SYMPLECTIC RESTRICTION VARIETIES AND GEOMETRIC BRANCHING RULES II

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To Joe Harris, in celebration of his sixtieth birthday

Abstract. In this paper, we introduce combinatorially defined subvarieties of symplectic flag varieties called *symplectic restriction varieties*. We study their geometric properties and compute their cohomology classes. In particular, we give a positive, combinatorial, geometric branching rule for computing the map in cohomology induced by the inclusion of the symplectic partial flag variety $SF(k_1, \dots, k_h; n)$ in the partial flag variety $F(k_1, \dots, k_h; n)$. These rules have many applications in algebraic geometry, combinatorics, symplectic geometry and representation theory.

Contents

1. Introduction

Let $n = 2m$ be an even integer. Let $0 < k_1 < k_2 < \cdots < k_h \leq m$ be an increasing sequence of positive integers. For ease of notation, set $k_0 = 0$ and $k_{h+1} = n$. Let V be an n-dimensional vector space over $\mathbb{C}$ and let Q be a non-degenerate skew-symmetric form on V. The symplectic isotropic partial flag variety $SF(k_1, \dots, k_h; n)$ parameterizes partial flags

$$W_1 \subset W_2 \subset \cdots \subset W_h,$$

where W_i is a k_i-dimensional isotropic subspace of V with respect to Q.

The purpose of this paper is to give a positive, geometric rule for computing the restriction coefficients of $SF(k_1, \dots, k_h; n)$. Since every isotropic linear space is in particular a linear space, there is a natural inclusion

$$i : SF(k_1, \dots, k_h; n) \to F(k_1, \dots, k_h; n)$$

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of the isotropic partial flag variety into the flag variety. This inclusion induces a map in cohomology

$$i^* : H^*(F(k_1, \dots, k_h; n), \mathbb{Z}) \to H^*(SF(k_1, \dots, k_h; n), \mathbb{Z}).$$

The cohomology groups of both $SF(k_1, \dots, k_h; n)$ and $F(k_1, \dots, k_h; n)$ have integral bases consisting of Schubert classes. We will index Schubert classes in $F(k_1, \dots, k_h; n)$ by pairs of sequences $(a_\bullet, c_\bullet)$ and Schubert classes in $SF(k_1, \dots, k_h; n)$ by triples of sequences $(\lambda_\bullet; \mu_\bullet; c_\bullet)$. We refer the reader to §2 below for the definitions of Schubert classes and our conventions. The image

$$i^*\sigma_{a_\bullet, c_\bullet} = \sum \gamma^{a_\bullet, c_\bullet}_{\lambda^j_\bullet; \mu^j_\bullet; c^j_\bullet} \sigma_{\lambda^j_\bullet; \mu^j_\bullet; c^j_\bullet}$$

of a Schubert class $\sigma_{a_\bullet, c_\bullet}$ in the cohomology of $F(k_1, \dots, k_h; n)$ under i^* is a linear combination of Schubert classes in the cohomology of $SF(k_1, \dots, k_h; n)$ with nonnegative integer coefficients. The coefficients $\gamma^{a_\bullet, c_\bullet}_{\lambda^j_\bullet; \mu^j_\bullet; c^j_\bullet}$ are called *symplectic restriction coefficients*. In [C2], we computed the orthogonal restriction coefficients. In [C3], we gave a positive, geometric rule for computing the restriction coefficients for the symplectic isotropic Grassmannian $SG(k, n)$. In this paper, we extend the rule to the setting of partial flag varieties, thereby completing the program of finding positive, geometric rules for the restriction coefficients of all classical flag varieties. The following theorem is the main theorem of the paper.

Theorem 1.1. *Algorithm 3.58 provides an explicit, geometric, combinatorial, positive rule for computing the symplectic restriction coefficients for $SF(k_1, \dots, k_h; n)$.*

We will repeat the main definitions from [C3] in order to make this paper as self-contained as possible. However; before tackling the additional combinatorial complications of the flag case, the reader may wish to consult [C3] for the Grassmannian case. As in [C3] and [C2], the rule given in this paper is obtained using a sequence of specializations. These specializations transform a general flag into an isotropic flag by increasing the dimension of the kernel of the restriction of Q to the flag elements. The rank of a skew-symmetric form is even. This imposes strong constraints on possible specializations in the type C case and makes the rule for the symplectic restriction coefficients more complicated than the orthogonal case given in [C2]. In the orthogonal case, one can degenerate a linear space to increase the dimension of the kernel by one. Consequently, the geometry of the specialization is far more controlled. In the symplectic case, one can only increase the dimension of the kernel by even integers. This difference accounts for the combinatorial complexity of the type C case such as the more complicated ordering of the integers in a sequence in §3 or the fact that there are five cases in Definition 3.37.

