

# SINGULARITIES OF GENERALIZED RICHARDSON VARIETIES

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ABSTRACT. Richardson varieties play an important role in intersection theory and in the geometric interpretation of the Littlewood-Richardson Rule for flag varieties. We discuss three natural generalizations of Richardson varieties which we call projection varieties, intersection varieties, and rank varieties. In many ways, these varieties are more fundamental than Richardson varieties and are more easily amenable to inductive geometric constructions. In this paper, we study the singularities of each type of generalization. Like Richardson varieties, projection varieties are normal with rational singularities. We also study in detail the singular loci of projection varieties in Type A Grassmannians. We use Kleiman's Transversality Theorem to determine the singular locus of any intersection variety in terms of the singular loci of Schubert varieties. This is a generalization of a criterion for any Richardson variety to be smooth in terms of the nonvanishing of certain cohomology classes which has been known by some experts in the field, but we don't believe has been published previously.

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## 1. INTRODUCTION

A *Richardson variety* is the intersection of two Schubert varieties in general position in a homogeneous variety  $G/P$ . Their cohomology classes encode information that plays a significant role in algebraic geometry, representation theory and combinatorics [15, 16, 32, 35]. In recent years, the study of the singularities of Richardson varieties has received a lot of interest. We refer the reader to [5] for general results about the singularities of Richardson varieties and to [30] for a detailed study of the singularities of Richardson varieties in Type A Grassmannians.

In this paper, we study three natural generalizations of Richardson varieties called intersection varieties, projection varieties and rank varieties. We extend several of the results of [30] pertaining to smoothness criteria, singular loci and multiplicities to these varieties in  $G/P$  for arbitrary semi-simple algebraic groups  $G$  and parabolic subgroups  $P$ . However, it is important to note that while in [30] the authors work over algebraically

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closed fields of arbitrary characteristic, we require the ground field to have characteristic zero.

The first generalization of Richardson varieties that we discuss is the *intersection varieties*. These varieties are simply the intersection of any finite number of general translates of Schubert varieties and they appear throughout the literature on Schubert calculus. We recall how Kleiman's Transversality Theorem [24] determines the singular locus of any intersection variety in terms of the singular loci of Schubert varieties. In Corollary 2.9, we characterize the smooth Richardson varieties in terms of vanishing conditions on certain products of cohomology classes for Schubert varieties. As an application, we show that a Richardson variety in the Grassmannian variety  $G(k, n)$  is smooth if and only if it is a Segre product of Grassmannians (see Corollary 2.13).

The second generalization of Richardson varieties that we discuss is the *projection varieties*. Given  $G/P$  as above, let  $Q \subset G$  be another parabolic subgroup containing  $P$ . Thus, we have the natural projection

$$\pi_Q : G/P \rightarrow G/Q.$$

A *projection variety* is the image of a Richardson variety  $R(u, v)$  under a projection  $\pi_Q$  with its reduced induced structure. Projection varieties naturally arise in inductive constructions such as the Bott-Samelson resolutions. For example, they are related to the stratifications used by Lusztig, Postnikov and Rietsch in the theory of total positivity [33, 38, 41] and Brown-Goodearl-Yakimov [6] in Poisson geometry. They generalize the (closed) positroid varieties defined by Knutson, Lam and Speyer in [25, Section 5.4] and they play a crucial role in the positive geometric Littlewood-Richardson rule for Type A flag varieties in [10]. Since the set of projection varieties is closed under the projection maps among flag varieties, projection varieties form a more fundamental class of varieties than Richardson varieties. Our first theorem about the singularities of projection varieties is the following, generalizing [25, Cor. 7.9 and Cor. 7.10].

**Theorem 1.1.** *Let  $G$  be a complex simply connected algebraic group and  $Q$  be a parabolic subgroup of  $G$ . Then all projection varieties in  $G/Q$  are normal and have rational singularities.*

In fact, we will prove a much more general statement (Theorem 3.3) about the restriction of Mori contractions to subvarieties satisfying certain cohomological properties. Since Richardson varieties satisfy these cohomological properties, it will follow that projections of Richardson varieties have rational singularities proving the theorem. By [29, Theorem 3], this implies that projection varieties are Cohen-Macaulay.

To define the third family of varieties related to Richardson varieties, we specialize to Type A ( $G = GL(n)$ ) Grassmannian projection varieties. In this case, we consider  $G/P$  to be the partial flag variety  $Fl(k_1, \dots, k_m; n)$  consisting of partial flags

$$V_1 \subset \dots \subset V_m,$$

where each  $V_i$  is a complex vector space of dimension  $k_i$ . Let  $G(k, n)$  be the Grassmannian variety of  $k$ -dimensional subspaces in an  $n$ -dimensional complex vector space  $V$ . We can

realize  $G(k, n)$  as  $G/Q$  where  $Q$  is a maximal parabolic subgroup of  $G$ . Let

$$\pi : Fl(k_1, \dots, k_m; n) \longrightarrow G(k_m, n)$$

denote the natural projection morphism defined by  $\pi(V_1, \dots, V_m) = V_m$ .

A *Grassmannian projection variety* in  $G(k_m, n)$  is the image  $\pi(R(u, v))$  of a Richardson variety  $R(u, v) \subset Fl(k_1, \dots, k_m; n)$  with its reduced induced structure. It is convenient to have a characterization of Grassmannian projection varieties without referring to the projection of a particular Richardson variety. We introduce rank sets and rank varieties to obtain such a characterization.

Fix an ordered basis  $e_1, \dots, e_n$  of  $V$ . If  $W$  is a vector space spanned by a consecutive set of basis elements  $e_i, e_{i+1}, \dots, e_j$ , let  $l(W) = i$  and  $r(W) = j$ . A *rank set*  $M$  for  $G(k, n)$  is a set of  $k$  vector spaces  $M = \{W_1, \dots, W_k\}$ , where each vector space is the span of (non-empty) consecutive sequences of basis elements and  $l(W_i) \neq l(W_j)$  and  $r(W_i) \neq r(W_j)$  for  $i \neq j$ . Observe that the number of vector spaces  $k$  is equal to the dimension of subspaces parameterized by  $G(k, n)$ . Two rank sets  $M_1$  and  $M_2$  are equivalent if they are defined with respect to the same ordered basis of  $V$  and consist of the same set of vector spaces.

Given a rank set  $M$ , we can define an irreducible subvariety  $X(M)$  of  $G(k, n)$  associated to  $M$  as follows. The *rank variety*  $X(M)$  is the subvariety of  $G(k, n)$  defined by the Zariski closure of the set of  $k$ -planes in  $V$  that have a basis  $b_1, \dots, b_k$  such that  $b_i \in W_i$  for  $W_i \in M$ . For example,  $G(k, n)$  is itself a rank variety corresponding with the rank set  $M = \{W_1, \dots, W_k\}$  where each  $W_i = \langle e_i, \dots, e_{n-k+i} \rangle$ .

In Theorem 4.5, we prove that  $X \subset G(k, n)$  is a projection variety if and only if  $X$  is a rank variety. In particular, the Richardson varieties in  $G(k, n)$  are rank varieties. The singular loci of rank varieties or equivalently of Grassmannian projection varieties can be characterized as follows.

**Theorem 1.2.** *Let  $X$  be a rank variety in  $G(k_m, n)$ .*

- (1) *There exists a partial flag variety  $F(k_1, \dots, k_m; n)$  and a Richardson variety*

$$R(u, v) \subset F(k_1, \dots, k_m; n)$$

*such that*

$$\pi|_{R(u, v)} : R(u, v) \longrightarrow X$$

*is a birational morphism onto  $X$ .*

- (2) *The singular locus of  $X$  is the set of points  $x \in X$  such that either  $\pi^{-1}|_{R(u, v)}(x) \in R(u, v)$  is singular or  $\pi^{-1}|_{R(u, v)}(x)$  is positive dimensional:*

$$X^{sing} = \{x \in X \mid \dim(\pi^{-1}|_{R(u, v)}(x)) > 0 \text{ or } \pi^{-1}|_{R(u, v)}(x) \in R(u, v)^{sing}\}.$$

- (3) *The singular locus of a projection variety is an explicitly determined union of projection varieties.*

In Lemma 4.12, we give a simple formula for the dimension of a rank variety. In Corollary 4.25, we relate the enumeration of rank varieties by dimension to a  $q$ -analog of the Stirling numbers of the second kind.

The organization of this paper is as follows. In Section 2, we will introduce our notation and study the singular loci of Richardson varieties and other intersection varieties in

homogeneous varieties. We recall Kleiman’s Transversality Theorem. This allows us to completely characterize the singular loci of intersection varieties in terms of the singular loci of Schubert varieties in Proposition 2.8. An example in the Grassmannian  $G(3, 8)$  is given showing that Richardson varieties can be singular at every  $T$ -fixed point. As corollaries, we discuss some special properties of intersection varieties in Grassmannians. In Section 3, we will prove a general theorem about the singularities of the image of a subvariety satisfying certain cohomological properties under a Mori contraction. This will immediately imply Theorem 1.1. In Section 4, we will undertake a detailed analysis of the singularities of Grassmannian projection varieties and rank varieties. In particular, the proof of Theorem 1.2 follows directly from Corollary 4.17.

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## 2. THE SINGULARITIES OF INTERSECTION VARIETIES AND RICHARDSON VARIETIES

In this section, we review the necessary notation and background for this article. In particular, we recall Kleiman’s Transversality Theorem and review its application to the singular loci of Richardson varieties and intersection varieties. For the convenience of the reader, we included the proofs of results such as Theorem 2.6 and Corollary 2.9 when our formulation differed from what is commonly available in the literature. For further background, we recommend [5, 17, 19, 20, 22, 43].

Given a projective variety  $X$  and a point  $p \in X$ , let  $T_p X$  denote the Zariski tangent space to  $X$  at  $p$ . Then,  $p$  is a *singular point* of  $X$  if  $\dim T_p X > \dim X$ , and  $p$  is *smooth* if  $\dim T_p X = \dim X$ . Let  $X^{\text{sing}}$  denote the set of all singular points in  $X$ .

Let  $G$  denote a simply connected, semi-simple algebraic group over the complex numbers  $\mathbb{C}$ . Fix a maximal torus  $T$  and a Borel subgroup  $B$  containing  $T$ . Let  $P$  denote a parabolic subgroup of  $G$  containing  $B$ . Let  $W = N(T)/T$  denote the Weyl group of  $G$ , and let  $W_P$  denote the Weyl group of  $P$ . We will abuse notation by considering any element  $u \in W$  to also represent a choice of element in the coset  $uT \subset G$ . In particular, we consider  $W \subset G$  via this choice. Let  $e_u = uP$ , since  $T \subset P$  this point is well defined in  $G/P$ . The points  $\{e_u : u \in W/W_P\}$  are the  $T$ -fixed points in  $G/P$ .

For an element  $u \in W/W_P$ , the *Schubert variety*  $X_u$  is the Zariski closure of the  $B$ -orbit of  $e_u = uP$  in  $G/P$ . Thus,  $X_u$  is the union of  $B$ -orbits  $Be_t$  for  $t \leq u$  in the Bruhat order on  $W/W_P$ . Since we are working over  $\mathbb{C}$ , the (complex) dimension of  $X_u$  is the length of  $u$  as an element of  $W/W_P$ .

The singular locus of  $X_u$  is also a  $B$ -stable subvariety of  $G/P$ , hence it is a union of Schubert varieties. The typical way of studying the singularities of Schubert varieties is in terms of the  $T$ -fixed points. In particular,  $p \in X_u$  is a smooth point if and only if there exists a  $t \in W/W_P$  such that  $p \in Be_t$  and  $e_t$  is a smooth point of  $X_u$ . There are many effective tools for determining if  $e_t$  is a smooth point in  $X_u$  and exactly which elements of

$W/W_P$  index the Schubert varieties which form the irreducible components of the singular locus of  $X_u$ , see [3, 4, 7, 8, 23, 31, 34, 37].

