s+1}$ *or* $a_s = b_j$ *for some* $s+1 \leq j \leq k$ *and* $b_j = b_{j-1} + 2 = b_{s+1} + j - s - 1$.

Then $\sigma_{a_\bullet;b_\bullet}$ *is not multi rigid.*

Remark 1.9. Observe that Condition (2) in Theorem 1.7 specializes to the conditions in Part (1) of Theorem 1.4 for classes of Grassmannian type. Condition (1) in Theorem 1.7 specializes to the condition in Part (1) of Theorem 1.5 for classes of quadric type. Similarly, Condition (1) in Theorem 1.8 specializes to the conditions in Part (2) of Theorem 1.4 for classes of Grassmannian type.

We now turn to examples of rigid Schubert varieties. One can speculate that provided that n is sufficiently large, a Schubert class $\sigma_{a_\bullet;b_\bullet}$ in the cohomology of $OG(k,n)$ is rigid if $\sigma_{a_\bullet}$ is rigid in $OG(s,n)$ (described in Theorem 1.4) and b_j is either equal to a_i for some i or $b_j = b_{j-1} + 1$ for $s < j \leq n$. Our next theorem shows that, in fact, a large subset of such classes are rigid.

Theorem 1.10. *Let* $\sigma_{a_\bullet;b_\bullet}$ *be a Schubert class in* $OG(k,n)$ *satisfying the following properties:*

(1) *Set* $\alpha_i = a_i - i$ *and write the sequence as* $(\tilde{\alpha}_1^{i_1}, \ldots, \tilde{\alpha}_t^{i_t})$. *Assume that there does not exist an index* $1 \leq u < t$ *such that* $0 < \tilde{\alpha}_u = \tilde{\alpha}_{u+1} - 1$.

(2) *Either* $b_j = a_s + k - j$ *and* $n > 2a_s + 2k - 2s + 1$; *or* $s = 1$, $b_j = k - j$ *if* $j < a_1 - 1$, $b_j = k - j + 1$ *if* $j \geq a_1 - 1$, *and* $n > 2k + a$.

Then $\sigma_{a_\bullet;b_\bullet}$ *is rigid.*

Theorem 1.10 and Theorem 1.7 suffice to characterize all the rigid Schubert classes in $OG(2,n)$ when $n > 8$. Theorem 6.11 will make this characterization explicit.

Remark 1.11. Robles and The in [RT, Theorem 8.1] have proved the multi rigidity of certain Schubert classes in maximal orthogonal Grassmannians $OG(k, 2k)$.

The organization of this paper is as follows. In §2, we will recall some basic facts concerning the rigidity of Schubert cycles in $G(k,n)$. In §3, we will recall the definition of restriction varieties and describe the combinatorial algorithm for computing their cohomology classes. In §4, we will prove Theorem 1.4 and Theorem 1.5. In §5, we will prove Theorem 1.7 and Theorem 1.8. Finally, in §6, we will prove Theorem 1.10.

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2. Preliminaries

In this section, we recall facts concerning the rigidity of Schubert classes in $G(k,n)$ and some maps between Grassmannians and orthogonal Grassmannians that we will use in the rest of the paper.

2.1. **Rigidity in** $G(k,n)$**.** The basic facts concerning rigidity and multi rigidity of Schubert cycles in $G(k,n)$ proved in [C1], [Ho2] and [RT] can be summarized in the following theorem.

Theorem 2.1. *Let $\sigma_{a_\bullet}$ be a Schubert class in $G(k,n)$. Let $\alpha_\bullet$ be the associated sequence defined by setting $\alpha_i = a_i - i$ and express it as $(\tilde{\alpha}_1^{i_1}, \dots, \tilde{\alpha}_t^{i_t})$ by grouping the equal parts.*

1. *By* [C1, Theorem 1.6]*, the class $\sigma_{a_\bullet}$ is not smoothable if there exists an index $1 \leq u < t$ such that $i_u > 1$ and $0 < \tilde{\alpha}_u$, or an index $1 \leq u < t$ such that $0 < \tilde{\alpha}_u < \tilde{\alpha}_{u+1} - 1$.*
2. *By* [C1, Theorem 1.3]*, the class $\sigma_{a_\bullet}$ is rigid if and only if there does not exist an index $1 \leq u < t$ such that $i_u = 1$ and $0 < \tilde{\alpha}_u = \tilde{\alpha}_{u+1} - 1$.*
3. *By* [Ho2] *and* [RT, §8.1 and Remark 6.3]*, the class $\sigma_{a_\bullet}$ is multi rigid if and only if $i_u \geq 2$ for every $1 < u < t$, $\tilde{\alpha}_u \leq \tilde{\alpha}_{u+1} - 2$ for every $1 \leq u \leq t-1$, $i_1 \geq 2$ unless $\tilde{\alpha}_1 = 0$ and $i_t \geq 2$ unless $\tilde{\alpha}_t = n-k$.*

Proof. Part (1) is [C1, Theorem 1.6] and Part (2) is [C1, Theorem 1.3].

Hong, Robles and The index Schubert classes by partitions $(p_1^{q_1}, \dots, p_r^{q_r})$, where the parts p_i are positive, bounded above by $n-k$ and strictly decreasing. The part p_i occurs with multiplicity q_i. The parts that are equal to zero are not recorded in their notation. There is a duality isomorphism τ between the Grassmannians $G(k,n)$ and $G(n-k,n)$ that takes Schubert varieties to Schubert varieties. When the partitions are depicted by Young diagrams, τ takes a Schubert variety with class depicted by a Young diagram Y to a Schubert variety with class depicted by the transpose of Y. Hong, Robles and The denote the transposed partition by $(p_1'^{q_1'}, \dots, p_r'^{q_r'})$. Then Hong's Theorem [Ho2], with an improvement by Robles and The [RT, §8.1 and Remark 6.3], asserts that a Schubert class in $G(k,n)$ is Schur rigid (hence, multi rigid by [B, §2.8.1], [RT, Theorem 7.1] or [W]) if $q_i, q_i' \geq 2$ for $i \geq 2$ and one of the four mutually exclusive and exhaustive conditions holds:

1. If $p_1 = n-k$ and $p_1' = k$, no further conditions are necessary.
2. If $p_1 = n-k$ and $p_1' < k$, then $q_1' \geq 2$.
3. If $p_1 < n-k$ and $p_1' = k$, then $q_1 \geq 2$.
4. If $p_1 < n-k$ and $p_1' < k$, then $q_1, q_1' \geq 2$.

Translating this statement to our notation, the condition $q_i \geq 2$ for $i \geq 2$ means that $i_u \geq 2$ for $1 < u < t$ and $i_t \geq 2$ if $\tilde{\alpha}_t < n-k$. The two conditions $q_i' \geq 2$ for $i \geq 2$ and $q_1' \geq 2$ if $p_1' < k$ translate to $\tilde{\alpha}_{i+1} - \tilde{\alpha}_i \geq 2$ for $1 \leq i < t$. In addition, the condition $q_1 \geq 2$ if $p_1 < n-k$ translates to the condition $i_1 \geq 2$ if $\tilde{\alpha}_1 > 0$. We conclude that the class $\sigma_{a_\bullet}$ is multi rigid if $i_u \geq 2$ for every $1 < u < t$, $\tilde{\alpha}_u \leq \tilde{\alpha}_{u+1} - 2$ for every $1 \leq u \leq t-1$, $i_1 \geq 2$ unless $\tilde{\alpha}_1 = 0$ and $i_t \geq 2$ unless $\tilde{\alpha}_t = n-k$.

It is easy to see that the results of Hong, Robles and The are sharp (see also [R]). To prove the converse, we have to show that if there exists an index $1 \leq u \leq t$ such that $i_u = 1$ and $\tilde{\alpha}_u \neq 0, n-k$ or an index $1 \leq u < t$ such that $\tilde{\alpha}_{u+1} - \tilde{\alpha}_u = 1$, then $\sigma_{a_\bullet}$ is not multi rigid. In fact, by the duality between $G(k,n)$ and $G(n-k,n)$, it suffices to show that if there exists an index $1 \leq u \leq t$ such that $i_u = 1$ and $\tilde{\alpha}_u \neq 0, n-k$, then $\sigma_{a_\bullet}$ is not multi rigid. First, assume that $i_t = 1$ and $\tilde{\alpha}_t < n-k$. Consider an irreducible quadric $Q_{a_k+1}^{a_k-2}$ in F_{a_k+1} of rank 3 whose singular locus is F_{a_k-2}. Such a quadric exists, since we are assuming that $\tilde{\alpha}_t < n-k$, hence $a_k + 1 \leq n$.

Let Z be the Zariski closure of the locus Z^0 of linear spaces W contained in $Q_{a_k+1}^{a_k-2}$ such that $\dim(W \cap F_{a_i}) = i$ for $1 \leq i < k$. Then Z is an irreducible variety whose cohomology class is $2\sigma_{a_\bullet}$. To prove the irreducibility, note that Z^0 admits a morphism $f : Z^0 \to G(k-1, F_{a_{k-1}})$ defined by sending W to $W \cap F_{a_{k-1}}$. The image is a dense Zariski open subset in the Schubert variety $\Sigma_{a_1,\ldots,a_{k-1}}$, hence is irreducible. The fiber of f over a $(k-1)$-dimensional vector space Λ is a rank three quadric in F_{a_k+1}/Λ. Hence, the fibers are all irreducible of the same dimension. By the Theorem on the Dimension of Fibers [S, Theorem I.6.8], we conclude that Z is irreducible. By intersecting with complementary dimensional Schubert cycles, it is immediate to see that the cohomology class of Z is $2\sigma_{a_\bullet}$.

Now assume that $1 \leq u < t$, $i_u = 1$ and $\tilde{\alpha}_u \neq 0$. Then we can use a similar construction to find an irreducible variety representing $2\sigma_{a_\bullet}$. Let $h = i_1 + \cdots + i_u$. Let $Q_{a_h+1}^{a_h-2}$ be a quadric of rank 3 in F_{a_h+1} singular along F_{a_h-2}. Let Y be the Zariski closure of the locus Y^0 of linear spaces W such that $\dim(W \cap F_{a_i}) = i$, for $1 \leq i \leq k$ and $i \neq h$, and W intersects $Q_{a_h+1}^{a_h-2}$ in an h-dimensional linear space. Then Y is irreducible and has class $2\sigma_{a_\bullet}$. Let Z^0 be the locus of linear spaces W' in $G(h, a_h+1)$ defined by requiring $\dim(W' \cap F_{a_i}) = i$ for $i < h$ and W' is contained in $Q_{a_h+1}^{a_h-2}$. By the previous paragraph, Z^0 is irreducible. The quasi-projective variety Y^0 admits a morphism $f : Y^0 \to Z^0$, where the fiber over a linear space Λ is isomorphic to a Zariski open subset in a Schubert variety in $G(k-h, V/\Lambda)$. Hence, Y is irreducible. As in the previous case, by intersecting with complementary dimensional Schubert classes, it is clear that the cohomology class of Y is $2\sigma_{a_\bullet}$. Therefore, $\sigma_{a_\bullet}$ is not multi rigid. This completes the proof of Part (3). □

Remark 2.2. The proof of Theorem 2.1 shows that when $\sigma_{a_\bullet}$ is not multi rigid in $G(k,n)$, then $2\sigma_{a_\bullet}$ can be represented by an irreducible subvariety. Furthermore, this subvariety is contained in $G(k, a_k)$ (when $i_u = 1$ with $u < t$ and $\tilde{\alpha}_u \neq 0$ or $\tilde{\alpha}_{u+1} - \tilde{\alpha}_u = 1$ with $1 \leq u < t$) or in the Fano variety of k-planes contained in a quadric of rank 3 in F_{a_k+1} (when $i_t = 1$ and $\tilde{\alpha}_t \neq n-k$). For the cases with $i_u = 1$, this is immediate by the construction in the proof. For the cases with $\tilde{\alpha}_{u+1} - \tilde{\alpha}_u = 1$ with $1 \leq u < t$, this follows easily by duality. The duality between $G(k,n)$ and $G(n-k,n)$ exchanges Schubert classes of the two types. If $a_k = n$, there is nothing to prove. The image in $G(k,n)$ of the variety constructed in $G(n-k,n)$ represents $2\sigma_{a_\bullet}$. If $a_k < n$, the construction applied to the dual Schubert class in $G(n-k,n)$ does not alter the first $n-a_k$ flag elements of dimensions $1, \ldots, n-a_k$, respectively. Hence, the image in $G(k,n)$ of the variety constructed in the proof under the duality is contained in $G(k, a_k)$.

2.2. **Maps between Grassmannians.** There are natural maps between Grassmannians and orthogonal Grassmannians. We will utilize these maps throughout the paper. First, since isotropic subspaces are in particular subspaces, the orthogonal Grassmannian $OG(k,n)$ naturally embeds in the Grassmannian $G(k,n)$ by inclusion

$$i : OG(k,n) \hookrightarrow G(k,n).$$

If $\Sigma_{a_\bullet}$ is a Schubert variety of Grassmannian type in $OG(k,n)$, then $i(\Sigma_{a_\bullet})$ is a Schubert variety in $G(k,n)$ with cohomology class $\sigma_{a_\bullet}$.