The rule in [C3] forms the base case and motivates many of the constructions in the flag case. When $h = 1$, the rule given in this paper reduces to the rule given in [C3]. However, the geometry and combinatorics of the symplectic flag variety $SF(k_1, \dots, k_h; n)$ are considerably more complicated than that of $SG(k, n)$. While the order of specializations is the same in both cases, in the flag case, one needs to keep track of all h flag elements as opposed to one linear space–resulting, necessarily, in more complicated combinatorial statements. The number of limits in the specializations is much larger (as large as $h + 1$ at each stage as opposed to 2). Most importantly, certain shortcuts that simplify the geometry and the combinatorics in the Grassmannian case are no longer available and need to be replaced by new algorithms such as Algorithm FixA2 (3.52). Even turning the initial problem into a form where degenerations can be applied inductively is not as straightforward in the flag case and requires a 'Pieri-like' rule. Consequently, the rule given here is not an immediate extension of the rule in [C3].

Theorem 1.1 has many applications, most notably to calculating moment polytopes and asymptotics of the restrictions of representations of SL_n to Sp_n. Let $j : G' \to G$ be an inclusion of complex, reductive, connected Lie groups. Choose Borel subgroups $B' \subset G'$ and $B \subset G$ such that $j(B') \subset B$. Then the inclusion $j : G'/B' \to G/B$ induces a map in cohomology $j^* : H^*(G/B) \to H^*(G'/B')$. The structure coefficients of this map in terms of Schubert bases are called *branching coefficients*. Finding positive rules for calculating branching coefficients is a central problem with a long history. We refer the reader to [P], [RR] and [R] for references, a detailed history and recent developments on computing branching coefficients. In the case of Sp_n and SL_n, the map j is given by sending an isotropic flag $F_\bullet$ to the pair $(F_\bullet, F_\bullet^\perp)$. Our theorem calculates all the branching coefficients of $j^* : H^*(F(1,2,\dots,n-1;n)) \to H^*(SF(1,2,\dots,m;n))$ for the classes that are pulled back from $F(1,2,\dots,m;n)$ under the natural projection that sends $(F_\bullet, F_\bullet^\perp)$ to $F_\bullet$.

Knowing the set of non-zero branching coefficients has important applications in symplectic geometry and representation theory. Let K and K' be maximal compact subgroups of G and G', respectively. To each non-vanishing branching coefficient, in [BS], Berenstein and Sjamaar associate an inequality satisfied by the K'-moment polytope of a K-coadjoint orbit. Moreover, the totality of these inequalities gives a sufficient set of inequalities for the moment polytope. Similarly, non-vanishing branching coefficients determine which irreducible representations of G' occur in the restriction of an irreducible representation of G asymptotically (see [BS], [GS], [He], [P]).

More importantly, we will introduce a new set of subvarieties of $SF(k_1,\dots,k_h;n)$ called *symplectic restriction varieties* and compute their cohomology classes in terms of the Schubert basis of $SF(k_1,\dots,k_h;n)$. The analogues of these varieties for orthogonal flag varieties were introduced in [C2]. In many ways, these varieties are more fundamental than Schubert varieties and have applications to many other geometric problems such as problems of smoothability and rigidity (see [C1], [C3], [C4]). Symplectic restriction varieties are defined by imposing rank conditions on a partial flag $(W_1,\dots,W_h)$ with respect to a not-necessarily-isotropic flag. They interpolate between the intersection of $SF(k_1,\dots,k_h;n)$ with a general translate of a Schubert variety in $F(k_1,\dots,k_h;n)$ and a Schubert variety in $SF(k_1,\dots,k_h;n)$. We will discuss their geometric properties in detail in §4.