Let  $w_0$  be the unique longest element in  $W$ . For  $v \in W/W_P$ , define the *opposite Schubert variety*, denoted  $X^v$ , as the Schubert variety  $w_0X_v$ . We caution the reader that some authors use  $X^v$  to denote the Schubert variety in the Poincaré dual class. The *Richardson variety*  $R(u, v) \in G/P$  is defined as the intersection of the two Schubert varieties  $X_u$  and  $X^v$ .  $R(u, v)$  is empty unless  $u \geq w_0v$  in the Bruhat order, in which case the dimension of  $R(u, v)$  is  $l(u) - l(w_0v)$ .

For much of the discussion, there is no reason to restrict to Richardson varieties. Intersection varieties provide a more natural set of varieties to consider. Let  $gX_u$  denote the translate of the Schubert variety  $X_u$  under the action of  $g \in G$  by left multiplication on  $G/P$ .

*Definition 2.1.* Let  $u_1, \dots, u_r \in W/W_P$ , and let  $g_\bullet = (g_1, \dots, g_r)$  be a general  $r$ -tuple of elements in  $G^r$ . The *intersection variety*  $R(u_1, \dots, u_r; g_\bullet)$  is defined as the intersection of the translated Schubert varieties  $g_iX_{u_i}$  in  $G/P$ :

$$R(u_1, \dots, u_r; g_\bullet) = g_1X_{u_1} \cap \dots \cap g_rX_{u_r}.$$

*Remark 2.2.* Richardson varieties are the special case of intersection varieties when  $r = 2$ . Let  $B^- = w_0Bw_0$  be the opposite Borel subgroup of  $G$  defined by the property that  $B \cap B^- = T$ . The  $G$ -orbit of  $(B, B^-)$  is a dense open orbit under the action of  $G$  on  $G/B \times G/B^-$ . Consequently, for any choice of a pair  $(g_1, g_2)$  in this orbit, the intersection of two translated Schubert varieties defined with respect to the pair is isomorphic. Hence, for  $r = 2$ , the generality condition simply means that  $(g_1, g_2)$  should belong to the dense open orbit. In particular,  $R(u_1, u_2; g_\bullet)$  is isomorphic to  $R(u_1, u_2)$  for general  $g_\bullet = (g_1, g_2)$ .

*Remark 2.3.* When  $r > 2$ , it is hard to characterize the  $g_\bullet$  that are sufficiently general. Two general intersection varieties  $R(u_1, \dots, u_r; g_\bullet)$  and  $R(u_1, \dots, u_r; g'_\bullet)$  are not necessarily isomorphic or even birational to each other. For example, let  $u \in W/W_P$  be the element indexing the divisor class for the Grassmannian  $G(2, 5)$ . Then  $R(u, u, u, u, u; g_\bullet)$  is an elliptic curve. As  $g_\bullet$  varies, all  $j$ -invariants occur in this family of elliptic curves. Since two elliptic curves with different  $j$ -invariants are not birational to each other, we get examples of intersection varieties  $R(u, u, u, u, u; g_\bullet)$  and  $R(u, u, u, u, u; g'_\bullet)$  in  $G(2, 5)$  that are not isomorphic (or even birational) to each other, see [9].

It is well-known that Richardson varieties are reduced and irreducible [40]. However, for  $r > 2$  the intersection varieties may be reducible. For example, if  $\sum l(u_i) = \dim(G/Q)$ , then the intersection variety  $R(u_1, \dots, u_r, g_\bullet)$  consists of finitely many points, where the number of points is given by the intersection number  $\prod_{i=1}^r [X_{u_i}]$  of the Schubert classes. Nevertheless, every connected component of an intersection variety is normal and has rational singularities [5, Lemma 4.1.2].

Kleiman's Transversality Theorem [24] is the key tool for characterizing the singular loci of intersection varieties. We recall the original statement of the theorem for the reader's convenience.

**Theorem 2.4** (Kleiman, [24]). *Let  $X$  be an integral algebraic scheme with a transitive action of the integral algebraic group  $G$ . Let  $f : Y \rightarrow X$  and  $g : Z \rightarrow X$  be two maps of integral algebraic schemes. For any point  $s \in G$ , let  $sY$  denote the  $X$ -scheme given by the map  $y \mapsto sf(y)$ .*

- (i) *There exists a dense Zariski open subset  $U$  of  $G$  such that for  $s \in U$ , the fibered product  $sY \times_X Z$  is either empty or equi-dimensional of dimension  $\dim(Y) + \dim(Z) - \dim(X)$ .*
- (ii) *Assume the characteristic of the ground field is zero. If  $Y$  and  $Z$  are smooth, then there exists a dense open subset  $V$  of  $G$  such that for  $s \in V$ , the fibered product  $(sY) \times_X Z$  is smooth.*

*Remark 2.5.* The proof of part (ii) in Kleiman's Theorem uses generic smoothness. This is the main reason why we are working over  $\mathbb{C}$ . Kleiman gives a specific example where (ii) fails if the ground field has positive characteristic and  $X$  is a Grassmannian variety [24, 9.Example].

There exist many variations of Kleiman's Theorem in the original paper and in the literature. Below we spell out the variation we need for Richardson varieties and intersection varieties in general. This variation is similar to the statement in [20, Theorem 17.22].

**Theorem 2.6.** *Let  $G$  be an algebraic group acting transitively on a smooth projective variety  $X$ . Let  $Y$  and  $Z$  be two subvarieties of  $X$ . Then for any general translate  $gY$  of  $Y$ , we have*

$$(gY \cap Z)^{sing} = ((gY)^{sing} \cap Z) \cup ((gY) \cap Z^{sing}).$$

*Proof.* Since  $Y$  and  $Z$  are subvarieties of  $X$ , both map into  $X$  by inclusion. Furthermore, for any  $g \in G$  the fibered product

$$(gY) \times_X Z = \{(gy, z) \in gY \times Z : gy = z\} = gY \cap Z.$$

Applying part (ii) of Kleiman's Transversality Theorem to the smooth loci

$$\begin{aligned} (gY)^{sm} &= gY - (gY)^{sing} \\ Z^{sm} &= Z - Z^{sing} \end{aligned}$$

we conclude that  $(gY)^{sm} \cap Z^{sm}$  is smooth, provided  $g$  is sufficiently general. Therefore,  $(gY \cap Z)^{sing} \subset ((gY)^{sing} \cap Z) \cup (gY \cap Z^{sing})$ .

Conversely, by part (i) of Kleiman's Transversality Theorem,  $Z$  intersects a general translate  $gY$  of  $Y$  properly, so we can assume

$$\dim((gY) \cap Z) = \dim(gY) + \dim(Z) - \dim(X).$$

We claim that a proper intersection of two varieties cannot be smooth at a point where one of the varieties is singular. This claim is verified by the following computation. Let

$p \in Z^{sing}$ . Then Lemma 2.7 below implies that

$$\begin{aligned} \dim(T_p(gY \cap Z)) &= \dim(T_p(gY) \cap T_p(Z)) \\ &\geq \dim(T_p(gY)) + \dim(T_p(Z)) - \dim(X) \quad (\text{since } X \text{ is smooth}) \\ &> \dim(gY) + \dim(Z) - \dim(X) \quad (\text{since } p \in Z^{sing}) \\ &= \dim(gY \cap Z). \end{aligned}$$

Therefore,  $p \in (gY \cap Z)^{sing}$ . Similarly,  $p \in (gY)^{sing}$  implies  $p \in (gY \cap Z)^{sing}$ . This proves  $((gY)^{sing} \cap Z) \cup (gY \cap Z^{sing}) \subset (gY \cap Z)^{sing}$ . Hence, we conclude

$$(gY \cap Z)^{sing} = ((gY)^{sing} \cap Z) \cup (gY \cap Z^{sing}).$$

□

**Lemma 2.7.** *Let  $Y$  and  $Z$  be two subvarieties of a smooth projective variety  $X$  and let  $p \in Y \cap Z$ . Then  $T_p(Y \cap Z) = T_p Y \cap T_p Z$  in  $T_p X$ .*

*Proof.* Since this is a local question, we may assume that  $X$  is affine space and  $Y$  and  $Z$  are affine varieties. Let  $I(Y)$  and  $I(Z)$  denote the ideals of  $Y$  and  $Z$ , respectively. Then  $I(Y \cap Z) = I(Y) + I(Z)$ . Let  $f_1, \dots, f_n$  be generators of  $I(Y)$  and  $g_1, \dots, g_m$  be generators of  $I(Z)$ . Then  $f_1, \dots, f_n, g_1, \dots, g_m$  generate  $I(Y) + I(Z)$ . The Zariski tangent spaces  $T_p(Y)$ ,  $T_p(Z)$  and  $T_p(Y \cap Z)$  are the kernels of the matrices

$$M = \left( \frac{\partial f_i}{\partial t_j}(p) \right), N = \left( \frac{\partial g_i}{\partial t_j}(p) \right), L = \begin{pmatrix} M \\ N \end{pmatrix},$$

respectively, where  $t_j$  denote the coordinates on affine space. It is now clear that the kernel of  $L$  is the intersection of the kernels of  $M$  and  $N$ . □

In the next proposition, we specialize Theorem 2.6, to  $X = G/P$ . Given a variety  $Y$  in  $G/P$ , let  $[Y]$  denote the cohomology class of  $Y$  in  $H^*(G/P, \mathbb{Z})$ .

**Proposition 2.8.** *The singular locus of the intersection variety  $R(u_1, \dots, u_r; g_\bullet)$  is*

$$\bigcup_{i=1}^r \left( g_i X_{u_i}^{sing} \cap R(u_1, \dots, u_r; g_\bullet) \right) = \bigcup_{i=1}^r \left( g_i X_{u_i}^{sing} \cap \bigcap_{j=1}^r g_j X_{u_j} \right).$$

*Hence, the singular locus of an intersection variety is a union of intersection varieties. Furthermore,  $R(u_1, \dots, u_r; g_\bullet)$  is smooth if and only if*

$$[X_{u_i}^{sing}] \cdot \prod_{j \neq i} [X_{u_j}] = 0$$

*for every  $1 \leq i \leq r$ .*

*Proof.* Applying Theorem 2.6 and induction on  $r$  to  $g_1 X_1 \cap \dots \cap g_r X_r$ , it follows that

$$(g_1 X_1 \cap \dots \cap g_r X_r)^{sing} = \bigcup_{i=1}^r \left( (g_i X_i)^{sing} \cap \bigcap_{j=1}^r g_j X_j \right)$$

provided that the tuple  $(g_1, \dots, g_r)$  is general in the sense of Kleiman's Transversality Theorem. In particular, taking  $X_i$  to be the Schubert variety  $X_{u_i} \in X = G/P$  we recover the first statement in Proposition 2.8.

Since the singular locus of a Schubert variety is a union of Schubert varieties corresponding to certain smaller dimensional  $B$ -orbits, we conclude that the singular locus of a Richardson variety is a union of Richardson varieties. In particular, by part (i) of Kleiman's Transversality Theorem, we may assume that all the intersections

$$g_i X_{u_i}^{sing} \cap \bigcap_{j \neq i} g_j X_{u_j}$$

are dimensionally proper. The cohomology class of this intersection is the cup product of the cohomology classes of each of the Schubert varieties. Hence, this intersection is empty if and only if its cohomology class is zero. We conclude that the singular locus of an intersection variety is empty if and only if the cohomology classes  $[X_{u_i}^{sing}] \cdot \prod_{j \neq i} [X_{u_j}] = 0$  for all  $1 \leq i \leq r$ . This concludes the proof of Proposition 2.8.  $\square$

Specializing to the case  $r = 2$ , we obtain the following characterization of the singular loci of Richardson varieties.