Let $k \leq m \leq \lfloor \frac{n}{2} \rfloor$ be an integer. Let W be an m-dimensional isotropic subspace of V. Then $G(k,W)$ naturally includes in $OG(k,n)$

$$\phi_{m,n} : G(k,W) \hookrightarrow OG(k,n).$$

Under this inclusion, a Schubert variety $\Sigma_{a_\bullet}$ in $G(k, W)$ maps to a Schubert variety in $OG(k, n)$. The Schubert variety in $OG(k, n)$ is of Grassmannian type provided that W is not maximal when n is even.

Let W be an m-dimensional vector space. Let $0 \leq s < k$ and $0 \leq t$ be two integers such that $s + t \leq m$. Let W' and W'' be two subspaces of W of dimensions s and t, respectively, such that $W' \cap W'' = 0$. Then there is a natural embedding

$$\zeta_{(k-s,t),(k,m)} : G(k - s, t) \hookrightarrow G(k, m)$$

by sending a linear space $\Lambda \in G(k - s, W'')$ to $Span(\Lambda, W') \in G(k, W)$. Under this map, a Schubert variety $\Sigma_{a_\bullet}$ in $G(k - s, t)$ embeds as a Schubert variety $\Sigma_{a'_\bullet}$ in $G(k, W)$, where $a'_i = i$ for $i \leq s$ and $a'_i = s + a_{i-s}$ for $s < i \leq k$.

3. Restriction varieties

A restriction variety is a subvariety of $OG(k, n)$ defined by rank conditions with respect to a flag that is not necessarily isotropic. Restriction varieties were introduced in [C2] to compute the map induced in cohomology via the inclusion $i : OG(k, n) \hookrightarrow G(k, n)$. In this section, we will recall the definition of restriction varieties and the algorithm for computing their cohomology classes. We refer the reader to [C2] for proofs and more details. Restriction varieties will allow us to exhibit explicit deformations of multiples of Schubert classes in orthogonal Grassmannians, thereby proving that these classes are not multi rigid.

Let Q be a non-degenerate, quadratic form on an n-dimensional vector space V. We denote an isotropic linear space of dimension n_j by L_{n_j}. In case $2n_j = n$, L_{n_j} and L'_{n_j} denote isotropic linear spaces in different connected components. Let $Q^{r_i}_{d_i}$ denote a sub-quadric of corank r_i obtained by restricting Q to a d_i-dimensional linear space. We denote the singular locus of $Q^{r_i}_{d_i}$ by $\text{Sing}(Q^{r_i}_{d_i})$. For convenience, we set $r_0 = 0$ and $d_0 = n$.

Definition 3.1. A *sequence* of linear spaces and quadrics $(L_\bullet, Q_\bullet)$ associated to $OG(k, n)$ is a totally ordered set

$$L_{n_1} \subsetneq L_{n_2} \subsetneq \cdots \subsetneq L_{n_s} \subsetneq Q^{r_{k-s}}_{d_{k-s}} \subsetneq \cdots \subsetneq Q^{r_1}_{d_1}$$

of isotropic linear spaces L_{n_j} (or possibly L'_{n_s} in case $2n_s = n$) and sub-quadrics $Q^{r_i}_{d_i}$ of Q such that

(1) $\text{Sing}(Q^{r_{i-1}}_{d_{i-1}}) \subseteq \text{Sing}(Q^{r_i}_{d_i})$ for every $1 < i \leq k - s$.
(2) $\dim(L_{n_j} \cap \text{Sing}(Q^{r_i}_{d_i})) = \min(n_j, r_i)$.
(3) Either $r_i = r_1 = n_{r_1}$ or $r_t - r_i \geq t - i - 1$ for every $t > i$. Moreover, if $r_t = r_{t-1} > r_1$ for some t, then $d_i - d_{i+1} = r_{i+1} - r_i$ for every $i \geq t$ and $d_{t-1} - d_t = 1$.

Remark 3.2. The first two conditions in the definition of a sequence mean that the singular loci of the quadrics are in as special a position as possible. The singular locus of a quadric in the sequence contains the singular locus of any larger dimensional quadric in the sequence. A linear space in the sequence is contained in (respectively, contains) the singular locus of a quadric in the sequence whose corank exceeds (respectively, is less than or equal to) the dimension of the linear space. The third condition is a technical condition which is included for accuracy but is not used in this paper. The reader can safely ignore it.

Remark 3.3. Naturally, the dimensions n_j, d_i, r_i must satisfy certain inequalities. Since the largest dimensional isotropic subspace of a non-degenerate quadratic form has at most half the dimension of the ambient space, we must have the inequality

$$2n_s \leq d_{k-s} + r_{k-s}.$$

Since the corank of a linear section of a smooth quadric is bounded by its codimension, we must have the inequality

$$r_i + d_i \leq r_{i-1} + d_{i-1} \leq n \text{ for every } 1 \leq i \leq k-s.$$

We will always assume that the invariants of a sequence satisfy these inequalities.

Definition 3.4. A sequence $(L_\bullet, Q_\bullet)$ associated to $OG(k,n)$ is *admissible* if the linear spaces and quadrics satisfy the following three properties:

(A1) $r_{k-s} \leq d_{k-s} - 3$.

(A2) For any $1 \leq j \leq s$, there does not exist $1 \leq i \leq k-s$ such that $n_j - r_i = 1$.

(A3) Let $x_i = \#\{j \mid n_j \leq r_i\}$. For every $1 \leq i \leq k-s$,

$$x_i \geq k - i + 1 - \lfloor \frac{d_i - r_i}{2} \rfloor.$$

Definition 3.5. Let $(L_\bullet, Q_\bullet)$ be an admissible sequence for $OG(k,n)$. A *restriction variety* $V(L_\bullet, Q_\bullet)$ is the subvariety of $OG(k,n)$ defined as the Zariski closure of the quasi-projective variety $V(L_\bullet, Q_\bullet)^0$ defined by

$$\{W \in OG(k,n) \mid \dim(W \cap L_{n_j}) = j, \dim(W \cap Q_{d_i}^{r_i}) = k-i+1, \dim(W \cap \mathrm{Sing}(Q_{d_i}^{r_i})) = x_i\}.$$

Example 3.6. When $d_i + r_i = n$ for every $1 \leq i \leq k-s$ in the admissible sequence $(L_\bullet, Q_\bullet)$, the corresponding restriction variety is the Schubert variety $\Sigma_{n_1,\dots,n_s;r_{k-s},\dots,r_1}$ in $OG(k,n)$. Moreover, every Schubert variety occurs this way.

Remark 3.7. The three conditions in the definition of an admissible sequence can be explained as follows. If $r_{k-s} > d_{k-s} - 3$, then the smallest dimensional quadric $Q_{d_{k-s}}^{r_{k-s}}$ in the sequence is reducible or non-reduced. Hence, $Q_{d_{k-s}}^{r_{k-s}}$ in the sequence can be replaced by an isotropic linear space to more accurately reflect the geometry. If $n_j = r_i + 1$, then the $(k-i+1)$-dimensional subspace contained in $Q_{d_i}^{r_i}$ either must intersect the singular locus of $Q_{d_i}^{r_i}$ in a j-dimensional subspace or it must be contained in the quadric everywhere singular along L_{n_j}. Hence, one of these possibilities better reflects the geometry. Finally, the definition of a restriction variety requires that the k-dimensional isotropic spaces intersect $Q_{d_i}^{r_i}$ in a subspace of dimension $k-i+1$. A linear space of dimension $k-i+1$ intersects the singular locus of $Q_{d_i}^{r_i}$ in a subspace of dimension at least $k-i+1-\lfloor \frac{d_i-r_i}{2} \rfloor$. Since the linear space is required to intersect the singular locus in a subspace of dimension x_i, the inequality in Condition (A3) must be satisfied.

Quadric diagrams provide a convenient combinatorial shorthand for depicting admissible sequences.

Notation 3.8. The *string of numbers* associated to a sequence $(L_\bullet, Q_\bullet)$ is a sequence of r_{k-s} non-decreasing positive integers followed by $n - r_{k-s}$ zeros such that $r_i - r_{i-1}$ of the positive integers are equal to i. The *quadric diagram* associated to the sequence $(L_\bullet, Q_\bullet)$ consists of the string of numbers associated to $(L_\bullet, Q_\bullet)$, where the n_j-th number in the string is followed by a bracket] and the d_i-th digit is followed by a brace }. If $2n_s = n$ and the sequence contains L'_{n_s}, the bracket after the n_s-th digit is decorated with a prime]′. The quadric diagram is called *admissible* if it is associated to an admissible sequence. We always order the brackets from left to right and the braces from right to left to conform with the indexing in the sequence $(L_\bullet, Q_\bullet)$.

Hence, the i-th brace will always refer to the i-th brace from the right and the j-th bracket will refer to the j-th bracket from the left.

Example 3.9. The quadric diagram associated to the sequence $L_1 \subset L_3 \subset L_5 \subset Q_8^3 \subset Q_{10}^1$ in $OG(5, 11)$ is 1]22]00]000}00}0. In this sequence, there are two quadrics of corank 1 and 3. Hence, the string of numbers is 122 followed by 8 zeros. The corank of the i-th quadric is equal to the number of positive integers in the string of numbers that are less than or equal to i. The dimension of the j-th isotropic linear space in the sequence is equal to the number of digits to the left of the j-th bracket. In this case, there are 1,3 and 5 digits to the left of the brackets. Hence, the isotropic subspaces in the sequence are L_1, L_3, L_5. Finally, the dimension of the span of the i-th quadric is equal to the number of digits to the left of the i-th brace. In this case, the first brace has 10 digits to its left. Hence, the largest dimensional quadric is Q_{10}^1. The second brace has 8 digits to its left. Hence, the second quadric is Q_8^3.

Remark 3.10. It is useful to reformulate Definition 3.4 in terms of quadric diagrams. Condition (A1) means that the leftmost brace must have at least three zeros to its left. For example, 11]00]00}00 or 33000}00}00}0 are admissible, but 1100}00 is not. Condition (A2) means that the two digits to the left of a bracket have to be equal. If there is only one digit to the left of a bracket, it has to be 1. For example, 1]22]33]0000}00}0}0 or 22]22]2000}00000}0 are admissible, but 2]22]000}000}0 or 1234]0000}0}0}0}0 are not admissible. Condition (A3) is the hardest to visually verify without resorting to some counting. The quantity $k - i + 1 - x_i$ is the number of brackets and braces to the left of the i-th brace (including the i-th brace) that have a zero or an integer larger than i to their left. The quantity $d_i - r_i$ is the total number of zeros and integers larger than i to the left of the i-th brace. Condition (A3) requires the latter number to be at least twice the former. For example, 10]00}0 violates the condition since there are three zeros to the right of the 1 and to the left of } but a total of two brackets and braces.

We are now ready to recall the algorithm that computes the class of a restriction variety in terms of quadric diagrams. Given a sequence $(L_\bullet, Q_\bullet)$, let $D(L_\bullet, Q_\bullet)$ be the diagram associated to the sequence. Given an admissible quadric diagram D, we associate two new quadric diagrams D^a and D^b. If these are not admissible, we modify them until they are admissible. We replace the diagram D with the collection of diagrams that result from this process. Geometrically, this process encodes a degeneration of the original restriction variety into a union of restriction varieties.

Definition 3.11. Let D be the quadric diagram associated to an admissible sequence $(L_\bullet, Q_\bullet)$. If there exists an index i such that $r_i - r_{i-1} < d_{i-1} - d_i$[1] in the sequence, let

$$\kappa = \max(i \mid r_i - r_{i-1} < d_{i-1} - d_i).^2$$

Let D^a be the quadric diagram obtained by changing the $(r_\kappa + 1)$-st integer in the string of numbers of D to κ. Denote this rightmost κ in D^a by κ_0. If there exists a bracket in D^a to the right of κ_0, let D^b be the quadric diagram obtained from D^a by moving the leftmost bracket to the right of κ_0 to the position immediately to the right of κ_0.

Example 3.12. To clarify, let us give some examples. Let $D =$ 233]0000}00}0}0. Then $\kappa = 1$. We change the 2 to 1 to obtain $D^a =$ 133]0000}00}0}0. We slide the first bracket in D^a to the right of the 1 we added to the immediate right of it to obtain $D^b =$ 1]330000}00}0}0. Note that in this case both D^a and D^b are admissible.

[1]In terms of the corresponding diagram, $r_i - r_{i-1}$ is equal to the number of digits that are equal to i and $d_{i-1} - d_i$ is equal to the number of integers between the i-th and $(i-1)$-st braces.

[2]In terms of the diagram, κ is the largest index i such that the number of digits equal to i is less than the number of digits between the i-th and $(i-1)$-st braces.

Example 3.13. Next let $D = 00]0]0000\}0$. Here $\kappa = 1$, so we turn the left most 0 into 1 to obtain $D^a = 10]0]0000\}0$. We slide the first bracket to the right of the 1 to its immediate right to obtain $D^b = 1]00]0000\}0$. Here note that D^b is admissible, but D^a fails Condition (A2). We have to turn D^a into an admissible quadric diagram.

Algorithm 3.14. [C2, Algorithm 3.8] We now state the algorithm for modifying D^a until it satisfies all the conditions in Definition 3.4.