The beauty of our approach is that, while the combinatorics of branching coefficients can be very complicated (and this is inevitably reflected in the combinatorial formulation of the rule), four basic geometric principles explain all the complexity. Our strategy for calculating the cohomology classes of symplectic restriction varieties is by specialization. We start with a general Schubert variety defined with respect to a flag $G_\bullet$. Since the flag is general, the dimension of the kernel of the restriction of Q to G_i is zero or one depending on whether i is even or odd. On the other hand, a Schubert variety in $SF(k_1,\dots,k_h;n)$ is defined with respect to an isotropic flag $F_\bullet$. The dimension of the kernel of the restriction of Q to F_i is i (respectively, $n-i$) if $i \le m$ (respectively, if $i > m$). The strategy is to transform the flag $G_\bullet$ to an isotropic flag by increasing the dimension of the kernel of the restriction of Q one flag element at a time. As we specialize the flag, the corresponding restriction variety specializes into a union of restriction varieties defined with respect to the limiting flag, each occurring with multiplicity one. The rule records the outcome of the specialization. We now explain the four basic facts about skew-symmetric forms that govern the order of the specialization and the limits that occur.

Let Q_d^r denote a d-dimensional vector space such that the restriction of Q has corank r. Let $\mathrm{Ker}(Q_d^r)$ denote the kernel of the restriction of Q to Q_d^r. Let L_j denote an isotropic subspace of dimension j with respect to Q. Let $L_j^\perp$ denote the set of $w \in V$ such that $w^T Q v = 0$ for all $v \in L_j$. The reader can easily verify the following four basic facts about skew-symmetric forms.

Evenness of rank. The rank of a skew-symmetric form is even. Hence, $d-r$ is even for Q_d^r. Furthermore, if $d=r$, then Q_d^r is isotropic.

The corank bound. Let $Q_{d_1}^{r_1} \subset Q_{d_2}^{r_2}$ and let $r_2' = \dim(\mathrm{Ker}(Q_{d_2}^{r_2}) \cap Q_{d_1}^{r_1})$. Then $r_1 - r_2' \leq d_2 - d_1$. In particular, $d+r \leq n$ for Q_d^r.

The linear space bound. The dimension of an isotropic subspace of Q_d^r is bounded above by $\lfloor \frac{d+r}{2} \rfloor$. Furthermore, an m-dimensional isotropic subspace L of Q_d^r satisfies $\dim(L \cap \mathrm{Ker}(Q_d^r)) \geq m - \lfloor \frac{d-r}{2} \rfloor$.

The kernel bound. Let L be an $(s+1)$-dimensional isotropic space such that $\dim(L \cap \mathrm{Ker}(Q_d^r)) = s$. If an isotropic linear subspace M of Q_d^r intersects $L - \mathrm{Ker}(Q_d^r)$, then M is contained in $L^{\perp}$.

Let us explain how these four principles dictate the order of the specialization and determine the limits that occur. Given a flag, we will specialize the smallest dimensional non-isotropic subspace Q_d^r, whose corank can be increased subject to the corank bound, keeping all other flag elements unchanged. We will replace Q_d^r with $\tilde{Q}_d^{r+2}$. The branching rule simply says that under this specialization, the limit $(W_1', \dots, W_h')$ of the one-parameter family of partial flags $(W_1, \dots, W_h)(t)$ satisfy the same rank conditions with the unchanged flag elements and $\dim(W_i' \cap \mathrm{Ker}(\tilde{Q}_d^{r+2})) = \dim(W_i \cap \mathrm{Ker}(Q_d^r))$ for $i < i_0$ and $\dim(W_i' \cap \mathrm{Ker}(\tilde{Q}_d^{r+2})) = \dim(W_i \cap \mathrm{Ker}(Q_d^r)) + 1$ for $i \geq i_0$. Furthermore, all of these cases occur with multiplicity one unless some of these loci lead to smaller dimensional varieties or the linear space bound is violated. These exceptions can be explicitly stated combinatorially (though sometimes resulting in cumbersome statements). See §3 and §5 for an explicit statement of the rule and for examples.