**Corollary 2.9.** *Let  $R(u, v) = X_u \cap X^v$  be a non-empty Richardson variety in  $G/P$ . Let  $X_u^{sing}$  and  $X_{sing}^v$  denote the singular loci of the two Schubert varieties  $X_u$  and  $X^v$ , respectively. Then the singular locus of  $R(u, v)$  is a union of Richardson varieties*

$$R(u, v)^{sing} = (X_u^{sing} \cap X^v) \cup (X_u \cap X_{sing}^v).$$

*In particular,  $R(u, v)$  is smooth if and only if the cohomology classes  $[X_u^{sing}] \cdot [X^v] = 0$  and  $[X_u] \cdot [X_{sing}^v] = 0$  in the cohomology ring  $H^*(G/P, \mathbb{Z})$ .*

*Remark 2.10.* Checking the vanishing conditions in Corollary 2.9 is very easy once the singular locus is determined since these conditions require only the product of pairs of Schubert classes to vanish. This is equivalent to testing the relations between pairs of elements in Bruhat order. However, in general for  $r > 2$ , checking the vanishing conditions in Proposition 2.8 is a hard problem which requires computing Schubert structure constants. There are many techniques for doing these computations; see for example [1, 2, 5, 10, 11, 12, 13, 16, 28] and references within those. It is an interesting open problem to efficiently characterize all triples  $u, v, w \in W/W_P$  such that  $[X_u][X_v][X_w] = 0$ . Purbhoo has given some necessary conditions for vanishing in [39].

Recall, that the typical way of studying the singularities of Schubert varieties in the literature relies on tests for the  $T$ -fixed points of the Schubert variety. Even for Richardson varieties  $R(u, v)$  in the Grassmannian, there may not be any torus fixed points in the smooth locus of  $R(u, v)$ . Below we give an example.

*Example 2.11.* Consider the Grassmannian variety  $G(3, 8)$  of 3-planes in  $\mathbb{C}^8$ . Fix a basis  $e_1, \dots, e_8$  of  $V$ . The  $T$ -fixed points in  $G(3, 8)$  are the subspaces spanned by three distinct basis elements  $\{e_{w_1}, e_{w_2}, e_{w_3}\}$ . These points are indexed by permutations  $w = [w_1, \dots, w_8] \in S_8$  such that  $w_1 < w_2 < w_3$  and  $w_4 < \dots < w_8$ . Such permutations could



be denoted simply by  $(w_1, w_2, w_3)$ . Furthermore, these permutations are in bijection with partitions that fit in a  $3 \times (8 - 3)$  rectangle. So,  $u = (4, 6, 8) = [4, 6, 8, 1, 2, 3, 5, 7]$  is a  $T$ -fixed point of  $G(3, 8)$  and it corresponds with the partition  $(2, 1, 0)$ . See [16] for more detail.

Let both  $u$  and  $v$  be the permutation  $u = (4, 6, 8)$ , or equivalently the partition  $(2, 1, 0)$  so that  $X_u$  is isomorphic to  $X^v$  but in opposite position. Consider  $R(u, v) \subset G(3, 8)$  defined in terms of flags  $F_\bullet = (F_1, \dots, F_8)$  and  $G_\bullet = (G_1, \dots, G_8)$  where  $F_i$  is the span of the first  $i$  basis elements and  $G_i$  is the span of the last  $i$  basis elements. The Schubert variety  $X_u$  is the set of 3-dimensional subspaces that intersect  $F_4, F_6$  and  $F_8$  in subspaces of dimensions at least 1, 2 and 3, respectively. Similarly,  $X^v$  is the set of 3-dimensional subspaces that intersect  $G_4, G_6$  and  $G_8$  in subspaces of dimensions at least 1, 2 and 3, respectively.

The singular locus of  $X_u \subset G(k, n)$  is the union of Schubert varieties indexed by all the partitions obtained from the partition corresponding with  $u$  by adding a maximal hook in a way that the remaining shape is still a partition [3, Thm 9.3.1]. Thus,  $X_{(4,6,8)}^{sing} = X_{(3,4,8)} \cup X_{(4,5,6)}$ . In terms of flags,  $\Lambda \in X_u$  is a singular point if  $\dim(\Lambda \cap F_4) \geq 2$  or  $\dim(\Lambda \cap F_6) = 3$ . Similarly,  $\Lambda$  is singular for  $X^v$  if  $\dim(\Lambda \cap G_4) \geq 2$  or  $\dim(\Lambda \cap G_6) = 3$ .

The  $T$ -fixed points of  $R(u, v)$  consist of subspaces that are spanned by  $e_{i_1} \in F_4, e_{i_2} \in F_6 \cap G_6$  and  $e_{i_3} \in G_4$ . We claim that each of these  $T$ -fixed points is singular in either  $X_u$  or  $X^v$ , so by Corollary 2.9 they are all singular in  $R(u, v)$ . The claim holds since any basis vector in  $F_6 \cap G_6$  is also contained in either  $F_4$  or  $G_4$ . We conclude that the smooth locus of  $R(u, v)$  does not contain any torus fixed points.

In Remark 4.15, we will characterize the Richardson varieties in the Grassmannian that contain a torus fixed smooth point.

More generally, when  $G/P$  is the Grassmannian  $G(k, n)$ , Corollary 2.9 implies a nice, geometric characterization of the smooth Richardson varieties. This characterization is essentially proved but not explicitly stated in [30].

*Definition 2.12.* The Segre product of  $r$  Grassmannians  $G(k_1, n_1) \times \dots \times G(k_r, n_r)$  in  $G(\sum k_i, n)$ , with  $n \geq \sum n_i$ , is the image under the direct sum map

$$(W_1, \dots, W_r) \mapsto (W_1 \oplus \dots \oplus W_r).$$

A Segre product of Grassmannians is a Richardson variety in  $G(k, n)$ . Specifically, the Segre product  $G(k_1, n_1) \times \dots \times G(k_r, n_r)$  is the intersection of the two opposite Schubert varieties  $X_u \cap X^v$  where

$$u = (n_1 - k_1 + 1, n_1 - k_1 + 2, \dots, n_1, \dots, \sum_{i=1}^r n_i - k_r + 1, \dots, \sum_{i=1}^r n_i)$$

and

$$v = (n - \sum_{i=1}^{r-1} n_i - k_r + 1, n - \sum_{i=1}^{r-1} n_i - k_r + 2, \dots, n - \sum_{i=1}^{r-1} n_i, \dots, n - k_1 + 1, \dots, n - 1, n).$$

This Richardson variety can also be realized as the intersection of two Schubert varieties with respect to partial flags. Let  $F_j = V_1 \oplus \dots \oplus V_j$ , where  $V_i$  is the  $n_i$  dimensional vector

space defining  $G(k_i, n_i)$ . Let  $E$  be a vector subspace of  $V$  complementary to  $F_r$ . Let  $G_j = E \oplus V_r \oplus \cdots \oplus V_{r-j+1}$ . Then we have

$$X_u = \{W \in G(k, n) \mid \dim(W \cap F_j) \geq \sum_{i=1}^j k_i\}$$

and

$$X^v = \{W \in G(k, n) \mid \dim(W \cap G_j) \geq \sum_{i=r-j+1}^r k_i\},$$

**Corollary 2.13.** *A Richardson variety in  $G(k, n)$  is smooth if and only if it is a Segre product of Grassmannians.*

*Remark 2.14.* By taking  $X^v$  to equal  $G(k, n)$ , we see that Corollary 2.13 generalizes the well-known fact that a Schubert variety  $X_u$  in  $G(k, n)$  is smooth if and only if  $X_u$  is a sub-Grassmannian.

*Proof.* A Segre product of Grassmannians is clearly smooth since it is the Segre embedding of the product of Grassmannians  $G(k_1, n_1) \times \cdots \times G(k_r, n_r)$  in  $G(k, n)$  with  $k = \sum k_i$  and  $n \geq \sum n_i$ . Conversely, suppose that  $R(u, v)$  is a smooth Richardson variety in  $G(k, n)$ . Observe that  $X_u^{sing} \cap X^v = \emptyset$  if and only if  $u_i + v_{n-i} \leq n$  for every  $1 \leq i \leq k$  for which  $u_{i+1} \neq u_i + 1$ . Similarly,  $X_u \cap X_{sing}^v = \emptyset$  if and only if  $v_i + u_{n-i} \leq n$  for every  $1 \leq i \leq k$  for which  $v_{i+1} \neq v_i + 1$ . Suppose first that  $u_{i+1} = u_i + 1$  for all  $1 \leq i \leq k - 1$ . Then the Richardson variety is isomorphic to a Schubert variety and is smooth if and only if it is a Grassmannian. The corollary now is immediate by induction on the number of times  $u_{i+1} \neq u_i + 1$ . Suppose  $u_{j+1} \neq u_j + 1$ . Then  $u_j + v_{n-j} \leq n$  and the Richardson variety is a product of two Richardson varieties in  $G(j, F_{u_j}) \times G(n - j, G_{v_{n-j}})$  and is smooth if and only if each factor is smooth.  $\square$

*Remark 2.15.* Finally, we note that the proof given in [30, Remark 7.6.6] for determining the multiplicities of Richardson varieties in minuscule partial flag varieties generalizes by induction to intersection varieties. Let  $R(u_1, \dots, u_r; g_\bullet)$  be an intersection variety in a minuscule partial flag variety. Then

$$\text{mult}_p(R(u_1, \dots, u_r; g_\bullet)) = \prod_{i=1}^r \text{mult}_p(g_i X_{u_i}).$$

### 3. PROJECTION VARIETIES

In this section, we prove that projection varieties have rational singularities. This claim follows from a general fact, which we prove below, about the images of certain subvarieties under Mori contractions. We refer the reader to [27] for more detail about Mori theory and rational singularities.

*Definition 3.1.* A variety  $X$  has *rational singularities* if there exists a resolution of singularities  $f : Y \rightarrow X$  such that  $f_* \mathcal{O}_Y = \mathcal{O}_X$  and  $R^i f_* \mathcal{O}_Y = 0$  for  $i > 0$ .

*Remark 3.2.* A variety with rational singularities is normal. Moreover, for every resolution  $g : Z \rightarrow X$ , we have  $g_*\mathcal{O}_Z = \mathcal{O}_X$  and  $R^i g_*\mathcal{O}_Z = 0$  for  $i > 0$  (see [27]).

The map  $\pi_Q : G/P \rightarrow G/Q$  is a Mori contraction. To deduce Theorem 1.1 we will apply the following general theorem.

**Theorem 3.3.** *Let  $X$  be a smooth, projective variety. Let  $\pi : X \rightarrow Y$  be a Mori contraction defined by the line bundle  $M = \pi^*L$ , where  $L$  is an ample line bundle on  $Y$ . Let  $Z \subset X$  be a normal, projective subvariety of  $X$  with rational singularities. Assume that*

- (1)  $H^i(Z, M^{\otimes n}|_Z) = 0$  for all  $i > 0$  and all  $n \geq 0$ .
- (2) The natural restriction map  $H^0(X, M^{\otimes n}) \rightarrow H^0(Z, M^{\otimes n}|_Z)$  is surjective for all  $n \geq 0$ .

*Then  $W = \pi(Z)$ , with its reduced induced structure, is normal and has rational singularities.*

*Proof.* Denote the restriction of the map  $\pi$  to  $Z$  also by  $\pi$ . To simplify notation, we will denote the restriction of the line bundles  $M$  to  $Z$  and  $L$  to  $W$  again by  $M$  and  $L$ .