Step 1. If D^a fails Condition (A3), discard it. D^a does not lead to any quadric diagrams.

Step 2. If D^a satisfies Condition (A3) but not Condition (A2), change the digit to the right of κ_0 in the sequence to κ and move the κ-th brace one position to the left. Repeat until you reach a sequence of brackets and braces that satisfies Condition (A2). Label the resulting sequence D^c. If D^c is admissible, we refer to it as an admissible quadric diagram derived from D^a. Otherwise, proceed to the next step.

Step 3. If D^a or D^c satisfies Conditions (A2) and (A3), but fails Condition (A1), replace D^a or D^c with two identical diagrams D^{a_1} and D^{a_2} obtained by replacing the leftmost brace (in D^a or D^c) with a bracket one position to the left and turning the digits equal to $k-s$ to 0. Let n_s be the number of digits to the left of the rightmost brace in D^{a_2}. If $2n_s = n$, then we use $]'$ instead of $]$ in D^{a_2}. We refer to D^{a_1} and D^{a_2} as quadric diagrams derived from D^a.

In Example 3.13, we first turn $D^a = 10]0]0000\}0$ into $11]0]000\}00$. This diagram still fails Condition (A2), so we repeat to obtain $11]1]00\}000$. Now condition (A2) is satisfied, but Condition (A1) fails. Since $n = 8 = 2 \cdot 4$, we obtain the two diagrams $00]0]0]0000$ and $00]0]0]'0000$. These are the two diagrams derived from D^a.

Example 3.15. Let $D = 000\}000\}000\}$, then $\kappa = 3$. We turn the left most 0 into 3 to obtain $D^a = 300\}000\}000\}$. In this case, there are no brackets to the left of the 3, so there is no D^b. The sequence D^a fails Condition (A1). Since n is odd, we replace D^a with two identical quadric diagrams $D^{a_1} = 00]0000\}000\}$ and $D^{a_2} = 00]0000\}000\}$.

Example 3.16. Let $D = 00]0000\}00\}0$. Then $D^a = 20]0000\}00\}0$ and $D^b = 2]00000\}00\}0$. Neither of these diagrams satisfy Condition (A2). We already know that we should replace D^a with $22]000\}000\}0$. Here is how to modify D^b.

Algorithm 3.17. [C2, Algorithm 3.9] If D^b does not satisfy Condition (A2), suppose it fails for the j-th bracket. Let i be the integer immediately to the left of the j-th bracket. Replace this i with $i-1$ and move $(i-1)$-st brace one position to the left. As long as the resulting sequence does not satisfy Condition (A2), repeat this process either until the resulting sequence is admissible (in which case this is the quadric diagram derived from D^b) or two braces occupy the same position. In the latter case, no quadric diagrams are derived from D^b.

In Example 3.16, we replace $D^b = 2]00000\}00\}0$ with $1]00000\}0\}00$, which is a quadric diagram. If our example had been $D = 00]0000\}0\}0$, then $D^b = 2]00000\}0\}0$. Replacing 2 with 1 and moving the rightmost brace to the left would produce $1]00000\}\}00$. Hence, in this case no quadric diagrams are derived from D^b.

We need one more definition. Assume $n_s > r_\kappa$ (i.e., there exists a bracket strictly to the right of the rightmost κ). If there exists an index i such that $r_i \geq n_{x_\kappa+1}$, let $y_\kappa = \max\{i \mid r_i \leq n_{x_\kappa+1}\}$. Otherwise (i.e., if $r_i < n_{x_\kappa+1}$ for every i), set $y_\kappa = k-s+1$. The integer y_κ is the positive integer that occurs immediately to the left of the first bracket that has an integer larger than κ to its left or $y_\kappa = k-s+1$ if this bracket is preceded by a zero. The condition $n_{x_\kappa+1} - r_\kappa - 1 = y_\kappa - \kappa$ plays an important role. A sequence satisfying this equality looks like $\cdots\kappa{+}1\ \kappa{+}2\ \cdots\kappa{+}l{-}1\ \kappa{+}l\ \kappa{+}l]\cdots$ or $\cdots\kappa+1\ \kappa+2\ \cdots\kappa+l-1\ 00]\cdots$, where we have drawn the part of the sequence starting with the left most $\kappa+1$ and ending with $(x_\kappa+1)$-st bracket. We are now ready to state the algorithm.

Algorithm 3.18. [C2, Algorithm 3.10] Let D be an admissible quadric diagram associated to the admissible sequence $(L_\bullet, Q_\bullet)$. If $r_i + d_i = n$ for every $1 \leq i \leq k-s$, then return D and stop. Otherwise, let D^a and D^b be the diagrams described above.

(1) If $n_{x_\kappa+1} - r_\kappa - 1 > y_\kappa - \kappa$ or $n_s \leq r_\kappa$ in D, then return the quadric diagrams that are derived from D^a.
(2) If D^a violates Condition (A3) in Definition 3.4, then return the quadric diagrams that are derived from D^b.
(3) Otherwise, return the quadric diagrams that are derived from both D^a and D^b.

We then have the following theorem that calculates the cohomology class of $V(L_\bullet, Q_\bullet)$.

Theorem 3.19. [C2, Theorem 5.12] *Let $V(L_\bullet, Q_\bullet)$ be a restriction variety. Then the cohomology class of $V(L_\bullet, Q_\bullet)$ satisfies*

$$[V(L_\bullet, Q_\bullet)] = \sum_h [V(L^h_\bullet, Q^h_\bullet)],$$

where the sum ranges over the restriction varieties corresponding to the admissible quadric diagrams derived from $D(L_\bullet, Q_\bullet)$ by Algorithm 3.18.

For our purposes, it is important to note that in the first two cases of Algorithm 3.18 (with the exception when $n = 2k$ and a quadric is replaced by two maximal dimensional linear spaces), a restriction variety is replaced by one restriction variety or two restriction varieties with the same cohomology class.

4. Classes of Grassmannian type or quadric type

In this section, we discuss the rigidity and multi rigidity of Schubert classes of Grassmannian type or quadric type in $OG(k,n)$ and prove Theorem 1.4 and Theorem 1.5.

Proposition 4.1. *Let $i : OG(k,n) \to G(k,n)$ denote the natural inclusion. Let $\Sigma_{a_\bullet}$ be a Schubert variety in $OG(k,n)$ of Grassmannian type with cohomology class $\sigma_{a_\bullet}$. The cohomology class of the Schubert variety $i(\Sigma_{a_\bullet})$ in the cohomology of $G(k,n)$ is also $\sigma_{a_\bullet}$.*

(1) *The class $\sigma_{a_\bullet}$ is rigid in $G(k,n)$ if and only if the class $\sigma_{a_\bullet}$ is rigid in $OG(k,n)$.*
(2) *The class $\sigma_{a_\bullet}$ is multi rigid in $G(k,n)$ if and only if the class $\sigma_{a_\bullet}$ is multi rigid for $OG(k,n)$.*

Proof. Since $\Sigma_{a_\bullet}$ is of Grassmannian type, it parameterizes isotropic linear spaces W such that $\dim(W \cap F_{a_i}) \geq i$ for some isotropic subspaces $F_{a_1} \subset \cdots \subset F_{a_k}$, where if n is even F_{a_k} is not maximal. If an arbitrary linear space W satisfies $\dim(W \cap F_{a_i}) \geq i$, then W is contained in the isotropic subspace F_{a_k}. Therefore, W is automatically isotropic. It follows that $i(\Sigma_{a_\bullet})$ is a Schubert variety in $G(k,n)$ with cohomology class $\sigma_{a_\bullet}$.

Let $\sigma_{a_\bullet}$ be rigid in $G(k,n)$. Let Y be a variety that represents $\sigma_{a_\bullet}$ in $OG(k,n)$. Then $i(Y)$ represents $\sigma_{a_\bullet}$ in $G(k,n)$. Since $\sigma_{a_\bullet}$ is rigid in $G(k,n)$, $i(Y)$ is a Schubert variety in $G(k,n)$. Since the Schubert variety $i(Y)$ in $G(k,n)$ is contained in $i(OG(k,n))$, Y must be a Schubert variety of Grassmannian type in $OG(k,n)$. If the largest linear space F_{a_k} in the definition of the Schubert variety $i(Y)$ is not isotropic, then there exists a linear subspace W of dimension

k contained in F_{a_k} such that $\dim(W \cap F_{a_i}) = i$ and W is not isotropic. This contradicts that $i(Y)$ is contained in $i(OG(k,n))$. Hence, Y is a Schubert variety in $OG(k,n)$ of Grassmannian type. We conclude that $\sigma_{a_\bullet}$ is rigid in $OG(k,n)$.

Conversely, suppose that $\sigma_{a_\bullet}$ is not rigid in $G(k,n)$. The proof of [C1, Theorem 1.3] constructs a deformation of the Schubert variety $\Sigma_{a_\bullet}$ in $G(k,a_k)$ to a variety Y which is not a Schubert variety. Embedding Y via the map $\phi_{a_k,n} : G(k,a_k) \hookrightarrow OG(k,n)$, we see that $\phi_{a_k,n}(Y)$ is a variety that represents $\sigma_{a_\bullet}$ but is not a Schubert variety. We conclude that $\sigma_{a_\bullet}$ is not rigid in $OG(k,n)$.

The argument for multi rigidity is almost identical. Let $\sigma_{a_\bullet}$ be multi rigid in $G(k,n)$. Suppose $m\sigma_{a_\bullet}$ can be represented by a subvariety Y in $OG(k,n)$. Then $i(Y)$ has cohomology class $m\sigma_{a_\bullet}$ in $G(k,n)$, hence it is a union of m Schubert varieties. Applying the argument in the second paragraph of the proof to each component of $i(Y)$, we conclude that each component of Y must be a Schubert variety of Grassmannian type in $OG(k,n)$. Hence, $\sigma_{a_\bullet}$ is multi rigid in $OG(k,n)$.

Conversely, if $\sigma_{a_\bullet}$ is not multi rigid in $G(k,n)$, the proof of Theorem 2.1 Part (3) constructs an irreducible subvariety Y that represents the class $2\sigma_{a_\bullet}$. Except when $i_t = 1$, the variety Y is defined inside $G(k,a_k)$ (see Remark 2.2). When $i_t = 1$ is the only condition violating multi rigidity, the deformation is defined in the Fano variety of k-planes of a quadric hypersurface of rank 3 in F_{a_k+1}. In particular, we can represent $2\sigma_{a_\bullet}$ by an irreducible variety Y in $G(k,a_k+1)$. If n is even, then by the definition of Schubert cycles of Grassmannian type, F_{a_k} is not maximal. Composing the variety Y with $\phi_{a_k+1,n}$, which embeds F_{a_k+1} into a maximal isotropic subspace, we obtain an irreducible variety representing $2\sigma_{a_\bullet}$ in $OG(k,n)$. Hence, $\sigma_{a_\bullet}$ is not multi rigid.

If n is odd, then the same argument applies when $i_t \neq 1$. In that case, the proof of Theorem 2.1 Part (3) constructs an irreducible variety Y in $G(k,a_k)$ representing $2\sigma_{a_\bullet}$. Composing Y with $\phi_{a_k,n}$, we obtain a variety in $OG(k,n)$ representing $2\sigma_{a_\bullet}$. When $i_t = 1$, the proof of Theorem 2.1 Part (3) constructs an irreducible subvariety Y in the Fano variety of a rank 3 quadric in F_{a_k+1}. By taking this quadric to be a rank 3 quadric contained in Q, we see that Y embeds in $OG(k,n)$ as the restriction variety associated to the sequence $L_{a_1} \subset \cdots L_{a_{k-1}} \subset Q^{a_k-2}_{a_k+1}$. It is easy to check that the conditions in Definitions 3.1 and 3.4 are satisfied for this sequence. The conditions in Definition 3.1 are obvious. We have that $d_1 = a_k + 1$ and $r_1 = a_k - 2$, hence $d_1 = r_1 + 3$ and Condition (A1) holds. Since $i_t = 1$, we have that $a_{k-1} < a_k - 1$. Hence, $a_k - 2 \geq a_j$ for every $1 \leq j \leq k-1$ and Condition (A2) holds. Finally, $x_1 = k - 1 > k - \frac{3}{2}$. Hence, Condition (A3) holds. By [C2, Theorem 5.12] (see Theorem 3.19), the class of this restriction variety is $2\sigma_{a_\bullet}$. To see this, note that when we run the algorithm with $\kappa = 1$, only D^a is formed. The quadric diagram D^a is not admissible since it violates Condition (A1) (note that the conditions (A2) and (A3) hold). Hence, the algorithm replaces D^a with two identical sequences associated to the Schubert variety $\sigma_{a_\bullet}$. We conclude that $\sigma_{a_\bullet}$ is not multi rigid in $OG(k,n)$. □

Remark 4.2. When n is even, if we consider Schubert cycles $\sigma_{a_\bullet}$ in $OG(k,n)$ that have $a_k = \frac{n}{2}$, then Part (1) of Proposition 4.1 remains true with the same proof. However, Part (2) of Proposition 4.1 may be false. Part (2) remains true with the same proof for cycles with $a_k = \frac{n}{2}$ provided that $i_t \geq 2$. However, when $i_t = 1$ and $a_k = \frac{n}{2}$, a cycle can be multi rigid in $OG(k,n)$, but fail to be multi rigid in $G(k,n)$. For example, in $OG(1,n)$ a maximal isotropic subspace is multi rigid. However, a half-dimensional linear space is not multi rigid in $G(1,n)$. This is the main reason to disallow maximal isotropic subspaces in the definition of Schubert cycles of Grassmannian type when n is even.