There are other methods for computing restriction coefficients. For example, Pragacz gave a positive rule for computing restriction coefficients for Lagrangian Grassmannians [Pr1], [Pr2]. It is also possible to compute restriction coefficients (in a non-positive way) by first computing the pullbacks of the tautological bundles from $F(k_1, \dots, k_h; n)$ to $SF(k_1, \dots, k_h; n)$ and then using localization or the theory of Schubert polynomials to express the Chern classes of these bundles in terms of Schubert classes. To the best of the author's knowledge, Algorithm 3.58 is the first positive, geometric rule for computing the restriction coefficients for all isotropic partial flag varieties $SF(k_1, \dots, k_h; n)$. In the author's experience, the positive rule given here is computationally much more efficient than their non-positive counterparts.

The organization of this paper is as follows. In §2, we will recall basic facts concerning the geometry of $SF(k_1, \dots, k_h; n)$. In §3, we will introduce symplectic diagrams, which are the main combinatorial objects of this paper. We will then state the rule combinatorially. This section is written so that a reader who skips all the remarks can read the rule without any reference to geometry. The remarks give the geometric intuition behind each combinatorial definition. In §4, we will introduce symplectic restriction varieties and discuss their basic geometric properties. In §5, we will interpret the combinatorial rule geometrically and prove that it computes the symplectic restriction coefficients.

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2. Preliminaries

In this section, we recall basic facts concerning the geometry of $SF(k_1, \dots, k_h; n)$.

Let V be an n-dimensional vector space over the complex numbers endowed with a non-degenerate skew-symmetric form Q. Since Q is non-degenerate, n must be even. Set $n = 2m$. A linear space $W \subset V$ is isotropic with respect to Q if for every $w_1, w_2 \in W$, $w_1^T Q w_2 = 0$. Given an isotropic space W, let the orthogonal complement $W^\perp$ be the set of vectors $v \in V$ such that $v^T Q w = 0$ for every $w \in W$. If the dimension of W is k, then the dimension of $W^\perp$ is $n - k$.

The partial flag variety $SF(k_1, \ldots, k_h; n)$ parameterizing partial flags $W_1 \subset \cdots \subset W_h$, where W_i is an isotropic subspace of V of dimension k_i, is a homogeneous variety for the symplectic group Sp_n. Let $SG(k, n)$ denote the symplectic isotropic Grassmannian parameterizing k-dimensional isotropic subspaces of V. There is a natural projection map

$$\pi_h : SF(k_1, \ldots, k_h; n) \to SG(k_h, n), \quad \text{mapping} \quad (W_1, \ldots, W_h) \mapsto W_h.$$

The fiber of π_h over a point W_h is the partial flag variety $F(k_1, \ldots, k_{h-1}; k_h)$. The geometric properties of $SF(k_1, \ldots, k_h; n)$ can be deduced by studying the geometry of $SG(k_h, n)$ and the projection map π_h. For example, the dimension of $SF(k_1, \ldots, k_h; n)$ is

$$\dim(SF(k_1, \ldots, k_h; n)) = nk_h - \frac{3k_h^2 - k_h}{2} + \sum_{i=1}^{h-1} k_i(k_{i+1} - k_i),$$

given by the sum of the dimensions of $SG(k_h, n)$ [C3] and $F(k_1, \ldots, k_{h-1}; k_h)$.

By Ehresmann's Theorem [E] (see also [Bo, IV, 14.12]), the cohomology of $SF(k_1, \ldots, k_h; n)$ is generated by the classes of Schubert varieties. Our indexing for Schubert varieties will take into account the projection map π_h. Let

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_s \leq m$$

be a sequence of increasing positive integers. Let

$$m > \mu_{s+1} > \mu_{s+2} > \cdots > \mu_{k_h} \geq 0$$

be a sequence of decreasing non-negative integers such that $\lambda_i \neq \mu_j + 1$ for any $1 \leq i \leq s$ and $s < j \leq k_h$. Then the Schubert varieties in $SG(k_h, n)$ are parameterized by pairs of admissible sequences $(\lambda_\bullet; \mu_\bullet)$ [C3].

Definition 2.1. A *coloring* $c_\bullet$ for a sequence $0 < k_1 < \cdots < k_h \leq m$ is a sequence of integers

$$c_1, c_2, \ldots, c_{k_h}$$

such that $k_i - k_{i-1}$ of the integers are equal to i for $1 \leq i \leq h$. A coloring for $SF(k_1, \ldots, k_h; n)$ or $F(k_1, \ldots, k_h; n)$ is a coloring for the corresponding sequences $k_1 < \cdots < k_h$.