Step 1. We first show that  $R^i \pi_*\mathcal{O}_Z = 0$  for  $i > 0$ . Since  $L$  is ample on  $W$ , by Serre's Theorem,  $R^i \pi_*\mathcal{O}_Z \otimes L^{\otimes n}$  is generated by global sections for  $n \gg 0$  ([21], II.5.17). Therefore, to show that  $R^i \pi_*\mathcal{O}_Z = 0$ , it suffices to show that  $H^0(W, R^i \pi_*\mathcal{O}_Z \otimes L^{\otimes n}) = 0$  for  $n \gg 0$ . Similarly, by Serre's Theorem,  $H^j(W, R^i \pi_*\mathcal{O}_Z \otimes L^{\otimes n}) = 0$  for all  $j > 0$  and all  $n \gg 0$  ([21], III.5.2). Since only finitely many sheaves  $R^i \pi_*\mathcal{O}_Z$  are non-zero, we may choose a number  $N$  such that for all  $n \geq N$ ,  $R^i \pi_*\mathcal{O}_Z \otimes L^{\otimes n}$  is globally generated and has no higher cohomology for all  $i$ .

Given a coherent sheaf  $\mathcal{F}$  on  $Z$ , the Leray spectral sequence expresses the cohomology of  $\mathcal{F}$  in terms of the cohomology of the higher direct image sheaves  $R^i \pi_*\mathcal{F}$  on  $W$ . More precisely, the spectral sequence has  $E_2^{p,q} = H^p(W, R^q \pi_*\mathcal{F})$  and abuts to  $H^{p+q}(Z, \mathcal{F})$ . We apply the spectral sequence to  $\mathcal{F} = M^{\otimes n}$  for  $n \geq N$ . Since  $M = \pi^*L$ , by the projection formula,  $R^i \pi_*M^{\otimes n} = R^i \pi_*\mathcal{O}_Z \otimes L^{\otimes n}$ . Since  $n$  is chosen so that  $H^p(W, R^q \pi_*\mathcal{O}_Z \otimes L^{\otimes n}) = 0$  for  $p > 0$ , we conclude that the spectral sequence degenerates at the  $E_2$  term. Consequently,  $H^0(W, R^i \pi_*\mathcal{O}_Z \otimes L^{\otimes n}) \cong H^i(Z, M^{\otimes n})$ . Since by assumption,  $H^i(Z, M^{\otimes n}) = 0$  for  $i > 0$  and  $n > 0$ , we conclude that  $H^0(W, R^i \pi_*\mathcal{O}_Z \otimes L^{\otimes n}) = 0$  for  $i > 0$  and  $n \geq N$ . Therefore,  $R^i \pi_*\mathcal{O}_Z = 0$  for  $i > 0$ .

Step 2. We next show that  $\pi_*\mathcal{O}_Z = \mathcal{O}_W$ . In particular, this implies that the Stein factorization of the map  $\pi : Z \rightarrow W$  is trivial ([21], III.11.5). Therefore, the fibers of the map  $\pi$  are connected and  $W$  is normal. Let  $\mathcal{F}$  be the cokernel of the natural injection from  $\mathcal{O}_W$  to  $\pi_*\mathcal{O}_Z$ . We thus obtain the exact sequence

$$(*) \quad 0 \rightarrow \mathcal{O}_W \rightarrow \pi_*\mathcal{O}_Z \rightarrow \mathcal{F} \rightarrow 0.$$

We want to show that  $\mathcal{F} = 0$ . Since  $L$  is ample,  $\mathcal{F} \otimes L^{\otimes n}$  is globally generated for  $n \gg 0$ . Hence, it suffices to show  $H^0(W, \mathcal{F} \otimes L^{\otimes n}) = 0$  for  $n \gg 0$ . Using the long exact sequence of cohomology associated to the exact sequence (\*) and the fact that  $H^1(W, L^{\otimes n}) = 0$  for  $n \gg 0$ , to conclude that  $\mathcal{F} = 0$ , it suffices to show that  $h^0(W, L^{\otimes n}) = h^0(W, \pi_*\mathcal{O}_Z \otimes L^{\otimes n})$  for  $n \gg 0$ .

Consider the exact sequence

$$0 \rightarrow I_W \rightarrow \mathcal{O}_Y \rightarrow \mathcal{O}_W \rightarrow 0.$$

Tensoring the exact sequence by  $L^{\otimes n}$  for  $n \gg 0$ , we get that the restriction map  $H^0(Y, L^{\otimes n}) \rightarrow H^0(W, L^{\otimes n})$  is surjective. Since the map  $\pi : X \rightarrow Y$  is a Mori contraction,  $\pi_*\mathcal{O}_X = \mathcal{O}_Y$ . Consequently, by the projection formula

$$H^0(Y, L^{\otimes n}) = H^0(Y, \pi_*\mathcal{O}_X \otimes L^{\otimes n}) = H^0(X, M^{\otimes n}).$$

The inclusion of  $Z$  in  $X$  and  $W$  in  $Y$  gives rise to the commutative diagram

$$\begin{array}{ccc} H^0(Y, L^{\otimes n}) = H^0(Y, \pi_*\mathcal{O}_X \otimes L^{\otimes n}) = H^0(X, M^{\otimes n}) & & \\ \downarrow & \swarrow & \\ H^0(W, L^{\otimes n}) & & H^0(Z, M^{\otimes n}) \\ \downarrow & \swarrow & \\ H^0(W, \pi_*\mathcal{O}_Z \otimes L^{\otimes n}) & & \end{array}$$

By assumption, the restriction map  $H^0(X, M^{\otimes n}) \rightarrow H^0(Z, M^{\otimes n})$  is surjective for all  $n \geq 0$ .  $H^0(Z, M^{\otimes n}) \cong H^0(W, \pi_*\mathcal{O}_Z \otimes L^{\otimes n})$ . In particular, it follows that  $H^0(Y, L^{\otimes n}) \rightarrow H^0(W, \pi_*\mathcal{O}_Z \otimes L^{\otimes n})$  is surjective. Therefore, the map  $H^0(W, L^{\otimes n}) \rightarrow H^0(W, \pi_*\mathcal{O}_Z \otimes L^{\otimes n})$  must be surjective. Since by the exact sequence (\*), it is also injective, we conclude that  $h^0(W, L^{\otimes n}) = h^0(W, \pi_*\mathcal{O}_Z \otimes L^{\otimes n})$  for  $n \gg 0$ . This concludes the proof that  $\mathcal{O}_W \cong \pi_*\mathcal{O}_Z$  and that  $W$  is normal.

Step 3. Finally, to conclude that  $W$  has rational singularities, we simply apply a theorem of Kollár. First, observe that since  $Z$  has rational singularities by assumption, for any desingularization  $\rho : U \rightarrow Z$ , we have that  $\rho_*\mathcal{O}_U \cong \mathcal{O}_Z$  and  $R^i\rho_*\mathcal{O}_U = 0$  for  $i > 0$ . Hence, considering  $\phi = \pi \circ \rho : U \rightarrow W$ , we have that  $\phi_*\mathcal{O}_U = \pi_*(\rho_*\mathcal{O}_U) = \mathcal{O}_W$  and  $R^i\phi_*\mathcal{O}_U = 0$  for  $i > 0$ . In particular, the Stein factorization of  $\phi$  is trivial and the geometric generic fiber of  $\phi$  is connected. Kollár's Theorem 7.1 in [26] then guarantees that  $W$  has rational singularities. This concludes the proof.  $\square$

*Remark 3.4.* The proof of Theorem 3.3 does not use the full strength of the hypotheses. In assumptions (1) and (2), it is not necessary to require the vanishing of higher cohomology and the surjectivity of the restriction map for all  $n \geq 0$ . It suffices to assume these only for sufficiently large  $n$ .

Theorem 3.3 reduces understanding singularities of certain subvarieties of flag varieties to the vanishing of cohomology. The higher cohomology groups of the restriction of NEF line bundles on flag varieties to Schubert or Richardson varieties vanish. Hence, Theorem 3.3 is a very useful tool in the context of Schubert geometry. For instance, Theorem 3.3 immediately implies Theorem 1.1.

*Proof of Theorem 1.1.* Let  $Q \subset G$  be a parabolic subgroup containing  $P$ . Let  $P_i$ ,  $i = 1, \dots, j$ , be the maximal parabolic subgroups containing  $Q$ . Let  $\pi_Q : G/P \rightarrow G/Q$  and

$\pi_{P_i} : G/Q \rightarrow G/P_i$  denote the natural projections. In Theorem 3.3 take

$$X = G/P \quad \text{and} \quad Y = G/Q.$$

Let  $L$  be the ample line bundle on  $Y$  defined by  $L = L_{i_1} \otimes L_{i_2} \otimes \cdots \otimes L_{i_j}$ , where  $L_{i_s}$  is the pull-back of the ample generator of the Picard group of  $G/P_i$  under the natural projection map  $\pi_{P_i}$ . Let  $M = \pi_Q^* L$ . Then  $M$  is NEF and defines the projection  $\pi_Q : X \rightarrow Y$ . Let  $Z$  be the Richardson variety  $R(u, v)$ . It is well-known that Richardson varieties are normal with rational singularities ([5] Theorem 4.1.1). Furthermore, the higher cohomology of the restriction of a NEF line bundle on the flag variety to a Richardson variety vanishes and the restriction map on global sections is surjective [5] Theorem 4.2.1 (ii) and Remark 4.2.2. Hence, all the assumptions of Theorem 3.3 are satisfied. We conclude that the projection variety  $\pi_Q(Z)$  is normal and has rational singularities.  $\square$

#### 4. SINGULARITIES OF GRASSMANNIAN PROJECTION VARIETIES

In this section, we discuss the singularities of projection varieties in Type A Grassmannians. We first characterize projection varieties in  $G(k, n)$  without reference to a projection from a flag variety. Given a projection variety, we then exhibit a minimal Richardson variety projecting to it. This Richardson variety is birational to the projection variety and the projection map does not contract any divisors. This description allows us to characterize the singular loci of projection varieties. We first begin by introducing some notation.

*Notation 4.1.* Let  $0 < k_1 < k_2 < \cdots < k_m < n$  be an increasing sequence of positive integers less than  $n$ . We set  $k_0 = 0$  and  $k_{m+1} = n$ . Let  $Fl(k_1, \dots, k_m; n)$  denote the partial flag variety parameterizing partial flags  $(V_1 \subset \cdots \subset V_m)$  of length  $m$  in  $V$  such that  $V_i$  has dimension  $k_i$ .