Proof of Theorem 1.4. Let $\sigma_{a_\bullet}$ be a Schubert class of Grassmannian type in $OG(k,n)$. By Proposition 4.1, $\sigma_{a_\bullet}$ is rigid in $OG(k,n)$ if and only if it is rigid in $G(k,n)$. By [C1, Theorem 1.3] (see Theorem 2.1 Part (2)), $\sigma_{a_\bullet}$ is rigid in $G(k,n)$ if and only if there does not exist an index $1 \leq u < t$ such that $i_u = 1$ and $0 < \tilde{\alpha}_u = \tilde{\alpha}_{u+1} - 1$. Part (1) of the theorem follows.

By Proposition 4.1, $\sigma_{a_\bullet}$ is multi rigid in $OG(k,n)$ if and only if it is multi rigid in $G(k,n)$. Since $\sigma_{a_\bullet}$ is of Grassmannian type, we must have that $\tilde{\alpha}_t < n-k$. By Hong's Theorem [Ho2] (see Theorem 2.1 Part (3)), $\sigma_{a_\bullet}$ is multi rigid in $G(k,n)$ if and only if $i_u \geq 2$ for every $1 < u \leq t$, $\tilde{\alpha}_u \leq \tilde{\alpha}_{u+1} - 2$, and $i_1 \geq 2$ unless $\tilde{\alpha}_1 = 0$. Part (2) of the theorem follows.

Suppose a Schubert class $\sigma_{a_\bullet}$ is smoothable in $OG(k,n)$. Let Y be a smooth subvariety of $OG(k,n)$ representing $\sigma_{a_\bullet}$. Then $i(Y)$ is a smooth subvariety of $G(k,n)$ representing $\sigma_{a_\bullet}$. By [C1, Theorem 1.6] (see Theorem 2.1 Part (1)), $\sigma_{a_\bullet}$ is not smoothable in $G(k,n)$ if there exists an index $1 \leq u < t$ such that $i_u > 1$ and $\tilde{\alpha}_u > 0$ or an index $1 \leq u < t$ such that $0 < \tilde{\alpha}_u < \tilde{\alpha}_{u+1} - 1$. We conclude that $\sigma_{a_\bullet}$ is not smoothable in $OG(k,n)$ under the same assumptions. Part (3) of the theorem follows. ☐

Remark 4.3. When n is even, if we consider Schubert cycles $\sigma_{a_\bullet}$ where $a_k = \frac{n}{2}$, Parts (1) and (3) of Theorem 1.4 remain true with the same proof but using Remark 4.2 instead of Proposition 4.1. Part (2) of Theorem 1.4 need not be true. However, if in addition we assume that $i_t \geq 2$, then Part (2) also holds.

Proof of Theorem 1.5. Let $\sigma_{b_\bullet}$ be a Schubert class of quadric type. We then have $b_1 < \frac{n}{2} - 1$. Consequently, the dimension of $F_{b_1}^{\perp}$ is greater than $\frac{n}{2} + 1$. Since the corank of a quadric is bounded by its codimension, for a quadric $Q^{r_1}_{n-b_1}$, we have that $r_1 \leq b_1 \leq n - b_1 - 3$. In particular, $Q^{r_1}_{n-b_1}$ is irreducible. Let V be the restriction variety defined by the sequence

$$Q^0_{n-b_1} \subset Q^0_{n-b_2} \subset \cdots \subset Q^0_{n-b_k}.$$

The i-th quadric in this sequence has the same dimension as the i-th quadric in the sequence defining a Schubert variety $\Sigma_{b_\bullet}$ but is smooth. It is easy to check that this sequence satisfies the conditions in Definitions 3.1 and 3.4. Since none of the quadrics are singular, the conditions in Definition 3.1 vacuously hold. Since there does not exist any isotropic subspaces in the sequence, Condition (A2) holds. The inequalities in conditions (A1) and (A3) improve when r_i decreases. Since these conditions hold for the sequence corresponding to the Schubert variety with class $\sigma_{b_\bullet}$, we conclude that they also hold for this sequence. By Algorithm [C2, Algorithm 3.10] (see Algorithm 3.18), the class of this restriction variety is the Schubert class $\sigma_{b_\bullet}$. The quadric diagram representing this class has the form

$$00\cdots00\}0\cdots0\}0\cdots0\}0\cdots0.$$

When we apply the algorithm to compute its class, the only quadric diagram that is formed is D^a. Since there are no brackets, D^a automatically satisfies Condition (A2) of Definition 3.4. Since the rank of the smallest dimensional quadric is at least three (recall that $Q^{r_1}_{n-b_1}$ satisfies $r_1 \leq n - b_1 - 3$), D^a also automatically satisfies Condition (A1). As we already observed, Condition (A3) holds since it already holds for the sequence associated to the Schubert variety. Hence, D^a is admissible and no new brackets are formed while running the algorithm. Consequently, when we run the algorithm, we never form D^b. By Theorem [C2, Theorem 5.12] (see Theorem 3.19), we conclude that the cohomology class of this restriction variety is the Schubert class $\sigma_{b_\bullet}$.

Note that the exception $t = 1$ and $\tilde{\beta}_1 = n - k$ (equivalently, $t = 1$ and $b_k = 0$) in the statement of the theorem corresponds to the fundamental class of $OG(k,n)$. In all other cases, we now show that the restriction variety constructed in the previous paragraph gives a non-trivial deformation of the Schubert variety. The linear spaces parameterized by V sweep out the quadric $Q^0_{n-b_k}$. Hence, if $b_k \neq 0$, the restriction variety cannot be projectively equivalent to a Schubert variety since for a Schubert variety the linear spaces sweep out a quadric of corank b_k. If $b_k = 0$, since $t \neq 1$, there exists b_u such that $b_u > k - u$. Let v be $\max\{u | b_u > k - u\}$. The smallest dimensional quadric that contains a v-dimensional subspace of every linear space parameterized by a Schubert variety $\Sigma_{b_\bullet}$ has corank b_v. In the restriction variety this quadric has the same dimension and has full rank. Therefore, we conclude that the restriction variety cannot be projectively equivalent to a Schubert variety. This concludes the proof that unless $t = 1$ and $b_k = 0$, a Schubert cycle of quadric type is not rigid. In fact, we have proved that such a class can always be represented by the intersection of a general Schubert variety in $G(k, n)$ with the orthogonal Grassmannian $OG(k, n)$ [C2, Example 4.13 and Proposition 6.2].

Suppose that a Schubert class $\sigma_{b_\bullet}$ can be represented by a smooth subvariety Y of $OG(k, n)$. If n is odd, let $\sigma_{a_\bullet}$ be the Schubert class of Grassmannian type defined by the sequence $a_i = \frac{n-1}{2} + i - k$. Notice that a Schubert variety representing such a class consists of k-dimensional subspaces contained in a fixed maximal isotropic subspace. In particular, such a Schubert variety is smooth and isomorphic to $G(k, \frac{n-1}{2})$. Similarly, if n is even, let $\sigma_{a_\bullet}$ be the Schubert class defined by the sequence $a_i = \frac{n}{2} + i - k$. A Schubert variety representing this class is also smooth and isomorphic to $G(k, \frac{n}{2})$.

In either case, by Kleiman's Transversality Theorem [K2], the intersection of Y with a general translate of a Schubert variety $\Sigma_{a_\bullet}$ is a smooth subvariety of $OG(k, n)$. The class of this intersection in $G(k, \lfloor \frac{n}{2} \rfloor)$ is given by $\sigma_{c_\bullet}$, where $c_i = \lfloor \frac{n}{2} \rfloor - b_i$. We thus conclude that the Schubert class $\sigma_{c_\bullet}$ is smoothable in the Grassmannian $G(k, \lfloor \frac{n}{2} \rfloor)$. Set $\gamma_i = c_i - i = \beta_i - \lfloor \frac{n+1}{2} \rfloor$ and express the sequence $\gamma_\bullet$ by grouping the equal parts as $(\tilde{\gamma}_1^{i_1}, \dots, \tilde{\gamma}_t^{i_t})$. If there is an index $1 \leq u < t$ such that $i_u > 1$ and $\tilde{\beta}_u > \lfloor \frac{n+1}{2} \rfloor$, then $i_u > 1$ and $\tilde{\gamma}_u > 0$. Similarly, if there exists an index $1 \leq u < t$ such that $\lfloor \frac{n+1}{2} \rfloor < \tilde{\beta}_u < \tilde{\beta}_{u+1} - 1$, then $0 < \tilde{\gamma}_u < \tilde{\gamma}_{u+1} - 1$. By [C1, Theorem 1.6] (see Theorem 2.1 Part (1)), $\sigma_{c_\bullet}$ cannot be represented by a smooth subvariety of $G(k, \lfloor \frac{n}{2} \rfloor)$ leading to a contradiction. This concludes the proof of Part (2) of Theorem 1.5. ☐

The proof yields the following corollary.

Corollary 4.4. *Let $\sigma_{b_\bullet}$ be a Schubert class of quadric type in $OG(k, n)$. If $t = 1$, then $\sigma_{b_\bullet}$ is smoothable.*

Proof. When $t = 1$, then $b_i = b_{i+1} + 1$ for $1 \leq i < k$. Hence, the restriction variety associated to the sequence $Q^0_{n-b_1} \subset Q^0_{n-b_2} \subset \cdots \subset Q^0_{n-b_k}$ is the orthogonal Grassmannian $OG(k, n - b_k)$. In particular, this restriction variety is smooth. Since the cohomology class of this restriction variety is the Schubert class $\sigma_{b_\bullet}$, we conclude that $\sigma_{b_\bullet}$ is smoothable. ☐

In fact, we conclude the following stronger corollary.

Corollary 4.5. *Let $\sigma_{b_\bullet}$ be a Schubert class of quadric type in $OG(k, n)$. Let $a_\bullet$ be the sequence defined by $a_i = n - b_i$. If the Schubert class $\sigma_{a_\bullet}$ is smoothable in $G(k, n)$, then $\sigma_{b_\bullet}$ is smoothable in $OG(k, n)$. In particular, if $b_i = b_{i+1} + 1$ for $i \geq 2$ and $b_1 - b_2 = 2$, then $\sigma_{b_\bullet}$ is smoothable.*

Proof. Note that the restriction variety representing $\sigma_{b_\bullet}$ constructed in the proof of Theorem 1.5 is $\Sigma_{a_\bullet} \cap OG(k, n)$, where $\Sigma_{a_\bullet}$ is a general Schubert variety with class $\sigma_{a_\bullet}$ in $G(k, n)$ [C2, Example 4.13 and Proposition 6.2]. If Y is a smooth subvariety representing $\sigma_{a_\bullet}$, then $\sigma_{b_\bullet}$ can also be represented by $Y \cap OG(k, n)$, possibly after replacing Y with a general translate so that the intersection is dimensionally proper. By Kleiman's Transversality Theorem [K2], $Y \cap OG(k, n)$

may be taken to be smooth, possibly after replacing Y with a general translate. Hence, $\sigma_{b_\bullet}$ is smoothable. In particular, if $b_i = b_{i+1} + 1$ for $i \geq 2$ and $b_1 - b_2 = 2$, then $\sigma_{a_\bullet}$ is the hyperplane class in $G(k, n)$. Hence, by Bertini's Theorem [H, Theorem III.10.9], $\sigma_{a_\bullet}$ is smoothable. We conclude in this case that $\sigma_{b_\bullet}$ is smoothable. □

5. Non-rigidity of Schubert classes

In this section, using restriction varieties, we get explicit deformations of multiples of Schubert classes in $OG(k, n)$ and prove Theorem 1.7 and Theorem 1.8.

Proof of Theorem 1.7. Let $\sigma_{a_\bullet, b_\bullet}$ be a Schubert class in the cohomology of $OG(k, n)$. First, assume that $\tilde{\beta}_1 < n - k$ and $b_{s+i_1} \neq a_j$ for any $1 \leq j \leq s$. We will explicitly construct a restriction variety that has class $\sigma_{a_\bullet; b_\bullet}$ but is not a Schubert variety. This will show that $\sigma_{a_\bullet, b_\bullet}$ is not rigid.