Example 2.2. The sequences $1, 1, 2, 2, 3, 3, 3$ and $1, 3, 3, 2, 3, 2, 1$ are two coloring schemes for $SF(2, 4, 7; n)$. Note that k_i is equal to the number of integers in the sequence that are less than or equal to i.

We parameterize Schubert classes in $SF(k_1, \ldots, k_h; n)$ by colored sequences $(\lambda_\bullet; \mu_\bullet; c_\bullet)$, where $(\lambda_\bullet; \mu_\bullet)$ is an admissible sequence for $SG(k_h, n)$ and $c_\bullet$ is a coloring for $SF(k_1, \ldots, k_h; n)$. Fix an isotropic flag

$$F_\bullet = F_1 \subset F_2 \subset \cdots F_m \subset F_{m-1}^\perp \subset \cdots F_1^\perp \subset V.$$

The Schubert variety $\Sigma_{\lambda_\bullet; \mu_\bullet; c_\bullet}(F_\bullet)$ is defined as the Zariski closure of the set of partial flags

$$\{(W_1, \ldots, W_h) \in SF(k_1, \ldots, k_h; n) \mid \dim(W_j \cap F_{\lambda_i}) = \#\{c_l \mid l \leq i, c_l \leq j\} \text{ for } 1 \leq i \leq s,$$
$$\dim(W_j \cap F_{\mu_i}^\perp) = \#\{c_l \mid l \leq i, c_l \leq j\} \text{ for } s < i \leq k_h\}.$$

The projection $\pi_h(\Sigma_{\lambda_\bullet;\mu_\bullet;c_\bullet})$ is a Schubert variety in $SG(k_h,n)$ with class $\sigma_{\lambda_\bullet;\mu_\bullet}$. The sequence $c_\bullet$ determines a permutation with $h-1$ descents sending an integer $k_{i-1} < \alpha = k_{i-1}+j \leq k_i$ to the position of the j-th number equal to i in the sequence $c_\bullet$. Over a general point W_h in the image of the projection $\pi_h(\Sigma_{\lambda_\bullet;\mu_\bullet;c_\bullet})$, the fiber is a Schubert variety in $F(k_1,\dots,k_{h-1};k_h)$ with class determined by the sequence $c_\bullet$. More explicitly, at a general point W_h of the image, the isotropic flag $F_\bullet$ defines a complete flag $G_\bullet$ on W_h. The Schubert variety in $F(k_1,\dots,k_{h-1};k_h)$ is defined by

$$\{(W_1,\dots,W_{h-1}) \mid \dim(W_i \cap G_j) \geq \#\{c_l \mid l \leq j, c_l \leq i\} \text{ for } 1 \leq i \leq h-1\}.$$

Definition 2.3. For $1 \leq u < h$, define $\mathrm{cdim}(u)$, the *codimension of the color* u, by

$$\mathrm{cdim}(u) = \sum_{1 \leq i \leq k_h,\ c_i \leq u} \#\{j > i \mid c_j = u+1\}.$$

Define $\mathrm{cdim}(c_\bullet)$, the codimension of a coloring, by

$$\mathrm{cdim}(c_\bullet) = \sum_{u=1}^{h-1} \mathrm{cdim}(u).$$

Define the dimension of a coloring $\dim(c_\bullet) = \sum_{u=1}^{h-1} k_u(k_{u+1}-k_u) \ - \mathrm{cdim}(c_\bullet)$.

Remark 2.4. The quantity $\dim(c_\bullet)$ is nothing other than the dimension of the Schubert variety in $F(k_1,\dots,k_{h-1};k_h)$ determined by the sequence $c_\bullet$. This can be seen by induction on the number of colors h. Let $c_\bullet^u$ be the two-color sequence obtained from $c_\bullet$ by deleting all integers greater than $u+1$ and changing all the integers less than u to u (keeping the integers equal to u and $u+1$ unchanged). The partial flag variety admits projections

$$F(k_1,\dots,k_{h-1};k_h) \xrightarrow{\rho_1} F(k_2,\dots k_{h-1};k_h) \xrightarrow{\rho_2} \cdots \xrightarrow{\rho_{h-2}} F(k_{h-1};k_h),$$