The cohomology of  $Fl(k_1, \dots, k_m; n)$  admits a  $\mathbb{Z}$ -basis generated by the classes of Schubert varieties. Schubert varieties in  $F(k_1, \dots, k_m; n)$  are parameterized by partial permutations in  $\mathfrak{S}_n$  with  $k_m$  entries and at most  $m$  descents at positions  $k_1, k_2, \dots, k_m$ . We record these permutations as a list of  $k_m$  distinct positive integers  $u = (u_1, u_2, \dots, u_{k_m})$  less than or equal to  $n$  such that  $u_j < u_{j+1}$  unless  $j = k_i$  for some  $1 \leq i \leq m$ . Given an entry  $u_i$  in the permutation, there exists a unique  $l$  such that  $k_{l-1} < i \leq k_l$ . We say that the *color*  $c_i$  of the entry  $u_i$  is  $l$ . Geometrically, the entries of the permutation record the dimensions of the elements in flag  $F_\bullet = (F_1, \dots, F_n)$  defining the Schubert variety  $X_u$  where a jump in dimension occurs and the corresponding color records the minimal  $j$  for which the dimension of  $V_j$  is required to increase:

$$X_u(F_\bullet) = \{(V_1, \dots, V_m) \in Fl(k_1, \dots, k_m; n) \mid \dim(V_j \cap F_{u_i}) \geq \#\{u_l \leq u_i \mid c_l \leq j\}\}.$$

It is convenient to assign a multi-index  $I_u(i) = (s_1^i, \dots, s_m^i)$  to each entry in a permutation by letting

$$s_j^i = \#\{u_l \leq u_i \mid c_l \leq j\}.$$

In particular, using the multi-indices, the definition of a Schubert variety can be expressed more compactly:

$$X_u(F_\bullet) = \{(V_1, \dots, V_m) \in Fl(k_1, \dots, k_m; n) \mid \dim(V_j \cap F_{u_i}) \geq s_j^i\}.$$

*Example 4.2.* Let  $u = (1, 4, 8, 3, 9, 2, 7)$  be a permutation for  $F(3, 5, 7; 9)$ . Then the color of the entries 1, 4, 8 is 1, the color of the entries 3, 9 is 2 and the color of the entries 2, 7 is 3. The multi-indices are  $I_u(1) = (1, 1, 1), I_u(2) = (2, 3, 4), I_u(3) = (3, 4, 6), I_u(4) = (1, 2, 3), I_u(5) = (3, 5, 7), I_u(6) = (1, 1, 2), I_u(7) = (2, 3, 5)$ . The corresponding Schubert variety parameterizes flags  $(V_1, V_2, V_3)$  such that  $V_3$  is required to intersect  $F_1, F_2, F_3, F_4, F_7, F_8, F_9$  in subspaces of dimension at least 1, 2, 3, 4, 5, 6, 7, respectively.  $V_2$  is required to intersect  $F_1, F_3, F_4, F_8, F_9$  in subspaces of dimension at least 1, 2, 3, 4, 5, respectively. Finally,  $V_1$  is required to intersect  $F_1, F_4, F_8$  in subspaces of dimension at least 1, 2, 3, respectively.

Recall that a *Grassmannian projection variety* is the projection of a Richardson variety  $R(u, v)$  in  $Fl(k_1, \dots, k_m; n)$  to  $G(k_m, n)$  under the natural projection map.

*Remark 4.3.* There may be many different ways of realizing the same projection variety as projections of different Richardson varieties. For example, take two successive projections of flag varieties

$$\mathcal{F}_1 = Fl(k_1, k_2, k_3; n) \xrightarrow{\pi_1} \mathcal{F}_2 = Fl(k_2, k_3; n) \xrightarrow{\pi_2} G(k_3, n).$$

Given a Richardson variety  $R(u, v) \subset \mathcal{F}_2$ ,  $\pi_1^{-1}(R(u, v)) = R(u', v')$  is a Richardson variety in  $\mathcal{F}_1$ . Hence, the projection variety  $\pi_2(R(u, v))$  may also be realized as the projection variety  $\pi_2 \circ \pi_1(R(u', v'))$ . In particular, the Grassmannian projection varieties all come from projections of Richardson varieties in the complete flag variety. This proves that rank varieties are examples of the (closed) positroid varieties defined in [25, Section 5]. Note, that lifting a Richardson variety in  $\mathcal{F}_1$  up to  $\mathcal{F}_2$  and then projecting it to  $G(k, n)$  need not be a birational map.

It is convenient to have a characterization the projection varieties without referring to the projection of a Richardson variety. Such a characterization can be obtained in terms of rank varieties. We recall the following notation from Section 1. Let  $V$  be an  $n$ -dimensional vector space. Fix an ordered basis  $e_1, \dots, e_n$  of  $V$ . Let  $W = [e_i, e_j]$  be the vector space spanned by a consecutive set of basis elements  $e_i, e_{i+1}, \dots, e_j$ . Let  $l(W) = i$  and  $r(W) = j$  be the smallest and largest index of the basis elements contained in  $W$ , respectively. A *rank set*  $M$  for  $G(k, n)$  is a set of  $k$  vector spaces  $M = \{W_1, \dots, W_k\}$ , where each vector space is the span of (non-empty) consecutive sequences of basis elements and  $l(W_i) \neq l(W_j)$  and  $r(W_i) \neq r(W_j)$  for  $i \neq j$ . Given a rank set  $M$ , the corresponding *rank variety*  $X(M)$  is the subvariety of  $G(k, n)$  defined by the Zariski closure of the set of  $k$ -planes in  $V$  that have a basis  $b_1, \dots, b_k$  such that  $b_i \in W_i$  for  $W_i \in M$ .

*Remark 4.4.* Alternatively, one can define a rank variety  $X(M)$  as the variety of  $k$ -planes that intersect any vector space  $W$  spanned by the ordered basis in a subspace of dimension at least the number of  $W_i \in M$  contained in  $W$

$$X(M) = \{\Lambda \in G(k, n) \mid \dim(\Lambda \cap W) \geq \#\{W_i \in M \mid W_i \subset W\} \\ \text{for every } W = \langle e_{i_1}, \dots, e_{i_s} \rangle\}.$$

These rank equations give rise to the terminology. In the sequel we will not need this fact, so we leave showing the equivalence of the two definitions to the reader.

We can characterize projection varieties in  $G(k, n)$  as rank varieties.

**Theorem 4.5.**  *$X \subset G(k, n)$  is a projection variety if and only if  $X$  is a rank variety.*

We will prove Theorem 4.5 in several steps. We begin by giving two algorithms. The first algorithm associates a Richardson variety  $R(u, v)(M)$  to every rank variety  $X(M)$  in  $G(k, n)$  in a minimal way such that the projection of  $R(u, v)(M)$  is  $X(M)$ . Given a Grassmannian projection variety, the second algorithm associates to it a rank set with the corresponding rank variety being equal to the original projection variety.

Throughout this proof, we fix two opposite flags  $F_\bullet = (F_1, \dots, F_n)$  and  $G_\bullet = (G_1, \dots, G_n)$  where  $F_i$  is the span of the first  $i$  basis elements  $e_1, \dots, e_i$  and  $G_i$  in  $G_\bullet$  is the span of the last  $i$  basis elements  $e_n, \dots, e_{n-i+1}$ . Then  $X_u = X_u(F_\bullet)$  and  $X_v = X_v(G_\bullet)$ .

*Algorithm 4.6. [Associating a Richardson variety to a rank variety.]* In this algorithm, given a rank set  $M = \{W_1, \dots, W_k\}$  and its rank variety  $X(M)$  in  $G(k_m, n)$ , we will associate a Richardson variety  $R(u, v)(M)$  in an appropriate flag variety  $F(k_1, \dots, k_m; n)$  such that the projection of  $R(u, v)$  is  $X(M)$ .

**Step 1: Associate a color to each vector space  $W_i$ .** Let  $m$  be the longest chain of subspaces

$$W_{j_1} \supsetneq W_{j_2} \supsetneq \dots \supsetneq W_{j_m},$$

where each  $W_{j_s} \in M$ . For a vector space  $W_i \in M$ , let  $m_i$  be the length of the longest chain

$$W_{j_1} \supsetneq W_{j_2} \supsetneq \dots \supsetneq W_{j_{m_i}} = W_i,$$

where  $W_{j_s} \in M$ . Assign  $W_i$  the color  $c_i = m - m_i + 1$ . From now on we decorate the vector spaces in the rank set with their color  $W_i^{c_i}$ .

**Step 2: Define two opposite Schubert varieties.** Let  $k_j$  be the number of vector spaces in the rank set that are assigned a color less than or equal to  $j$ . We define two Schubert varieties in  $F(k_1, \dots, k_m; n)$ . Recall that  $r(W_i)$  is the index of the basis element with the largest index in  $W_i$ . Let  $u$  be the permutation defined by the numbers  $r(W_i)$  listed so that those corresponding to vector spaces of color  $c$  all occur before those of color  $c + 1$  and among those of the same color the numbers are increasing. Similarly, recall that  $l(W_i)$  is the index of the basis element with the smallest index in  $W_i$ . Let  $v$  be the permutation defined by the numbers  $n - l(W_i) + 1$  listed so that those corresponding to vector spaces of color  $c$  all occur before those of color  $c + 1$  and among those of the same color the numbers are increasing. Let the *minimal Richardson variety*  $R(u, v)(M)$  associated to the rank set  $M$  be the Richardson variety  $X_u \cap X_v = X_u(F_\bullet) \cap X_v(G_\bullet)$  in  $F(k_1, \dots, k_m; n)$ .

*Remark 4.7.* The requirement that  $l(W_i) \neq l(W_j)$  and  $r(W_i) \neq r(W_j)$  for  $i \neq j$  guarantees that the numbers  $r(W_i)$  and  $n - l(W_i) + 1$  are all distinct. Moreover, both permutations have at most  $m$  descents at places  $k_1, \dots, k_m$  by construction. Therefore, Algorithm 4.6 produces well-defined partial permutations  $u$  and  $v$  for  $F(k_1, \dots, k_m; n)$ .

*Algorithm 4.8. [Associating a rank set to a Richardson variety.]* Let  $R(u, v) = X_u \cap X^v = X_u(F_\bullet) \cap X_v(G_\bullet)$  be a Richardson variety in  $F(k_1, \dots, k_m; n)$ . In this algorithm, we associate a rank set  $M(R(u, v))$  to  $R(u, v)$  such that the projection of  $R(u, v)$  to  $G(k_m, n)$  is the rank variety  $X(M(R(u, v)))$ .

**Step 0.** Recall that each  $u_i$  in the permutation  $u$  is assigned a color  $c_i$ , where  $c_i = l$  if  $k_{l-1} < i \leq k_l$ , and a multi-index  $I_u(i) = (s_1^i, \dots, s_m^i)$ , where  $s_j^i = \#\{u_l \leq u_i \mid c_l \leq j\}$ . Similarly, each  $v_i$  in the permutation  $v$  is assigned a color  $d_i$  and a multi-index  $I_v(i) = (t_1^i, \dots, t_m^i)$ . Given an entry  $u_j$  in a permutation, recall  $u^{-1}(u_j) = j$  denotes the index of the entry. The color, index and multi-index are assigned to  $u_i$  or  $v_i$  in a permutation for once and for all and do NOT vary during the algorithm.

**Step 1.** Let  $U_{k_m} = \{u_i : 1 \leq i \leq k_m\}$  be the initial set of entries in the permutation  $u$  and let  $V_{k_m} = \{v_i : 1 \leq i \leq k_m\}$  be the analogous the set of entries for  $v$ . Let  $M_0$  be the empty set. At each stage, we will remove an element from each of  $U_{k_m}$  and  $V_{k_m}$  and add a vector space to  $M_0$  until we exhaust  $U_{k_m}$  and  $V_{k_m}$ .

**Initial Step.** Let

$$\alpha = \min_{u_i \in U_{k_m}} (u_i).$$

Let  $c = c_{u^{-1}(\alpha)}$  be the color of the entry  $\alpha$ . For each  $d \geq c$ , let

$$\beta^d = \max_{v_i \in V_{k_m} \text{ s.t. } d_i \leq d} (v_i).$$

Let  $\beta$  be the minimum  $\beta^d$  over all  $d \geq c$ . Let  $W_1 = F_\alpha \cap G_\beta$ , where  $F_\bullet$  and  $G_\bullet$  are the two flags defining the Schubert varieties  $X_u$  and  $X^v$ , respectively as always. Set  $U_{k_m-1} = U_{k_m} - \{\alpha\}$ ,  $V_{k_m-1} = V_{k_m} - \{\beta\}$  and  $M_1 = \{W_1\}$ .