By assumption, we have that $b_{s+i_1} = b_{s+i_1-1} - 1 = b_{s+i_1-2} - 2 = \cdots = b_{s+1} - i_1 + 1$. Since for a Schubert variety $b_i \neq a_j - 1$ for any $1 \leq j \leq s$, we conclude that either $a_j < b_{s+i_1}$ or $a_j > b_{s+1} + 1$ for every $1 \leq j \leq s$. In particular, if we represent the Schubert variety with class $\sigma_{a_\bullet; b_\bullet}$ by a quadric diagram, then the diagram contains the string of numbers

$$\cdots \_\, k - s - i_1 + 1 \quad k - s - i_1 + 2 \ \cdots \ k - s - 1 \quad k - s \ 0 \ 0 \,\_ \cdots$$

and the diagram does not have any brackets in this range except possibly at the two underlined places. Consider the restriction variety V defined by the following sequence

$$L_{a_1} \subset \cdots \subset L_{a_s} \subset Q^{b_{s+1}-1}_{n-b_{s+1}} \subset Q^{b_{s+2}-1}_{n-b_{s+2}} \subset \cdots \subset Q^{b_{s+i_1}-1}_{n-b_{s+i_1}} \subset \cdots \subset Q^{b_k}_{n-b_k}.$$

Notice that this sequence differs from the sequence defining the Schubert variety with class $\sigma_{a_\bullet; b_\bullet}$ only in that the ranks of the quadrics $Q^{b_{s+i_1}-1}_{n-b_{s+i_1}}, \ldots, Q^{b_{s+i_1}-1}_{n-b_{s+i_1}}$ are one more than the corresponding quadrics in the sequence associated to the Schubert variety. In particular, the quadric diagram associated to V contains a string of numbers

$$\cdots \_\, k - s - i_1 + 2 \quad k - s - i_1 + 3 \ \cdots \ k - s \quad 0 \ 0 \ 0 \,\_ \cdots$$

and does not contain any brackets except possibly at the two underlined places. The conditions in Definitions 3.1 obviously hold for this sequence. Conditions (A1) and (A3) of Definition 3.4 hold since they hold for the sequence corresponding to the Schubert variety and the inequalities can only improve if r_i is decreased keeping all other quantities the same. The inequalities $a_j < b_{s+i_1}$ or $a_j > b_{s+1} + 1$ guarantee that Condition (A2) holds. We conclude that the sequence is admissible.

By Theorem [C2, Theorem 5.12] (see Theorem 3.19), the cohomology class of V is $\sigma_{a_\bullet; b_\bullet}$. To calculate the class of V, we run the algorithm successively for κ equal to $k - s - i_1 + 1, k - s - i_1 + 2, \ldots, k - s$. At each stage of the algorithm, we have that $n_{x_\kappa + 1} - r_\kappa + 1 > y_\kappa - \kappa$. Therefore, the algorithm returns only quadric diagrams derived from D^a. On the other hand, D^a is always admissible. Since Conditions (A1) and (A3) hold for the sequence corresponding to the Schubert variety, they also hold for all the intermediate sequences. Condition (A2) holds since $a_j < b_{s+i_1}$ or $a_j > b_{s+1} + 1$ for every $1 \leq j \leq s$. It follows that the cohomology class of V is $\sigma_{a_\bullet; b_\bullet}$. However, V is not isomorphic to a Schubert variety. The smallest dimensional linear space that contains an $(s + i_1)$-dimensional subspace of every k dimensional subspace parameterized by the Schubert variety has corank b_{i_1}. This corank for V is $b_{i_1} - 1$. We conclude that $\sigma_{a_\bullet; b_\bullet}$ is not rigid.

Next, assume that there exists an index $1 \leq u < t$ such that $j_u = 1$, $\tilde{\alpha}_u = \tilde{\alpha}_{u+1} - 1$ and $a_{j_1+\cdots+j_u} \neq b_i$ for any $s < i \leq k$. For simplicity, set $h = \sum_{i=1}^{u} j_i$. If we represent the Schubert class $\sigma_{a_\bullet;b_\bullet}$ by a quadric diagram, then the diagram contains $\cdots\gamma\gamma]\gamma\gamma]\cdots$, where $0 \leq \gamma \leq k-s$ and the brackets depicted here are those of index h and $h+1$. By the proof of [C1, Theorem 1.3], there exists a subvariety Y of $G(h+1, F_{a_{h+1}})$ with cohomology class $\sigma_{a_1,\ldots,a_h,a_{h+1}}$ that parameterizes $(h+1)$-dimensional subspaces $\Lambda \subset F_{a_{h+1}}$ that satisfy $\dim(\Lambda \cap F_{a_i}) \geq i$ for $i < h$ but is not a Schubert variety. Let Z be the Zariski closure of the following quasi-projective variety

$$\{W \in OG(k,n) \mid W \cap F_{a_{h+1}} \in Y, \dim(W \cap F_{a_j}) = j, \text{for } j \neq h, \dim(W \cap F_{b_i}^{\perp}) = i\}.$$

Then the class of Z is $\sigma_{a_\bullet;b_\bullet}$ since specializing Y to a Schubert variety specializes Z to a Schubert variety. Furthermore, Z is not a Schubert variety. Therefore, the class $\sigma_{a_\bullet;b_\bullet}$ is not rigid.

Finally, assume that for the sequence $(a_\bullet; b_\bullet)$, $\#\{j|a_j \leq b_{s+1}\} = s + b_{s+1} - \frac{n-3}{2}$ and $b_{s+1} = a_h$. Consider the restriction variety V associated to the sequence

$$L_{a_1} \subset \cdots \subset L_{a_{h-1}} \subset L_{a_h+1} \subset L_{a_{h+1}} \subset \cdots \subset Q^{b_{s+1}-1}_{n-b_{s+1}} \subset \cdots \subset Q^{b_k}_{n-b_k}.$$

Notice that this sequence differs from the sequence defining the Schubert variety in that the dimension of the h-th isotropic linear space is one larger and the corank of the smallest dimensional quadric is one smaller. We claim that this is an admissible sequence. As usual, the conditions in Definition 3.1 are clear. Condition (A1) of Definition 3.4 holds because increasing the rank of the smallest dimensional quadric only improves the inequality in Condition (A1). Condition (A2) needs to be checked only for L_{a_h+1} and $Q^{b_{s+1}-1}_{n-b_{s+1}}$. The ranks of all other quadrics and the dimensions of all other isotropic linear spaces remain unchanged. Since $a_h = b_{s+1}$, $a_h + 1 = b_{s+1} - 1 + 2$ and Condition (A2) holds. Finally, we have to check Condition (A3) for the smallest dimensional quadric. By assumption, for the sequence representing the Schubert variety, we have that $x'_{k-s} = s + b_{s+1} - \frac{n-3}{2}$. For the new sequence, we have that $x_{k-s} = x'_{k-s} - 1$. Since the corank of the smallest dimensional quadric is also one less, we see that the inequality in Condition (A3) holds. We conclude that this sequence is admissible.

This restriction variety V has cohomology class $\sigma_{a_\bullet;b_\bullet}$. To compute the class, we run the algorithm with $\kappa = k - s$. In this case, D^a violates Condition (A3) of Definition 3.4. Hence, the only diagrams are derived from D^b which is admissible (being the diagram associated to the Schubert variety with class $\sigma_{a_\bullet;b_\bullet}$). Therefore, the cohomology class of V is $\sigma_{a_\bullet;b_\bullet}$. As in the previous cases, it is clear that this restriction variety is not a Schubert variety. Therefore, $\sigma_{a_\bullet,b_\bullet}$ is not rigid. This concludes the proof of the theorem. ☐

Example 5.1. To make the proof of Theorem 1.7 more concrete, we give several examples. The quadric diagram associated to the Schubert class $\sigma_{1;3,1}$ in $OG(3,9)$ is 1]22000}00}0. The proof of Part (1) shows that the restriction variety associated to the sequence 1]20000}00}0 has the same class. Similarly, the quadric diagram associated to the Schubert class $\sigma_{2;5,4,3,0}$ in $OG(5,13)$ is 22]234000}0}0}000}. The restriction variety associated to the quadric diagram 22]340000}0}0}000} has the same cohomology class.

Example 1.6 gives an example of Part (3) of Theorem 1.7. The quadric diagram associated to the Schubert class $\sigma_{1;1}$ in $OG(2,5)$ is 1]000}0. The restriction variety associated to the sequence 00]00}0 also has the same class. More generally, the quadric diagram associated to the Schubert class $\sigma_{1,3,5;3,1}$ in $OG(5,11)$ is 1]22]00]000}00}0. The restriction variety associated to the sequence 1]200]0]000}00}0 has the same class.

Proof of Theorem 1.8. Since a multi rigid class is rigid, any Schubert class $\sigma_{a_\bullet;b_\bullet}$ satisfying one of the conditions of Theorem 1.7 cannot be multi rigid. To prove the theorem, we have to show that a Schubert cycle satisfying one of the two conditions in the theorem is not multi rigid.

Let $\sigma_{a_\bullet;b_\bullet}$ be a Schubert cycle. First, assume that there exists an index $1 \leq u \leq t$ such that either $i_u = 1$ and $0 < \tilde{\alpha}_u < \frac{n}{2}$, or an index $1 \leq u \leq t$ such that $\tilde{\alpha}_u = \tilde{\alpha}_{u+1} - 1$. Set $h = j_1 + \cdots + j_u$ and further assume that $a_h \neq b_i$ for any $1 \leq i \leq s$. Then, by Theorem 1.4, there exists an irreducible subvariety Y representing twice the Schubert class of Grassmannian type $2\sigma_{a_\bullet}$ in $OG(s,n)$. First, assume that either n is even or $i_t \neq 1$. Let Z be the Zariski closure of the following quasi-projective variety

$$\{W \in OG(k,n) \mid (W \cap F_{a_s}) \in Y, \dim(W \cap F_{b_i}^{\perp}) = i\}.$$

Then it is easy to see that Z is an irreducible variety representing $2\sigma_{a_\bullet;b_\bullet}$. When n is odd and $i_t = 1$, the definition of Z has to be slightly altered. Let Y be the variety constructed in the proof of Theorem 1.4 representing the class $2\sigma_{a_\bullet}$. Let Z be the closure of the quasi-projective variety

$$\{W \in OG(k,n) \mid (W \cap Q_{a_s+1}^{a_s-2}) \in Y, \dim(W \cap F_{b_i}^{\perp}) = i\}.$$

It is left to the reader to check that Z is an irreducible variety representing $2\sigma_{a_\bullet;b_\bullet}$.

Next, assume that $a_{s-1}+1 < a_s < \frac{n}{2}$ and $a_s > b_{s+1}$. Then we can construct an irreducible restriction variety that represents the class $2\sigma_{a_\bullet;b_\bullet}$. Let V be the restriction variety defined by the following sequence

$$L_{a_1} \subset \cdots \subset L_{a_{s-1}} \subset Q_{a_s+1}^{a_s-2} \subset Q_{n-b_{s+1}}^{b_{s+1}} \subset \cdots \subset Q_{n-b_k}^{b_k}.$$

Note that this sequence differs from the sequence defining the Schubert variety $\Sigma_{a_\bullet;b_\bullet}$ only in having the isotropic subspace L_{a_s} replaced by the quadric $Q_{a_s+1}^{a_s-2}$. We leave the easy verification that this sequence is admissible to the reader. By [C2, Proposition 4.16], V is irreducible. By [C1, Theorem 5.12], the cohomology class of V is $2\sigma_{a_\bullet;b_\bullet}$. Running the algorithm, we have $\kappa = k - s + 1$. Since there are no brackets to the left of κ_0 in D^a, only D^a leads to new quadric diagrams. Note that D^a is not admissible. It satisfies Conditions (A2) and (A3), but fails Condition (A1). Therefore, the algorithm replaces D^a by two (identical since $a_s < \frac{n}{2}$) quadric diagrams associated to the Schubert class $\sigma_{a_\bullet;b_\bullet}$. We conclude that $\sigma_{a_\bullet;b_\bullet}$ is not multi rigid.

Finally, assume that $a_{s-1}+1 < a_s < \frac{n}{2}$, $a_s = b_j$ and $b_j = b_{j-1}+2 = b_{s+1}+j-s-1$. Consider the following sequence of linear spaces and quadrics

$$L_{a_1} \subset \cdots L_{a_{s-1}} \subset Q_{a_s+1}^{a_s-2} \subset Q_{n-b_{j+1}-j+s-1}^{a_s-2} \subset \cdots \subset Q_{n-b_{j+1}-1}^{a_s-2} \subset Q_{n-b_{j+1}}^{b_{j+1}} \subset \cdots \subset Q_{n-b_k}^{b_k}$$

As usual let $x_i = \#\{a_j \mid a_j \leq r_i\}$. Let V be the Zariski closure of the locus in $OG(k,n)$ defined by

$$\{W \in OG(k,n) \mid \dim(W \cap L_{a_j}) = j, \dim(W \cap Q_{d_i}^{r_i}) = k-i+1, \dim(W \cap Sing(Q_{d_i}^{r_i})) = x_i\}.$$

Using the Theorem on the Dimension of Fibers [S, Theorem I.6.8], it is easy to check that V is irreducible. The cohomology class of V is $2\sigma_{a_\bullet;b_\bullet}$. Degenerating $Q_{a_s+1}^{a_s-2}$ into a union of two isotropic subspaces of dimension a_s, we see that the cohomology class of V is twice the cohomology class of the variety defined with respect to

$$L_{a_1} \subset \cdots L_{a_{s-1}} \subset L_{a_s} \subset Q_{n-b_{j+1}-j+s-1}^{a_s-2} \subset \cdots \subset Q_{n-b_{j+1}-1}^{a_s-2} \subset Q_{n-b_{j+1}}^{b_{j+1}} \subset \cdots \subset Q_{n-b_k}^{b_k}.$$

By the running the algorithm, the reader can check that the latter variety is homologous (in fact, equal) to a Schubert variety. We conclude that $\sigma_{a_\bullet;b_\bullet}$ is not multi rigid. This concludes the proof of the theorem. $\square$

Remark 5.2. Combining the proofs of Part (2) of Theorem 1.7 and Part (2) of Theorem 1.8, we conclude the following variation on Part (2) of Theorem 1.7. The class $\sigma_{a_\bullet;b_\bullet}$ is not rigid if there exists an index $1 \leq u < t$ such that $j_u = 1$, $0 < \tilde{\alpha}_u = \tilde{\alpha}_{u+1} - 1$, and if $b_i = a_{j_1+\cdots+j_u}$, then $b_{h-1} - b_h = x_{h-1} - x_h + 1$ for every $h > i$.