**The Inductive Step.** Suppose we have defined  $M_t$  and are left with two subsets  $U_{k_m-t}$  and  $V_{k_m-t}$  of the entries from the permutations  $u$  and  $v$ . Let

$$\alpha = \min_{u_i \in U_{k_m-t}} (u_i).$$

Let  $c = c_{u^{-1}(\alpha)}$  as above. For each  $d \geq c$ , let

$$\beta^d = \max_{v_i \in V_{k_m-t} \text{ s.t. } d_i \leq d} (v_i).$$

Let  $\beta$  be the minimum  $\beta^d$  over all  $d \geq c$  for which  $t_d^{v^{-1}(\beta^d)} \geq k_d - s_d^{u^{-1}(\alpha)} + 1$ . Since the inequality  $t_m^{v^{-1}(\beta^m)} \geq k_m - s_m^{u^{-1}(\alpha)} + 1$  is satisfied, such a  $\beta$  must exist. Let

$$W_{t+1} = F_\alpha \cap G_\beta.$$

Set

$$\begin{aligned} U_{k_m-t-1} &= U_{k_m-t} - \{\alpha\}, \\ V_{k_m-t-1} &= V_{k_m-t} - \{\beta\} \end{aligned}$$

and  $M_{t+1} = M_t \cup \{W_{t+1}\}$ .

**Step 2.** The inductive loop in Step 1 terminates when  $t = k_m$ . The rank set associated to the Richardson variety  $R(u, v)$  is  $M(R(u, v)) = M_{k_m}$ .



*Remark 4.9.* Every vector space  $W_i$  formed during the algorithm occurs as  $F_\alpha \cap G_\beta$ . Hence,  $W_i$  is the span of the consecutive set of basis elements  $e_{n-\beta+1}, \dots, e_\alpha$ . Since each  $\alpha$  and each  $\beta$  occur only once during the algorithm,  $l(W_i) \neq l(W_j)$  and  $r(W_i) \neq r(W_j)$  for  $i \neq j$ . Therefore,  $M(R(u, v))$  is a rank set.

Before proceeding to prove Theorem 4.5, we give some examples of Algorithm 4.6 and Algorithm 4.8.

*Example 4.10.* Let  $M$  be the rank set  $W_1 = [e_1, e_7], W_2 = [e_2, e_6], W_3 = [e_3, e_4], W_4 = [e_4, e_5], W_5 = [e_6, e_8]$  so  $X(M) \subset G(5, 8)$ . Then  $(m_1, \dots, m_5) = (1, 2, 3, 3, 1)$  so the colors of the vector spaces are  $W_1^3, W_2^2, W_3^1, W_4^1, W_5^3$  as indicated by the superscripts. Hence, the corresponding minimal Richardson variety is contained in  $Fl(2, 3, 5; 8)$  and has defining permutations  $u = (4, 5, 6, 7, 8)$  and  $v = (5, 6, 7, 3, 8)$ .

Conversely, suppose we begin with the Richardson variety associated to the permutations  $u = (4, 5, 6, 7, 8)$  and  $v = (5, 6, 7, 3, 8)$  in  $Fl(2, 3, 5; 8)$ . Construct the following tables of associated data:

$i$	$u_i$	$c_i$	$I_u(i)$	$i$	$v_i$	$d_i$	$I_v(i)$
1	4	1	(1, 1, 1)	1	5	1	(1, 1, 2)
2	5	1	(2, 2, 2)	2	6	1	(2, 2, 3)
3	6	2	(2, 3, 3)	3	7	2	(2, 3, 4)
4	7	3	(2, 3, 4)	4	3	3	(0, 0, 1)
5	8	3	(2, 3, 5)	5	8	3	(2, 3, 5).

In Algorithm 4.8, the vector spaces that are formed are  $W_1 = F_4 \cap G_6 = [e_3, e_4], W_2 = F_5 \cap G_5 = [e_4, e_5], W_3 = F_6 \cap G_7 = [e_2, e_6], W_4 = F_7 \cap G_8 = [e_1, e_7]$  and  $W_5 = F_8 \cap G_3 = [e_6, e_8]$ . We elaborate on the computation of  $W_2$ . For  $t = 2$ , we have  $\alpha = 5$  and  $c = 1$ , so  $\beta^1 = 5, \beta^2 = 7, \beta^3 = 8$ . The minimum among the  $\beta^d$ 's is 5 with  $d = 1$ . The tricky condition

$$t_d^{v^{-1}(\beta^d)} \geq k_d - s_d^{u^{-1}(\alpha)} + 1$$

is satisfied since  $1 = t_1^1 < k_1 - s_1^2 + 1 = 2 - 2 + 1 = 1$ , hence  $\beta = 5$ . Observe that we recover the initial rank set.

*Example 4.11.* Let  $R(u, v)$  be the Richardson variety in  $F(2, 4; 7)$  associated to the permutations  $u = (4, 6, 2, 7)$  and let  $v = (2, 7, 3, 5)$ . Construct the following tables of associated data:

$i$	$u_i$	$c_i$	$I_u(i)$	$i$	$v_i$	$d_i$	$I_v(i)$
1	4	1	(1, 2)	1	2	1	(1, 1)
2	6	1	(2, 3)	2	7	1	(2, 4)
3	2	2	(0, 1)	3	3	2	(1, 2)
4	7	2	(2, 4)	4	5	2	(1, 3).

In Algorithm 4.8, the vector spaces that are formed when computing  $M(R(u, v))$  are  $W_1 = F_2 \cap G_7 = [e_1, e_2], W_2 = F_4 \cap G_5 = [e_3, e_4], W_3 = F_6 \cap G_2 = [e_6],$  and  $W_4 = F_7 \cap G_3 = [e_5, e_7]$ .

If we begin with the rank set  $M = \{W_1 = [e_1, e_2], W_2 = [e_3, e_4], W_3 = [e_6], W_4 = [e_5, e_7]\}$ , then the colors of the vector spaces assigned in the Algorithm 4.6 are  $W_1^2, W_2^2, W_3^1, W_4^2$ . Hence, Algorithm 4.6 assigns the Richardson variety  $R(u', v')$  in  $F(1, 4; 7)$ , where  $u' =$

$(6, 2, 4, 7)$  and  $v' = (2, 3, 5, 7)$ . Observe that  $R(u', v')$  is different from  $R(u, v)$ . In fact, they are not subvarieties of the same flag variety. The projection of  $R(u, v)$  to  $X(M(R(u, v)))$  has positive dimensional fibers, whereas the projection map from  $R(u', v')$  to  $X(M(R(u, v)))$  is birational.

We are now ready to start the proof of Theorem 4.5. We first determine the dimension of rank varieties.

**Lemma 4.12.** *The rank variety  $X(M)$  associated to a rank set  $M$  is an irreducible subvariety of  $G(k, n)$  of dimension*

$$\dim(X(M)) = \sum_{i=1}^k \dim(W_i) - \sum_{i=1}^k \#\{W_j \in M \mid W_j \subseteq W_i\}.$$

*Proof.* This is a special case of Lemma 3.29 in [10] and follows easily by induction on  $k$ . When  $k = 1$ , the rank variety is projective space of dimension  $\dim(W_1) - 1$ , as claimed in the lemma. Let  $W_1$  be the vector space with minimal  $l(W_i)$  in  $M$ . Omitting  $W_1$  gives rise to a rank variety  $X(M')$  in  $G(k-1, n)$ . There is a dominant morphism from a dense open subset of  $X(M)$  to  $X(M')$  and the fibers are open subsets in a projective space of dimension

$$\dim(W_1) - \#\{W_i \in M \mid W_i \subseteq W_1\}.$$

The lemma follows by induction. □

Next we show that the projection of  $R(u, v)$  is  $X(M(R(u, v)))$ .

**Lemma 4.13.** *Let  $R(u, v)$  be a Richardson variety in  $F(k_1, \dots, k_m; n)$ . Let  $\pi$  denote the projection to  $G(k_m, n)$ . Then  $\pi(R(u, v)) = X(M(R(u, v)))$ .*

*Proof.* Let  $(V_1, \dots, V_m) \in R(u, v)$ . We first prove that  $\pi(R(u, v)) \subset X(M(R(u, v)))$ . It suffices to check that  $V_m$  satisfies all the rank conditions imposed by  $M(R(u, v))$ . The basic linear algebra fact is that

$$\dim(V_r \cap F_{u_i} \cap G_{v_j}) \geq s_r^i + t_r^j - k_r$$

since  $V_r$  intersects  $F_{u_i}$  and  $G_{v_j}$  in subspaces of dimension at least  $s_r^i$  and  $t_r^j$ , respectively. Furthermore, since  $V_r \subset V_m$ , we conclude that

$$\dim(V_m \cap F_{u_i} \cap G_{v_j}) \geq \max_{1 \leq r \leq m} s_r^i + t_r^j - k_r.$$

Now notice that  $M(R(u, v))$  is constructed so that  $\dim(V_m \cap F_{u_i} \cap G_{v_j})$  precisely equals  $\max_{1 \leq r \leq m} s_r^i + t_r^j - k_r$ . Hence, the rank conditions imposed by  $M(R(u, v))$  are satisfied by  $V_m$  for  $(V_1, \dots, V_m) \in R(u, v)$ .

Next, we show that the projection of  $R(u, v)$  is onto  $X(M(R(u, v)))$ . Since  $R(u, v)$  is a projective variety and  $\pi$  is a morphism, the image  $\pi(R(u, v))$  is a projective variety. Hence, it suffices to show that a general point of  $X(M(R(u, v)))$  is in the image of  $\pi$ . There is a dense open set of  $X(M(R(u, v)))$  consisting of  $k$ -planes  $\Lambda$  such that

$$\dim(\Lambda \cap W_i) = \#\{W_j \in M \mid W_j \subseteq W_i\}$$

for every  $i$  and

$$\dim(\Lambda \cap W_i \cap W_j) = \#\{W_t \in M \mid W_t \subset W_i \cap W_j\}$$

for every  $i, j$ . Fix such a  $k$ -plane  $\Lambda$  that has a basis  $(b_1, \dots, b_k)$  with  $b_i \in W_i$ . Let  $\tau_s u$  be the truncation of the permutation  $u$  obtained by taking the first  $k_s$  numbers  $(u_1, \dots, u_{k_s})$  in the permutation  $u$ . Similarly, let  $\tau_s v$  be the corresponding truncation of  $v$ . For every  $1 \leq s < m$ , construct a sequence of vector spaces  $W_1^s, \dots, W_{k_s}^s$  by running the Algorithm 4.8 with the permutations  $\tau_s u$  and  $\tau_s v$  for the flag variety  $F(k_1, \dots, k_s; n)$ . Inductively, we define a point of the Richardson variety  $R(u, v)$  as follows. Let  $V_m = \Lambda$ . For every  $W_i^{m-1}$  let  $b_i^{m-1} = \sum_{b_j \in W_i^{m-1}} b_j$ . Let  $V_{m-1}$  be the span of the vectors  $b_i^{m-1}$ . Continuing by descending induction, let  $b_i^s = \sum_{b_j^{s+1} \in W_i^s} b_j^{s+1}$ . Let  $V^s$  be the vector space spanned by the vectors  $b_i^s$ . In this way, we obtain a partial flag  $(V_1, \dots, V_m)$ . By construction, it is easy to see that this partial flag lies in both Schubert varieties  $X_u$  and  $X_v$ , hence in the Richardson variety  $R(u, v)$ . Furthermore,  $\pi((V_1, \dots, V_m)) = \Lambda$ . We conclude that  $\pi$  is surjective. This concludes the proof.  $\square$

**Lemma 4.14.** *Let  $M_0$  be a rank set for  $G(k, n)$ . Let  $R(u, v)(M_0)$  be the associated minimal Richardson variety assigned by Algorithm 4.6. Let  $M(R(u, v))(M_0)$  be the rank set associated to  $R(u, v)(M_0)$  by Algorithm 4.8. Then  $M(R(u, v)(M_0)) = M_0$ .*