6. Rigidity of Schubert classes

In this section, we prove Theorem 1.10. We begin by proving a variant of [C1, Lemma 4.1], which says that if a Schubert variety $\Sigma_{a_\bullet;b_\bullet}$ does not admit any non-trivial small deformations, then the Schubert class $\sigma_{a_\bullet;b_\bullet}$ is rigid.

Lemma 6.1. *If Y represents a Schubert class $\sigma_{a_\bullet;b_\bullet}$ in $OG(k,n)$, then there exists a flat deformation $\pi : \mathcal{Y} \to B$ over a smooth curve B and a point $p_0 \in B$ such that $\pi^{-1}(p)$ is isomorphic to Y for $p_0 \neq p \in B$ and $\pi^{-1}(p_0)$ is isomorphic to a Schubert variety $\Sigma_{a_\bullet;b_\bullet}$.*

Proof. The Grassmannian $OG(k,n)$ admits a Bruhat decomposition into affine cells such that the Zariski closure of each of these cells is a Schubert variety [Bo, IV.14.12]. If Y is a subvariety of $OG(k,n)$ representing the Schubert class $\sigma_{a_\bullet;b_\bullet}$, then by the action of a one-parameter subgroup $\psi : \mathbb{C}^* \to SO(n)$, Y can be projected onto the Schubert variety $\Sigma_{a_\bullet;b_\bullet}$ in the decomposition. One can see this as follows. Let U_1 be the smallest dimensional affine stratum whose closure contains Y. If the closure of U_1 is equal to Y, then Y is already a Schubert variety and there is nothing to prove. Otherwise, pick a point p in U_1 not contained in Y. Since U_1 is isomorphic to affine space, we can project Y away from p. Projection from a point is given by a one-parameter subgroup. As $t \to \infty$, the flat limit of Y under the projection is contained in the boundary of the cell U_1. Since the class of Y is a Schubert cycle which is an indecomposable class, the flat limit is irreducible and contained in the closure of a smaller dimensional cell U_2. If the closure of U_2 is not equal to the projection of Y, we can repeat the process. By induction on the dimension of the cells U_i, eventually the projection of Y is equal to the closure of one of the cells. Since the process does not change the cohomology class of the variety, we conclude that the projection of Y is equal to the Schubert variety with the same cohomology class. Finally, by composing these projections, we obtain a one-parameter subgroup that projects Y onto the Schubert variety with the same class.

Let $\mathcal{Y}^* \to \mathbb{C}^*$ be the family of varieties $\psi(t)Y$. The varieties $\psi(t)Y$ are all isomorphic, hence this family is flat over $\mathbb{C}^*$. By the properness of the Hilbert scheme [H, Proposition III.9.8], the flat limit exists over $t = 0$. Let $\psi : \mathcal{Y} \to \mathbb{C}$ be the induced family. Since the Schubert class is indecomposable, the support of the flat limit has to be irreducible and supported on the Schubert variety $\Sigma_{a_\bullet;b_\bullet}$. Both Y and the Schubert variety have the same cohomology class, hence the central fiber $\mathcal{Y}_0$ is generically reduced. Since the Schubert variety is normal [Br], by Hironaka's Theorem [Hi], [Ko, Theorem 2], we conclude that the central fiber is reduced and the limit is the Schubert variety. Therefore, we can let $\psi : \mathcal{Y} \to \mathbb{C}$ be the family whose existence is claimed in the lemma. This concludes the proof. □

Lemma 6.2. *Let $\sigma_{a_\bullet;b_\bullet}$ be a Schubert class with $s < k$. Let Y be a variety representing the class $\sigma_{a_\bullet;b_\bullet}$ in $OG(k,n)$. Let Z be the variety swept out by the projective linear spaces $\mathbb{P}^{k-1}$ in $\mathbb{P}^{n-1}$ parameterized by Y. Then Z is a quadric of dimension $n - b_k - 2$.*

Proof. The dimension and the degree of Z are determined by the cohomology class of Y. Let $i : OG(k,n) \to G(k,n)$ be the natural inclusion. Let S_l be a general Schubert variety in $G(k,n)$ parameterizing k-dimensional linear spaces that intersect a vector space of dimension l. Let $[X]$ denote the cohomology class of X. By definition, the projective linear spaces parameterized by a Schubert variety $\Sigma_{a_\bullet;b_\bullet}$ are contained in $\mathbb{P}F_{b_k}^\perp \cap Q$ which is a quadric of dimension $n-b_k-2$ in $\mathbb{P}^{n-1}$. Hence, in the cohomology of $G(k,n)$, $[Y] \cup [S_{b_k+1}] = 0$. We conclude that Z is disjoint from a general linear space $\mathbb{P}^{b_k}$ in $\mathbb{P}^{n-1}$. It follows that the dimension of Z is at most $n-b_k-2$. By semi-continuity, it is clear that the dimension is at least $n-b_k-2$. Hence, we conclude that the dimension of Z is $n-b_k-2$.

The degree of Z can be similarly computed. Suppose a general linear space $\Lambda \cong \mathbb{P}^{b_k+1}$ intersects Z in d points $p_1, \dots, p_d$. We may assume that no isotropic linear space contains two of these points. Consider the locus of isotropic subspaces parameterized by Y that intersect Λ. Then, by assumption, this locus has at least d irreducible components depending on the choice of p_i. On the other hand, in the cohomology of $OG(k,n)$, $[S_{b_k+2} \cap OG(k,n)] \cup \sigma_{a_\bullet;b_\bullet} = 2\sigma_{a'_\bullet,b'_\bullet}$, where $a'_1 = 1$, $a'_{i+1} = a_i + 1$ if $a_i \leq b_k$ and $a'_{i+1} = a_i$ if $a_i > b_k$ and $b'_j = b_{j-1}$ for $s < j \leq k$. To see this observe that the linear space defining S_{b_k+2} intersects the quadric $Q^{b_k}_{n-b_k} = F_{b_k}^\perp \cap Q$ in two points p and q. Any isotropic linear space in the intersection $S_{b_k+2} \cap \Sigma_{a_\bullet;b_\bullet}$ has to contain one of the points p and q and satisfy the rank conditions imposed by $\Sigma_{a_\bullet;b_\bullet}$. If we specify one of the points p or q, it is immediate to turn these conditions into Schubert conditions given by $(a'_\bullet, b'_\bullet)$. Let v_p be the isotropic vector corresponding to p. An isotropic linear space containing v_p must have $a'_1 = 1$ and must be contained in $T_p Q^{b_k}_{n-b_k}$. The linear space $T_p Q^{b_k}_{n-b_k}$ contains L_{a_i} if $a_i \leq b_k$ and intersects L_{a_i} for $a_i > b_k$ and $F_{b_j}^\perp$ for $j < k$ in codimension one linear spaces. Hence, any isotropic linear space in the intersection $S_{b_k+2} \cap \Sigma_{a_\bullet;b_\bullet}$ must satisfy Schubert conditions $a'_{i+1} = a_i + 1$ if $a_i \leq b_k$ (with respect to the span of L_{a_i} and v_p) and $a'_{i+1} = a_i$ if $a_i > b_k$ (with respect to the span of $L_{a_i} \cap T_p Q^{b_k}_{n-b_k}$ and v_p) and $b'_j = b_{j-1}$ (with respect to the span of v_p and $F^\perp_{b_{j-1}} \cap T_p Q^{b_k}_{n-b_k}$). The fact that any isotropic linear space satisfying these Schubert conditions is contained in $S_{b_k+2} \cap \Sigma_{a_\bullet;b_\bullet}$ is a tautology. Since Schubert classes are indecomposable, we conclude that the intersection of Y with S_{b_k+2} can have at most two components. Hence, d is at most 2. Since by Lemma 6.1, Y degenerates to a Schubert variety, the degree of Z is at least two. We conclude that the degree of Z is two. This concludes the proof of the lemma. $\square$

The proof of Theorem 1.10 will be by induction. In view of Lemma 6.1, the next proposition allows us to prove the rigidity of base cases.

Proposition 6.3. *Let $r \geq 1$ and let Q be a smooth quadric hypersurface of dimension at least 2. Then the cone over the Segre embedding of $\mathbb{P}^r \times Q$ does not admit any small deformations.*

Proof. We will first show that the Segre embedding of $\mathbb{P}^r \times Q$ is not a hyperplane section of a projective variety other than a cone. By [L, Corollary 1(b)], it suffices to show that $H^1(\mathbb{P}^r \times Q, T_{\mathbb{P}^r \times Q}(-1)) = 0$, where $T_{\mathbb{P}^r \times Q}$ denotes the tangent bundle of $\mathbb{P}^r \times Q$. Let p and q denote the two projections $p : \mathbb{P}^r \times Q \to \mathbb{P}^r$ and $q : \mathbb{P}^r \times Q \to Q$. Then, by [H, II.8.Ex.3], $T_{\mathbb{P}^r \times Q} \cong p^* T_{\mathbb{P}^r} \oplus q^* T_Q$, where $T_{\mathbb{P}^r}$ and T_Q denote the tangent bundles of $\mathbb{P}^r$ and Q, respectively. We conclude that

$$H^1(\mathbb{P}^r \times Q, T_{\mathbb{P}^r \times Q}(-1)) \cong H^1(\mathbb{P}^r \times Q, (p^* T_{\mathbb{P}^r})(-1)) \oplus H^1(\mathbb{P}^r \times Q, (q^* T_Q)(-1)).$$

By the Euler sequence, we have

$$0 \to \mathcal{O}_{\mathbb{P}^r \times Q}(-1,-1) \to \bigoplus_{i=1}^{n+1} \mathcal{O}_{\mathbb{P}^r \times Q}(0,-1) \to (p^* T_{\mathbb{P}^r})(-1) \to 0.$$

By the Kodaira Vanishing Theorem, $H^1(\mathbb{P}^r \times Q, \mathcal{O}_{\mathbb{P}^r \times Q}(0,-1)) = H^2(\mathbb{P}^r \times Q, \mathcal{O}_{\mathbb{P}^r \times Q}(-1,-1)) = 0$. Therefore, $H^1(\mathbb{P}^r \times Q, (p^* T_{\mathbb{P}^r})(-1)) = 0$.

Similarly, let $i : Q \to \mathbb{P}^t$ be the inclusion of Q in the projective space spanned by Q. Then the standard exact sequence for the normal bundle yields the exact sequence

$$0 \to (q^*T_Q)(-1) \to (q^*(i^*T_{\mathbb{P}^t}))(-1) \to \mathcal{O}_{\mathbb{P}^r \times Q}(-1,1) \to 0.$$

Using the Euler sequence for $T_{\mathbb{P}^t}$ and the Kodaira Vanishing Theorem, we conclude that $H^1(\mathbb{P}^r \times Q, (q^*T_Q)(-1)) = 0$. Therefore, $H^1(\mathbb{P}^r \times Q, T_{\mathbb{P}^r \times Q}(-1)) = 0$. Consequently, $\mathbb{P}^r \times Q$ cannot occur as the hyperplane section of a projective variety other than a cone.

To conclude the proof of the proposition, suppose that the cone over the Segre embedding of $\mathbb{P}^r \times Q$ admits a small deformation to a projective variety X. Then a hyperplane section of X is a deformation of $\mathbb{P}^r \times Q$. An analogous but easier cohomology calculation shows that $H^1(\mathbb{P}^r \times Q, T_{\mathbb{P}^r \times Q}) = 0$. Therefore, a small deformation of $\mathbb{P}^r \times Q$ is isomorphic to $\mathbb{P}^r \times Q$. Since the Picard group of $\mathbb{P}^r \times Q$ is discrete, the hyperplane section of X is the Segre embedding of $\mathbb{P}^r \times Q$. By the previous paragraph, we conclude that X itself must be the cone over the Segre embedding of $\mathbb{P}^r \times Q$. This concludes the proof. Note that the proposition would be false if $r = 0$ or if Q has dimension 1. ☐

Corollary 6.4. *Let $\sigma_{a_\bullet;b_\bullet}$ be a Schubert class in $OG(k,n)$ such that $s = k-1 > 0$, $n > 2k+3$ and $a_{k-1} = b_k = k$. Then $\sigma_{a_\bullet;b_\bullet}$ is rigid.*

Proof. In the sequence $a_\bullet$, suppose $a_i = i$ for $1 \leq i < h$ and $a_i = i+1$ for $h \leq i \leq k-1$. First, observe that in the minimal embedding of $OG(k,n)$, the Schubert variety $\Sigma_{a_\bullet;b_\bullet}$ is the cone over the Segre embedding of $\mathbb{P}^{k-h} \times Q^0_{n-2k}$. Every k-plane W parameterized by the Schubert variety is required to intersect the isotropic plane $F_{a_{k-1}=k}$ in a $(k-1)$-dimensional subspace W'. The line consisting of k-dimensional subspaces containing W' and contained in the span of W and F_k is contained in the Schubert variety. Hence, the Schubert variety is a cone with vertex at the point corresponding to F_k. The intersection of the Schubert variety with a general codimension one Schubert variety (defined with respect to a linear space Λ_{n-k-1} of dimension $n-k-1$) parameterizes isotropic k planes that are spanned by a $(k-1)$-dimensional linear space W', which is contained in F_k and satisfies $\dim(W' \cap F_{a_i}) \geq i$, and by an isotropic vector in $F_k^\perp \cap \Lambda_{n-k-1}$, which is disjoint from F_{a_k}. Hence, this intersection is isomorphic to $\mathbb{P}^{k-h} \times Q^0_{n-2k}$. Since the Schubert varieties are projectively normal [Br], in the minimal embedding of $OG(k,n)$, $\Sigma_{a_\bullet;b_\bullet}$ is embedded as a cone over the Segre embedding of $\mathbb{P}^{k-h} \times Q^0_{n-2k}$.