*Proof.* This is clear, hence left to the reader.  $\square$

*Proof of Theorem 4.5.* We are now ready to prove Theorem 4.5. By Lemma 4.13, every projection variety is a rank variety. By Lemma 4.14, every rank variety arises as a projection variety. These statements together imply that rank varieties are projection varieties and vice versa.  $\square$

*Remark 4.15.* As a first application, we can determine the torus fixed points in the smooth locus of a Richardson variety in  $G(k, n)$ . The rank set  $M$  associated to a Richardson variety consists of  $k$  vector spaces  $W_1, \dots, W_k$  such that they all have color one (equivalently, there are no containment relations among different vector spaces  $W_i$  and  $W_j$ ). We can assume that these vector spaces are ordered in increasing order by  $l(W_i)$ . Equivalently, we can order the vector spaces  $W_i$  by  $r(W_i)$  in increasing order. Since  $W_i \not\subset W_j$  for  $i \neq j$ , this leads to the same order. The torus fixed points are  $k$ -dimensional subspaces that are spanned by  $k$  distinct basis elements  $e_{i_1}, \dots, e_{i_k}$  with  $i_1 < i_2 < \dots < i_k$ . For a particular torus fixed point to be contained in the rank variety, we must have  $e_{i_j} \in W_j$ . To see this, note that

$$(1) \quad \dim(\text{Span}(e_{i_1}, \dots, e_{i_k}) \cap \text{Span}(W_1, W_2, \dots, W_j)) \geq j$$

and

$$(2) \quad \dim(\text{Span}(e_{i_1}, \dots, e_{i_k}) \cap \text{Span}(W_k, W_{k-1}, \dots, W_j)) \geq k - j + 1.$$

If  $e_{i_j} \notin W_j$ , then either  $e_{i_1}, \dots, e_{i_j} \in [e_1, e_{l(W_j)-1}]$  or  $e_{i_j}, \dots, e_{i_k} \in [e_{r(W_j)+1}, e_n]$ . The first case contradicts the second inequality and the second case contradicts the first inequality. Now we are ready to characterize the torus fixed points in the smooth locus of  $X(M)$ . They are spanned by  $e_{i_1}, \dots, e_{i_k}$  with  $i_1 < i_2 < \dots < i_k$  and  $e_{i_j} \in W_j$  such that:

- (1) If  $l(W_{j+1}) > l(W_j) + 1$ , then  $e_{i_j} \notin W_{j+1}$ ; and
- (2) If  $r(W_{j-1}) < r(W_j) - 1$ , then  $e_{i_j} \notin W_{j-1}$

To see that these are necessary and sufficient conditions, simply use Corollary 2.9 and the description of singularities of Schubert varieties in Grassmannians. In particular, if  $u_{i+1} > u_i + 1$  for  $1 \leq i < k$  and  $v_{i+1} > v_i + 1$  for  $1 \leq i < k$ , then the Richardson variety  $R(u, v)$  has a torus fixed point in its smooth locus if and only if every vector space  $W_j$  in the corresponding rank set contains a basis element  $e_{i_j}$  which is not contained in any of the other vector spaces in the rank set.

**Theorem 4.16.** *Let  $M$  be a rank set for  $G(k, n)$ . Let  $R(u, v)(M)$  be the Richardson variety associated to  $M$  by Algorithm 4.6. Let*

$$\pi : R(u, v)(M) \longrightarrow X(M)$$

*be the corresponding projection morphism. Then  $R(u, v)(M)$  is birational to  $X(M)$  under  $\pi$  and the exceptional locus of  $\pi$  has codimension at least 2.*

*Proof.* Let  $[\Lambda] \in X(M)$  be a  $k$ -dimensional subspace such that

$$\dim(\Lambda \cap W_i) = \#\{W_s \mid W_s \subseteq W_i\}$$

for every  $i$  and

$$\dim(\Lambda \cap W_i \cap W_j) = \#\{W_s \in M \mid W_s \subseteq W_i \cap W_j\}$$

for every  $i$  and  $j$ . The set of such  $\Lambda$  form a dense, Zariski open subset  $U$  of  $X(M)$ . To see that  $U$  is not empty take a vector space  $\Lambda$  spanned by vectors  $\sum_{e_j \in W_i} \alpha_j^i e_j$ , where the collection of coefficients  $\alpha_j^i$  are algebraically independent. Then it is clear that  $\Lambda$  is in  $U$ . The inverse of  $\pi$  can be defined over  $U$  as follows. Let  $W_{i_s}$  be  $1 \leq s \leq k_i - k_{i-1}$  be the vector spaces in  $M$  that are assigned the color  $i$ . Let  $\Lambda_i$  be the span of the vector spaces  $\Lambda \cap W_{i_s}$  with  $1 \leq s \leq k_i - k_{i-1}$ . Note that by construction  $\Lambda_i$  is a subspace of  $\Lambda$  of dimension  $k_i$  containing  $\Lambda_{i-1}$  and contained in  $\Lambda_{i+1}$ . It follows that the partial flag  $(\Lambda_1, \dots, \Lambda_k = \Lambda)$  is the inverse image of  $\Lambda$  under the projection map  $\pi$ . Hence  $\pi$  is birational.

We now bound the dimension of the exceptional locus. Note that the fiber dimension of  $\pi$  is positive if and only if at least one of the vector spaces  $V_j$  intersects  $W_i^{c_i}$  with  $c_i < j$  in a subspace of dimension greater than  $\#\{W_s \in M \mid W_s \subseteq W_i\}$ . We can stratify the rank variety into loci where such intersections happen and compare the decrease in the dimension of the image of  $\pi$  with the increase in the dimension of the fibers of  $\pi$ . In fact, by stratifying the rank variety successively, it suffices to carry out the calculation when  $j = c + 1$  and  $c_i = c$ . Let  $W_i$  and  $W_j$  be two vector spaces with colors  $c$  and  $c + 1$ , respectively, such that  $W_i \cap W_j \neq \emptyset$ . Define a new rank set  $M'(W_i, W_j)$  as follows.

Step 1. Let  $W'_j = W_j \cap W_i$ . List all the vector spaces  $W_1, W_2, \dots, W_r$  of color  $c + 1$  in  $M$  that contain  $W'_j$  ordered so that  $l(W_1) < l(W_2) < \dots < l(W_r)$ . Form a new *collection* of vector spaces  $M'$  by replacing  $W_1, \dots, W_r$  in  $M$  with  $W'_j$  and the spans  $\overline{W_1 W_2}, \overline{W_2 W_3}, \dots, \overline{W_{r-1} W_r}$ . Note that  $M'$  is not necessarily a rank set since two of the vector spaces may coincide or the least or largest index basis elements in two of the vector spaces may coincide.

Step 2. Let  $W - e_{r(W)}$  (respectively,  $W - e_{l(W)}$ ) denote the vector space spanned by the set of all the basis elements in  $W$  but  $e_{r(W)}$  (respectively,  $e_{l(W)}$ ). As long as there are two vector spaces  $W_1 \subseteq W_2$  in  $M'$  with  $r(W_1) = r(W_2)$ , replace  $W_2$  in  $M'$  with the vector

space  $W_2 - e_{r(W_2)}$  keeping its label the same and relabel the new collection of vector spaces  $M'$ . If there are no such vector spaces, as long as there are vector spaces  $W_1 \subseteq W_2$  in  $M'$  with  $l(W_1) = l(W_2)$ , replace  $W_2$  in  $M'$  with the vector space  $W_2 - e_{l(W_2)}$  and relabel the new collection of vector spaces  $M'$ . The procedure terminates when  $M'$  is a rank set or when one of the vector spaces consists only of the zero vector. In the former case, set  $M'(W_i, W_j) = M'$ . In the latter case, set  $M'(W_i, W_j) = \emptyset$ . We call this process the *normalization* of the set of vector spaces  $M'$ .

Observe that Step 2 leads to isomorphic subvarieties (see [10] for a discussion of the normalization algorithm). We include it in order to apply the dimension formula in Lemma 4.12 without modification. The fiber of  $\pi$  over  $M'(W_i, W_j)$  has dimension one. Note that the locus where  $\pi$  has higher dimensional fibers can be obtained by repeated applications of Steps 1 and 2. In order to estimate the dimension of the exceptional locus, it suffices to compare the dimension of  $X(M)$  to  $X(M'(W_i, W_j))$ . There are two cases to consider. If  $W_i \subset W_j$ , then  $\dim(W_i) \leq \dim(W_j) - 2$  since  $W_i$  does not contain  $l(W_j)$  and  $r(W_j)$ . Using the dimension formula given in Lemma 4.12 and the fact that Step 2 can only decrease the value of the expression, we see that  $\dim(X(M'(W_i, W_j))) \leq \dim(X(M)) - r - 2$ . In particular, this dimension is at least three less. Hence, the exceptional locus has codimension at least two.

If  $W_i \not\subset W_j$ , then we may assume that  $l(W_j) < l(W_i)$  and  $r(W_j) < r(W_i)$ . By the algorithm assigning colors, we know that there exists  $W_t$  of color  $c + 1$  containing  $W_i$  such that  $l(W_j) < l(W_t) < l(W_i)$ . We conclude that  $\dim(W_j \cap W_i) \leq \dim(W_j) - 2$ . By the dimension formula given in Lemma 4.12, it follows that  $\dim(X(M'(W_i, W_j))) \leq \dim(X(M)) - r - 2$ . In particular, this dimension is at least three less. Hence, the exceptional locus has codimension at least two.

This concludes the discussion that the exceptional locus has codimension at least two.  $\square$

The following corollary states a more precise version of Theorem 1.2.

**Corollary 4.17.** *Let  $X(M)$  be a rank variety. Let  $\pi : R(u, v)(M) \rightarrow X(M)$  be the projection from the minimal Richardson variety associated to  $M$ . Then the singular locus of  $X(M)$  is given by*

$$X(M)^{sing} = \{x \in X \mid \pi^{-1}(x) \in R(u, v)^{sing} \text{ or } \dim(\pi^{-1}(x)) \geq 1\}.$$

*In particular, the singular locus of  $X(M)$  is a union of projection varieties.*

*Remark 4.18.* More generally, the singular locus of any projection variety in an arbitrary  $G/P$  is a union of projection varieties. However, it is more complicated to determine the singular locus as above. It is an interesting open problem to find an explicit characterization in general.

The basic observation that allows us to characterize the singular loci of projection varieties is the following.