Since $k-1 > 0$ and $n > 2k+3$, by Proposition 6.3, a small deformation of the cone over the Segre embedding of $\mathbb{P}^{k-h} \times Q^0_{n-2k}$ is again a cone over the Segre embedding of $\mathbb{P}^{k-h} \times Q^0_{n-2k}$. By Lemma 6.1, we conclude that any variety Y representing the Schubert class $\sigma_{a_\bullet;b_\bullet}$ has to be a cone over $\mathbb{P}^{k-h} \times Q^0_{n-2k}$. The vertex of the cone determines a k-dimensional isotropic subspace F_k. By Theorem 1.4, a line in $OG(k,n)$ parameterizes linear spaces that contain a fixed $(k-1)$-dimensional isotropic space and are contained in a fixed $(k+1)$-dimensional isotropic space. Since Y is a cone with vertex at the point corresponding to F_k, we conclude that every linear space parameterized by Y must intersect F_k in a subspace of dimension at least $k-1$. Furthermore, the span of F_k with any linear space parameterized by Y must be isotropic. By Lemma 6.2, the linear spaces sweep out a quadric of codimension k. Hence, every linear space parameterized by Y must be contained in $F_k^\perp$. Finally, to conclude that Y is a Schubert variety, notice that for W parameterized by Y, the map $W \mapsto W \cap F_k$ defines a morphism to $OG(k-1,n)$ away from the cone point of Y. It is easy to see that this morphism is the projection from the cone point of Y followed by the first projection on $\mathbb{P}^{k-h} \times Q^0_{n-2k}$. Since by Theorem 1.4 the Schubert class $\sigma_{a_\bullet}$ is rigid in $OG(k-1,n)$, we conclude that Y must be defined by requiring

$W \subset F_k^{\perp}$ and $\dim(W \cap F_{a_i}) \geq i$ for $1 \leq i \leq k-1$ for a set of isotropic subspaces F_{a_i}. Hence, Y is a Schubert variety. We conclude that $\sigma_{a_\bullet;b_\bullet}$ is rigid. □

The next proposition gives some additional base cases.

Proposition 6.5. *Let $\sigma_{a_\bullet;b_\bullet}$ be a Schubert cycle in $OG(k,n)$ such that $s = k-1$, $a_{k-1} = b_k = k-1$ and $n > 2k+1$. Then $\sigma_{a_\bullet,b_\bullet}$ is rigid.*

Proof. Let $i : OG(k,n) \to G(k,n)$ be the natural inclusion. Then, the image of the Schubert variety $i(\Sigma_{a_\bullet;b_\bullet})$ has cohomology class $2\sigma_{1,2,\dots,k-1,n-k}$. In the Plücker embedding of $G(k,n)$, $i(\Sigma_{a_\bullet;b_\bullet})$ is a smooth quadric R of dimension $n-2k$. Since a deformation of a quadric of dimension $n-2k$ is again a quadric of the same dimension, by Lemma 6.1, we conclude that, for any variety Y representing the class $\sigma_{a_\bullet;b_\bullet}$, $i(Y)$ is a quadric of dimension $n-2k$ and cohomology class $2\sigma_{1,2,\dots,k-1,n-k}$. By assumption, the dimension of the quadric is at least two. The next lemma proves that the linear space spanned by such a quadric is contained in $G(k,n)$. Since this linear space necessarily has the form of k-planes containing a fixed $(k-1)$-plane and contained in an $(n-k+1)$-dimensional linear space, we conclude that every linear space parameterized by Y contains a fixed $(k-1)$-dimensional linear space Λ. Since the Schubert variety consisting of isotropic k-planes containing Λ has the same dimension as Y and contains Y, we conclude that Y is equal to the Schubert variety. Therefore, $\sigma_{a_\bullet;b_\bullet}$ is rigid. □

Lemma 6.6. *Let R be a smooth quadric of dimension $d \geq 2$ contained in the Plücker embedding of $G(k,n)$. Suppose that the cohomology class of R is $2\sigma_{1,2,\dots,k-1,k+d}$. Then the linear space spanned by R is contained in $G(k,n)$.*

Proof. First, suppose that d, the dimension of R, is equal to 2. A smooth quadric surface has two rulings by lines. A line in the Grassmannian $G(k,n)$ consists of linear spaces that contain a fixed $(k-1)$-dimensional linear space contained in a fixed $(k+1)$-dimensional linear space. Suppose that two of the lines in one of the rulings on R are defined by the partial flags $U_{k-1} \subset U_{k+1}$ and $W_{k-1} \subset W_{k+1}$. A line in the opposite ruling determined by the flag $V_{k-1} \subset V_{k+1}$ on R has to intersect these two lines. Therefore, U_{k+1} and W_{k+1} must intersect in a linear space of dimension at least $(k-1)$ containing V_{k-1}.

If $\dim(U_{k+1} \cap W_{k+1}) = k-1$, then every line in the other ruling must have V_{k-1} as part of the partial flag defining the line. Hence, every point of R is contained in the locus of k-planes that contain V_{k-1}. The locus of k-planes that contain V_{k-1} is a linear space in $G(k,n)$ that contains the span of R.

If $U_{k+1} = W_{k+1}$, then every k-plane parameterized by the quadric R is contained in U_{k+1}. Therefore, R is contained in the linear Schubert variety parameterizing k-dimensional subspaces of U_{k+1}. In this case, the cohomology class of R cannot be $2\sigma_{1,2,\dots,k-1,k+d}$ unless $d = 1$.

Therefore, we may assume that $\dim(U_{k+1} \cap W_{k+1}) = k$ and that the span of U_{k+1} and W_{k+1} has dimension $k+2$. We may also assume that the dimension of intersection of U_{k-1} and W_{k-1} is at least $k-2$. Otherwise, the flag V_{k+1} defining a line of the opposite ruling would have to be equal to the span of U_{k-1} and W_{k-1}. Hence, R would be contained in the linear space parameterizing k-planes contained in the span of U_{k-1} and W_{k-1}. Consequently, R is contained in the embedding of $G(2,4)$ in $G(k, Span(U_{k+1}, W_{k+1}))$ given by $\Lambda \mapsto Span(\Lambda, U_{k-1} \cap W_{k-1})$. Since $G(2,4)$ is a quadric fourfold in $\mathbb{P}^5$ under the Plücker embedding, the quadric surface R has to be a codimension two linear section of $G(2,4)$. The class of this surface in $G(2,4)$ is $\sigma_{2,3} + \sigma_{1,4}$ contrary to our assumptions. We conclude that we must have $\dim(U_{k+1} \cap W_{k+1}) = k-1$. In particular, R is contained in the linear space parameterizing k-planes that contain a fixed $(k-1)$-dimensional linear space.

Now suppose $d > 2$. If the linear space spanned by R is not contained in $G(k,n)$, we can find a $\mathbb{P}^3$ not contained in $G(k,n)$ such that $R \cap \mathbb{P}^3$ is a quadric surface whose span is not contained in $G(k,n)$. Since $2\sigma_{1,2,\dots,k-1,k+d+1} \cdot \sigma^{d-2}_{n-k,n-k+2,\dots,n} = 2\sigma_{1,2,\dots,k-1,k+2}$ [F, Chapter 14], this quadric surface has class $2\sigma_{1,2,\dots,k-1,k+2}$. We thus contradict the discussion of the case $d = 2$. This concludes the proof of the lemma. □

We now build on Corollary 6.4 and Proposition 6.5 to prove the rigidity of other classes by induction.

Proposition 6.7. *Suppose that $\sigma_{a_\bullet;b_\bullet}$ is a Schubert class such that $a_s = b_k = s$ and $b_j = s+k-j$. If $n > 2k+1$, then $\sigma_{a_\bullet;b_\bullet}$ is rigid.*

Proof. We will prove this proposition by induction on $k-s$. If $k = s$, then $\sigma_{a_\bullet}$ is the class of a point, which is clearly multi rigid (in particular, rigid). If $k = s+1$, then the proposition reduces to Proposition 6.5. Suppose the proposition holds for $k - s < \gamma$. Let Y be a variety representing the cohomology class $\sigma_{a_\bullet;b_\bullet}$. For the rest of the argument, it is more convenient to use projective language. By Lemma 6.2, the projective linear spaces parameterized by Y sweep out a quadric Q_Y of dimension $n-s-2$.

Fix a general point $p \in Q_Y$. Consider the linear spaces parameterized by Y that contain p. The cohomology class of this locus is $\sigma_{a'_\bullet;b'_\bullet}$, where $a'_{s+1} = b_k = s+1$ and $b'_j = s+1+k-j$. By induction on $k-s$, this locus is a Schubert variety parameterizing isotropic k-dimensional subspaces that contain a fixed $(s+1)$-dimensional isotropic space. In particular, the linear spaces parameterized by Y and that contain p, sweep out a quadric of dimension $n-s-3$ and corank $s+1$ contained in Q_Y. It follows that the corank of Q_Y is at least s. Since the codimension of Q_Y in the quadric defined by the quadratic form Q is s, the corank of Q_Y is at most s. We conclude that the corank of Q_Y is exactly s.

For a general point p of Q_Y, the linear spaces parameterized by Y that contain p also contain the singular locus of Q_Y. By the upper semi-continuity of the dimension of intersection of the linear spaces parameterized by Y with the singular locus of Q_Y, we conclude that every k-plane parameterized by Y contains the singular locus of Q_Y. Since the locus of k-planes contained in Q_Y and containing the singular locus of Q_Y is a Schubert variety with class $\sigma_{a_\bullet;b_\bullet}$, we conclude that Y is a Schubert variety. Hence, $\sigma_{a_\bullet;b_\bullet}$ is rigid. □

Lemma 6.8. *Let $\sigma_{a;b_\bullet}$ be a Schubert cycle such that $s = 1$, $a_\bullet = a_1 = a$, $b_i = k-i$ if $k-i < a-1$, and $b_i = k-i+1$ if $k-i \geq a-1$. If $n > 2k+a$, then $\sigma_{a;b_\bullet}$ is rigid.*

Proof. We prove the lemma by induction on a. If $a = 1$, then this lemma reduces to Proposition 6.7. Suppose $a = 2$. Let Y be a variety representing $\sigma_{a_\bullet;b_\bullet}$. Fix a general $(n-1)$-dimensional linear space Λ. The locus of k-dimensional linear spaces parameterized by Y that are contained in Λ has cohomology class $\sigma_{1;k-1,k-2,\dots,2,1}$ in $OG(k,n-1)$. By Proposition 6.7, this locus is a Schubert variety. Hence, it parameterizes k-dimensional isotropic linear spaces that contain a fixed vector v and are contained in the orthogonal complement of v in $Q \cap \Lambda$. Let Λ' be a general codimension one linear space containing v. The linear spaces parameterized by Y that are contained in Λ' similarly must all contain some vector v' and be contained in the orthogonal complement of v' in $Q \cap \Lambda'$. By considering linear spaces parameterized by Y that are contained in $\Lambda \cap \Lambda'$, we conclude that $v = v'$. In particular, every k-dimensional isotropic linear space containing v must be contained in Y. Now take a general pencil of codimension one linear spaces Λ_t. This yields a one parameter family of vectors v_t such that every k-dimensional isotropic linear space containing v_t is contained in Y. The degree of the curve on Q_Y swept out by the points corresponding to the vectors v_t is determined by the cohomology class $\sigma_{a_\bullet;b_\bullet}$. In particular, this curve must be a line in Q_Y. (Note that the class of the locus of isotropic subspaces that intersect a curve of degree d on the quadric Q is $d\sigma_{2;k-1,\dots,2,0}$ as can be easily seen by intersecting with the Poincaré dual Schubert class.) We conclude that Y contains every isotropic linear space that intersects a codimension two isotropic subspace. Hence, Y is a Schubert variety.