**Lemma 4.19.** *Let  $f : X \rightarrow Y$  be a birational morphism of normal, projective varieties such that the exceptional locus  $E$  of  $f$  (i.e., the locus in  $X$  where  $f$  fails to be an isomorphism) has codimension at least two. Then  $Y^{sing} = f(X^{sing} \cup E)$ .*

*Proof.* The map  $f$  gives an isomorphism between  $X - E$  and  $Y - f(E)$ . Hence,  $(Y - f(E))^{sing} = f((X - E)^{sing})$ . Consequently, the content of the lemma is that  $f(E) \subset Y^{sing}$ . It is well-known that  $Y$  is badly singular along  $f(E)$ . For example,  $Y$  cannot even be  $\mathbb{Q}$ -factorial along  $f(E)$ . To see this, note that by Zariski's Main Theorem, the fibers of  $f$  over the points of  $f(E) \subset Y$  are positive dimensional. Since the question is local on  $Y$ , by replacing  $Y$  by a Zariski open neighborhood containing  $y$ , we may assume that  $f(E) = y$ . Let  $C$  be a curve in the fiber of  $f$  over  $y$ . Let  $D$  be a divisor on  $X$  associated to a section of a very ample line bundle  $A$ . Since the exceptional locus of  $f$  has codimension at least 2,  $f(D)$  is a Weil-divisor on  $Y$  containing  $y$ . Suppose  $f(D)$  were  $\mathbb{Q}$ -Cartier at  $y$ . Then  $mf(D)$  would be the class of a line bundle  $M$  for some  $m > 0$ . Let  $L$  be an ample line bundle on  $Y$  such that  $L \otimes M^{-1}$  is ample. Then  $f^*(L \otimes M^{-1})$  is the pull-back of an ample line bundle by a birational map, hence it is NEF. In particular, it has non-negative degree on the curve  $C$ . However, the degree of  $f^*L$  on  $C$  is zero and the degree of  $f^*(M^{-1}) = A^{\otimes -m}$  is negative. We thus get a contradiction. We conclude that  $f(D)$  cannot be  $\mathbb{Q}$ -Cartier at  $y$ . This concludes the proof of the lemma.  $\square$

*Proof of Corollary 4.17.* Consider the map  $\pi : R(u, v) \rightarrow X(M)$ . Since  $X(M)$  is normal and the map is birational, by Zariski's Main Theorem,  $\pi$  is an isomorphism over the locus

$$U = \{x \in X(M) \mid \dim(\pi^{-1}(x)) = 0\}.$$

Since the exceptional locus of  $\pi$  has codimension at least two, by the previous lemma,  $X(M)$  is singular along  $X(M) - U$ . Since over  $U$  the map  $\pi$  is an isomorphism,  $x \in U$  is singular if and only if  $\pi^{-1}(x)$  is singular in  $R(u, v)$ . This concludes the proof of the Corollary.  $\square$

Note that Corollary 2.9 and Corollary 4.17 explicitly determine the irreducible components of the singular locus of projection varieties.

*Example 4.20.* We give a simple example showing how to find the singular locus of a projection variety. Let  $M$  be the rank set

$$W_1 = [e_1, e_6], W_2 = [e_3, e_4], W_3 = [e_5, e_{10}], W_4 = [e_7, e_8].$$

Let  $M_1$  be the rank set

$$W_1^1 = [e_3], W_2^1 = [e_4], W_3^1 = [e_5, e_{10}], W_4^1 = [e_7, e_8].$$

Let  $M_2$  be the rank set

$$W_1^2 = [e_1, e_6], W_2^2 = [e_3, e_4], W_3^2 = [e_7], W_4^2 = [e_8].$$

Then  $X(M_1)$  and  $X(M_2)$  are the loci over which  $\pi$  has positive dimensional fibers. Let  $M_3$  be the rank set

$$W_1^3 = [e_1, e_5], W_2^3 = [e_3, e_4], W_3^3 = [e_5, e_6], W_4^3 = [e_7, e_8].$$

Let  $M_4$  be the rank set

$$W_1^4 = [e_3, e_4], W_2^4 = [e_5, e_6], W_3^4 = [e_6, e_{10}], W_4^4 = [e_7, e_8].$$

By Corollary 2.9,  $X(M_3)$  and  $X(M_4)$  are the loci where  $\pi^{-1}$  is singular. We conclude that  $X(M)$  is singular along  $\bigcup_{i=1}^4 X(M_i)$ .

Finally, we obtain a generalization of Corollary 2.13.

*Definition 4.21.* Let  $j \leq k$  and let  $m \leq n - k + j$ . Let  $V$  be an  $n$ -dimensional vector space and let  $T$  and  $U$  be  $m$  and  $(k - j)$ -dimensional subspaces of  $V$  such that  $T \cap U = \{0\}$ . A *linearly embedded sub-Grassmannian*  $G(j, m)$  in  $G(k, n)$  is the image of  $\phi : G(j, T) \hookrightarrow G(k, V)$  under the map  $\phi : W \mapsto W \oplus U$ . A *Segre product of linearly embedded sub-Grassmannians* is a product of linearly embedded sub-Grassmannians followed by the Segre embedding

$$G(j_1, m_1) \times \cdots \times G(j_r, m_r) \hookrightarrow G(k_1, n_1) \times \cdots \times G(k_r, n_r) \hookrightarrow G\left(\sum k_i, n\right).$$

as described in Definition 2.12.

*Remark 4.22.* A linearly embedded sub-Grassmannian  $G(j, m) \subset G(k, n)$  is a smooth Schubert variety with class  $\sigma_{(n-k)k-j, (n-m+k-j)j}$ . In fact, every smooth Schubert variety in  $G(k, n)$  is a linearly embedded sub-Grassmannian.

**Corollary 4.23.** *Let  $X$  be a rank variety with rank set  $M$ . The following are equivalent.*

- (1)  $X$  is smooth.
- (2)  $X$  is a Segre product of linearly embedded sub-Grassmannians.
- (3)  $M$  is a union of 1-dimensional subspaces and rank sets on disjoint intervals which correspond with sub-Grassmannians after quotienting out by the 1-dimensional subspaces.

For example,  $G(2, 4)$  is the smooth rank variety with rank set  $\{[e_1, e_3], [e_2, e_4]\}$ . In  $G(7, 12)$ , the rank set

$$M = \{[e_1, e_5], [e_2, e_6], [e_3, e_7], [e_4], [e_8, e_{10}], [e_9, e_{11}], [e_{12}]\}$$

corresponds with a smooth rank variety isomorphic to the product of  $G(3, 6) \times G(2, 4)$ .

*Proof.* Let  $X$  be a Segre product of linearly embedded sub-Grassmannians

$$G(j_1, m_1) \times \cdots \times G(j_s, m_s) \hookrightarrow G(k_1, n_1) \times \cdots \times G(k_s, n_s) \hookrightarrow G(k, n).$$

Then  $X$  is smooth. We need to show that we can realize  $X$  as a rank variety. The Segre product of rank varieties  $X(M_i)$  is a rank variety corresponding to the concatenation of the corresponding rank set. A Schubert variety in  $G(k_i, n_i)$  is a rank variety since it is a Richardson variety in the Grassmannian. Since a linearly embedded sub-Grassmannian is a Schubert variety, we conclude that a Segre product of linearly embedded sub-Grassmannians is a smooth rank variety.

Conversely, suppose that  $X$  is a smooth rank variety. We show that  $X$  has to be a Segre product of linearly embedded sub-Grassmannians. Consider the corresponding rank set  $M$ . If  $M$  contains only one vector space, then  $X(M)$  is projective space and the corollary holds. If the color of all the vector spaces in the rank set is one, then  $X(M)$  is a Richardson variety in the Grassmannian and the corollary holds by Corollary 2.13. Now we will do induction on the number of vector spaces defining  $M$ . We may assume that there are some vector spaces in  $M$  assigned a color larger than one. Let  $W$  be a subspace in the rank set that is assigned the color 1. If  $\dim(W) = 1$ , then we can replace every vector space  $W_i$  in the rank set  $M$  by  $W_i/W$ . We obtain a new rank set  $M'$  for

$G(k-1, n-1)$  with one fewer vector space. The map  $f : X(M') \rightarrow X(M)$  sending  $\Lambda \in X(M')$  to the span of  $\Lambda$  and  $W$  is an isomorphism between  $X(M')$  and  $X(M)$ . By induction,  $X(M')$  is a Segre product of linearly embedded sub-Grassmannians. It follows that  $X(M)$  is a Segre product of linearly embedded Grassmannians. If  $\dim(W) > 1$ , we show that  $X(M)$  is singular. Take a vector space  $W'$  of color two containing  $W$ . Define a new set of vector spaces  $M'$  by replacing  $W'$  and  $W$  with the two vector spaces  $[e_{l(W)}, e_{r(W)-1}]$ ,  $[e_{l(W)+1}, e_{r(W)}]$ . If  $M'$  is a rank set, stop. The fiber of  $\pi$  over this locus is positive dimensional. Hence,  $X(M)$  is singular. If  $M'$  is not a rank set, normalize the set of vectors to obtain a rank set  $M''$ . Note that  $M''$  is non-empty and the fiber of  $\pi$  over  $X(M'')$  is positive dimensional. Hence,  $X(M)$  is singular. This concludes the proof.

The equivalence of (2) and (3) follows from the fact that  $G(k, n)$  is itself a rank variety corresponding with  $M = \{W_1, \dots, W_k\}$  where each  $W_i = [e_i, e_{n-k+i}]$ . □

There is a nice way to enumerate all the rank varieties in  $G(k, n)$  using the Stirling numbers of the second kind. In fact, if we  $q$ -count the rank varieties according to dimension, we get a well known  $q$ -analog of the Stirling numbers [14, 18, 36, 42].

Define the generating function

$$g[k, n] = \sum_M q^{\dim(X(M))}$$

where the sum is over all rank sets  $M$  for  $G(k, n)$ . Set  $g[k, n] = 0$  for  $k > n$ ,  $g[0, n] = 0$  for  $n > 0$  and  $g[0, 0] = 1$ . Let  $[k] = 1 + q + \dots + q^k$ .

**Lemma 4.24.** *The polynomials  $g[k, n]$  satisfy the recurrence*

$$g[k, n] = g[k, n-1] + [n-k+1] \cdot g[k-1, n-1]$$

for  $1 \leq k \leq n$ .

*Proof.* Every rank set  $M$  in  $G(k, n)$  is either a rank set in  $G(k, n-1)$  or it contains a subspace of the form  $[e_i, e_n]$ . In the latter case removing this subspace leaves a rank set  $M'$  in  $G[k-1, n-1]$  which does not include a subspace whose left endpoint is  $i$ . Observe that  $\dim(X(M)) - \dim(X(M'))$  equals  $n-i$  minus the number of subspaces in  $M'$  with left endpoint larger than  $i$ . Furthermore, for each  $0 \leq d \leq n-k$ , we can add a subspace with right endpoint  $n$  to  $M'$  to get a rank set in  $G(k, n)$  of dimension  $d + \dim(X(M))$  by choosing the left endpoint to be the  $d$ -th largest value in  $\{1, 2, \dots, n\} - \{r(W) : W \in M'\}$ . □

Recall that the Stirling numbers of the second kind  $S(n, k)$  count the number of set partitions of  $\{1, \dots, n\}$  into  $k$  nonempty blocks. Let  $S[n, k]$  be the  $q$ -analog of  $S(n, k)$  defined by the recurrence

$$S[n, k] = q^{k-1}S[n-1, k-1] + [k]S[n-1, k]$$

with initial conditions  $S[0, 0] = 1$ ,  $S[n, 0] = 0$  for  $n > 0$ , and  $S[n, k] = 0$  for  $k > n$ . One can show that  $S[n, k]$  is divisible by  $q^{\binom{k}{2}}$ . Then, simple algebraic manipulations prove the following corollary to Lemma 4.24.

**Corollary 4.25.** *For  $1 \leq k \leq n$ , we have  $g[k, n] = S[n+1, n-k+1] \cdot q^{-\binom{n-k+1}{2}}$ .*



Below are the 25 rank sets for  $G(2, 4)$  listed by dimension. Thus  $g[2, 4] = 6 + 8q + 7q^2 + 3q^3 + q^4$ . Here  $(34, 123)$  means the rank set consisting of two subspaces spanned by  $\langle e_3, e_4 \rangle$  and  $\langle e_1, e_2, e_3 \rangle$ .

*dim ranksets*

0 : (2, 1), (3, 1), (4, 1), (3, 2), (4, 2), (4, 3)  
 1 : (23, 1), (34, 1), (3, 12), (4, 12), (2, 123), (34, 2), (4, 23), (3, 234)  
 2 : (234, 1), (23, 12), (34, 12), (4, 123), (2, 1234), (3, 1234), (34, 23)  
 3 : (234, 12), (34, 123), (23, 1234)  
 4 : (234, 123)

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