Suppose by induction on a that any variety representing $\sigma_{a;b_\bullet}$ is a Schubert variety for $a < j$. Let $a = j$. Let Y represent $\sigma_{a;b_\bullet}$. Let Λ be a general codimension one linear space. Then the locus of k-dimensional isotropic linear spaces parameterized by Y that are contained in Λ is a Schubert variety of k-dimensional isotropic linear spaces that intersect an isotropic linear space Γ of dimension $a-1$. By taking another general codimension one linear space containing Γ, we conclude that any isotropic linear space intersecting Γ is contained in Y. Repeating the construction for another general linear space Λ', we obtain another linear space Γ' of dimension $a-1$ such that every isotropic linear space intersecting Γ' is contained in Y. By induction, Γ and Γ' have to intersect in a linear space of dimension $a-2$ and have to span an isotropic linear space Ψ of dimension a. By the same argument, it is clear that every isotropic linear space that intersects Ψ is contained in Y. This proves that Y is a Schubert variety. Hence, $\sigma_{a;b_\bullet}$ is rigid. This concludes the induction and the proof of the proposition. □

Proposition 6.9. *Let $\sigma_{a_\bullet;b_\bullet}$ be a Schubert cycle such that $s = k-1$, $b_k = a_{k-1}$, $n > 2a_{k-1}+3$. Let $\alpha_i = a_i - i$ and group the equal terms to express the sequence $\alpha_\bullet$ as $(\tilde{\alpha}_1^{i_1}, \dots, \tilde{\alpha}_t^{i_t})$. Assume that there does not exist an index $1 \leq u < t$ such that $0 < \tilde{\alpha}_u = \tilde{\alpha}_{u+1} - 1$. Then $\sigma_{a_\bullet;b_\bullet}$ is rigid.*

Proof. We will prove the proposition by induction on a_{k-1}. If $a_{k-1} = k-1$, then the proposition reduces to Proposition 6.5. If $a_{k-1} = k$, then the proposition reduces to Corollary 6.4. Suppose that the proposition is true if $a_{k-1} < \gamma$. Let Y be a variety representing the class $\sigma_{a_\bullet;b_\bullet}$. Let h be the index such that $a_i = i$ for $i < h$ and $a_i > i$ for $i \geq h$. Take a general Schubert variety S in $G(k,n)$ with class $\sigma_{n-k,\dots,n-h-1,n-h,n-h+2,\dots,n-1,n}$. By the argument in the proof of Lemma 6.2, the projective linear spaces parameterized by $Y \cap S$ sweep out a quadric $Q_{Y\cap S}$ of dimension $n-b_k-3$. Even when Y is a Schubert variety, the corank of this quadric is $b_k - 1$. Hence, by semi-continuity the corank for Y is at most $b_k - 1$. Therefore, there exists a smooth quadric of dimension $n-4$ containing $Y \cap S$. Hence, $Y \cap S$ is a subvariety of $OG(k, n-2)$. The cohomology class of $Y \cap S$ in $OG(k, n-2)$ is $\sigma_{a'_\bullet;b'_\bullet}$, where $a'_i = a_i$ if $i < h$ and $a'_i = a_i - 1$ if $i > h$ and $b'_k = b_k - 1$.

By induction on a_{k-1}, $Y \cap S$ is a Schubert variety in $OG(k, n-2)$. When $a'_{k-1} = k$, the variety is a cone. Inductively, we conclude that $Y \cap S$ is singular along a Schubert variety with class $\sigma_{a''_\bullet}$, where $a''_\bullet$ is the sequence of length k obtained from $a'_\bullet$ by adding the largest integer less than a'_{k-1} not contained in the sequence $a'_\bullet$. It follows that Y has to be singular along a variety with class $\sigma_{a^*_\bullet}$, where $a^*_\bullet$ is the sequence of length k obtained from $a_\bullet$ by adding the largest integer less than a_{k-1} not contained in $a_\bullet$. By [C1, Proposition 3.1], we conclude that there exists a distinguished isotropic linear space Λ of dimension a_k such that Y is singular along every k-dimensional linear space parameterized by Y that is contained in Λ.

Now by induction it is easy to see that Y is a Schubert variety. First, by induction, it is clear that the quadric Q_Y swept out by the linear spaces parameterized by Y is singular along $\mathbb{P}\Lambda$. Let W be a linear space such that $[W] \in Y$ and $\dim(W \cap \Lambda) = k-1$. Then, inductively every k-dimensional linear space containing $W \cap \Lambda$ is also in Y. Let U be the open set in Y parameterizing k-dimensional linear spaces that intersect Λ in a subspace of dimension $k-1$. Then the map $W \mapsto W \cap \Lambda$ gives a morphism from U to a variety in $OG(k-1, n)$ with class $\sigma_{a_\bullet}$. By Theorem 1.4, this Schubert class is rigid; hence, the variety is a Schubert variety. We conclude that there exist an isotropic partial flag such that the k-dimensional linear spaces parameterized by Y satisfy $\dim(W \cap F_{a_i}) \geq i$. We conclude that Y is a Schubert variety. Hence, $\sigma_{a_\bullet;b_\bullet}$ is rigid. □

Corollary 6.10. *Let $\sigma_{a_\bullet;b_\bullet}$ be a Schubert cycle such that $s \geq 1$, $b_j = a_s + k - j$, and $n > 2a_s + 2k - 2s + 1$. Let $\alpha_i = a_i - i$ and group the equal terms to express the sequence $\alpha_\bullet$ as $(\tilde{\alpha}_1^{i_1}, \dots, \tilde{\alpha}_t^{i_t})$. Assume that there does not exist an index $1 \leq u < t$ such that $0 < \tilde{\alpha}_u = \tilde{\alpha}_{u+1} - 1$. Then $\sigma_{a_\bullet;b_\bullet}$ is rigid.*

Proof. The proof is almost identical to the proof of Proposition 6.7. We induct on $k-s$. If $k-s=1$, then the proposition reduces to Proposition 6.9. Suppose the corollary is true by induction for $k-s<\gamma$. Let Y be a variety representing the class $\sigma_{a_\bullet;b_\bullet}$. Let Q_Y be the quadric swept out by the projective $(k-1)$-planes parameterized by Y. Let p be a general point of Q_Y. The class of the locus of linear spaces parameterized by Y that contain p has cohomology class $a'_1 = 1$, $a'_{i+1} = a_i + 1$ for $1 \leq i \leq s$ and $b'_j = b_{j-1}$ for $s+1 < j \leq k$. By induction on $k-s$, this is a Schubert variety. By the argument in Proposition 6.7, Q_Y has corank a_s. By Theorem 1.4 and induction, it is easy to see that there is a partial flag $F_{a_1} \subset \cdots \subset F_{a_s}$, where F_{a_s} is the singular locus of Q_Y, such that Y contains the locus parameterizing k-dimensional linear spaces W that are contained in Q_Y and satisfy $\dim(W \cap F_{a_i}) = i$. We conclude that Y is a Schubert variety and $\sigma_{a_\bullet;b_\bullet}$ is rigid. □

We are now ready to prove Theorem 1.10.

Proof of Theorem 1.10. Under the assumptions of the theorem, $\sigma_{a_\bullet;b_\bullet}$ is rigid by Corollary 6.10 or Lemma 6.8. This concludes the proof. □

We have already characterized all the rigid Schubert classes in $OG(1,n)$ in Example 1.3. The next theorem characterizes all the rigid Schubert classes in $OG(2,n)$ when $n > 8$.

Theorem 6.11. *A Schubert class $\sigma_{a_\bullet;b_\bullet}$ in $OG(2,n)$ with $n > 8$ is rigid if and only if one of the following holds:*

1. *$\sigma_{a_\bullet;b_\bullet} = \sigma_{a_1,a_2}$ and either $a_1 = 1$ or $a_2 \neq a_1 + 2$.*
2. *$\sigma_{a_\bullet;b_\bullet} = \sigma_{b_\bullet} = \sigma_{1,0}$.*
3. *$\sigma_{a_\bullet;b_\bullet} = \sigma_{a;a}$ and $n > 2a+3$.*
4. *$\sigma_{a_\bullet;b_\bullet} = \sigma_{a;0}$.*
5. *If $n = 2k$, then $\sigma_{a_\bullet;b_\bullet} = \sigma_{b_\bullet} = \sigma_{k-1,0}$ or $\sigma_{a;b} = \sigma_{a;k-1}$ with $a \neq k-2$.*

Proof. First, the Schubert class $\sigma_{a_\bullet;b_\bullet}$ may be of quadric type. By Theorem 1.5, the class is not rigid unless it is the identity element of the cohomology ring. In that case, the class is rigid since it can only be represented by the orthogonal Grassmannian itself. Hence, if the class is of quadric type in $OG(2,n)$, then it is rigid if and only if it has the form $\sigma_{a_\bullet;b_\bullet} = \sigma_{b_\bullet} = \sigma_{1,0}$.

Second, the Schubert class may be defined with respect to isotropic linear spaces alone. In that case, by Theorem 1.4 and Remark 4.2, the Schubert class σ_{a_1,a_2} is rigid if and only if either $a_1 = 1$ or $a_2 \neq a_1 + 2$. If $n = 2k$, there is an involution of the quadric that interchanges the two connected components of the space of maximal isotropic linear spaces. Hence, we do not change the rigidity of a Schubert class if we change the connected component of a maximal isotropic space used in the definition of the Schubert variety. Consequently, when $n = 2k$, by Theorem 1.4 and Remark 4.2, $\sigma_{a;k-1}$ is rigid if and only if $a \neq k-2$.

Finally, we can assume that the Schubert class is defined with respect to one isotropic linear space and one quadric. By Theorem 1.7 (1), $\sigma_{a;b}$ is not rigid unless $b = 0$ or $b = a$. Furthermore, by Theorem 1.7 (3), $\sigma_{a;a}$ is not rigid if $n = 2a+3$. Conversely, by Proposition 6.9, the classes $\sigma_{a;a}$ are rigid if $n > 2a+3$. Similarly, by Lemma 6.8, the classes $\sigma_{a;0}$ are rigid, provided that

$n > 4 + a$. The last condition is automatic if $n > 8$. If $n = 2k$ is even, by the same lemma, the class $\sigma_{b_\bullet} = \sigma_{k-1,0}$ is rigid. This concludes the proof of the theorem. □

References

[Bo] A. Borel, *Linear algebraic groups.* Springer-Verlag, New York, 1991.

[BL] S. Billey and V. Lakshmibai, *Singularities of Schubert varieties.* Springer, 2000.

[Br] M. Brion, *Lectures on the geometry of flag varieties*, in Topics in cohomological studies of algebraic varieties, Trends Math., Birkhäuser, Basel, 2005, pp. 33–85.

[B] R. Bryant, *Rigidity and quasi-rigidity of extremal cycles in Hermitian symmetric spaces.* Princeton University Press Ann. Math. Studies AM–153, 2005.

[C1] I. Coskun, Rigid and non-smoothable Schubert classes, *J. Differential Geom.*, **87** no. 3 (2011), 493–514.

[C2] I. Coskun, Restriction varieties and geometric branching rules, *Adv. Math.*, **228** no. 4 (2011), 2441–2502.

[F] W. Fulton, *Intersection Theory,* Springer-Verlag, Berlin Heidelberg, 1998.

[H] R. Hartshorne, *Algebraic geometry.* Springer, 1977.

[HRT] R. Hartshorne, E. Rees, and E. Thomas, Nonsmoothing of algebraic cycles on Grassmann varieties. *Bull. Amer. Math. Soc.*, **80** no.5 (1974), 847–851.

[Hi] H. Hironaka, A note on algebraic geometry over ground rings. The invariance of Hilbert characteristic functions under the specialization process. *Illinois J. Math.*, **2** (1958), 355–366.

[Ho1] J. Hong, Rigidity of smooth Schubert varieties in Hermitian symmetric spaces. *Trans. Amer. Math. Soc.*, **359** (2007), 2361–2381.

[Ho2] J. Hong, Rigidity of singular Schubert varieties in $Gr(m,n)$. *J. Differential Geom.*, **71** no. 1 (2005), 1–22.

[GH] P. Griffiths and J. Harris, *Principles of Algebraic Geometry.* Wiley Interscience, 1978.

[K1] S. Kleiman, Geometry on Grassmannians and applications to splitting bundles and smoothing cycles. *Inst. Hautes Études Sci. Publ. Math.*, **36** (1969), 281–297.

[K2] S. Kleiman, The transversality of a general translate. *Compos. Math.*, **28** no.3 (1974), 287–297.

[KL] S. Kleiman and J. Landolfi, Geometry and deformation of special Schubert varieties. *Compos. Math.*, **23** no. 4 (1971), 407–434.

[Ko] J. Kollár, Flatness criteria. *J. Algebra*, **175** (1995), 715–727.

[L] S. L'vovsky, Extensions of projective varieties and deformations, I. *Michigan Math. J.*, **39** (1992), 41–51.

[R] C. Robles, Schur flexibility of cominuscule Schubert classes, preprint.

[RT] C. Robles and D. The, Rigid Schubert varieties in compact Hermitian symmetric spaces, *Selecta Math.*, **18** no. 2 (2012), 717–777.

[S] I.R. Shafarevich, *Basic Algebraic Geometry I*, Springer-Verlag, 1994.

[W] M. Walters, Geometry and uniqueness of some extreme subvarieties in complex Grassmannians. *Ph. D. thesis,* University of Michigan, 1997.

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