where the generic fiber of ρ_u is the Grassmannian $G(k_u,k_{u+1})$. The restriction of a Schubert variety with class $c_\bullet$ to the generic fiber of the projection ρ_u is a Schubert variety in $G(k_u,k_{u+1})$ associated to the sequence $c_\bullet^u$, which has codimension $\mathrm{cdim}(u)$ (see [BC] and [C5] for more details about the geometry of type A flag varieties). The claim now follows by induction on h. Combining this discussion with [C3, Proposition 4.21], we conclude that a Schubert variety with class indexed by $(\lambda_\bullet;\mu_\bullet;c_\bullet)$ has dimension

$$\sum_{i=1}^{s}(\lambda_i - i) + \sum_{j=s+1}^{k_h}(n-\mu_j+1-2j+\#\{\lambda_i \mid \lambda_i \leq \mu_j\}) + \dim(c_\bullet).$$

We denote Schubert varieties in the ordinary flag variety $F(k_1,\dots,k_h;n)$ by pairs $(a_\bullet,c_\bullet)$, where $a_\bullet$ is a sequence of increasing positive integers $0 < a_1 < \cdots < a_{k_h} \leq n$ and $c_\bullet$ is a coloring. The corresponding Schubert variety $\Sigma_{a_\bullet,c_\bullet}(F_\bullet)$ is the Zariski closure of the locus

$$\{(W_1,\dots,W_h) \mid \dim(W_j \cap F_{a_i}) = \#\{c_l \mid l \leq i, c_l \leq j\} \text{ for } 1 \leq j \leq h\}.$$

3. The combinatorial rule

In this section, we define combinatorial objects called colored symplectic diagrams, which represent the main geometric objects, the symplectic restriction varieties, of this paper. We then describe an algorithm for computing their cohomology classes in terms of Schubert cycles. We will give some geometric motivation for the combinatorial definitions in remarks. However, the main geometric content of the paper is contained in the next two sections. The combinatorially minded reader may skip these remarks to read the rule without any reference to geometry. The geometrically minded reader may wish to look ahead to the next two sections for the geometric reasons behind the combinatorics.

Remark 3.1. Let V be a vector space of dimension $n = 2m$ and let Q be a non-degenerate skew-symmetric form on V. Let $e_1, \dots, e_m, f_1, \dots, f_m$ be a basis of V that diagonalizes the skew-symmetric form Q, with $f_i = e_i^*$. Initially, the algorithm starts with a partial flag defined with respect to the ordered basis $e_1, f_1, e_2, f_2, \dots, e_m, f_m$. The goal of the algorithm is to turn this flag into a flag defined with respect to the ordered basis $e_1, e_2, \dots, e_m, f_m, \dots, f_2, f_1$ via a sequence of specializations that at each stage swap two of the basis elements. The combinatorial algorithm records the transformation of the flags under these specializations. The translation between diagrams and ordered bases of V is given in §4.2. The crucial geometric information is the dimension of the kernel of the restriction of Q to each flag element and how these kernels intersect. Colored symplectic diagrams are a combinatorial shorthand for recording this data.

A colored (admissible) symplectic diagram for $SF(k_1, \dots, k_h; n)$ is an (admissible) symplectic diagram for $SG(k_h, n)$ (defined in [C3]) together with a coloring for $SF(k_1, \dots, k_h; n)$. For the convenience of the reader, we now recall the definition of an admissible symplectic diagram for $SG(k_h, n)$.

Let $0 \leq s \leq k_h$ be an integer. A *sequence of n natural numbers of type s* is a sequence of n natural numbers such that every number is less than or equal to $k_h - s$. We write the sequence from left to right. We refer to the position after the i-th number in the sequence as the *i-th position*. For example, 1 1 2 0 0 0 0 0 and 3 0 0 2 0 1 0 0 are two sequences of 8 natural numbers of types 1 and 0, respectively, for $k_h = 3$.

Definition 3.2. A *sequence D of brackets and braces of type s* for $SG(k_h, n)$ consists of a sequence of n natural numbers of type s, s brackets] ordered from left to right and $k_h - s$ braces } ordered from *right to left* such that:

(1) Every bracket or brace occupies a positive position and each position is occupied by at most one bracket or brace.
(2) Every bracket is to the left of every brace.
(3) Every positive integer greater than or equal to i is to the left of the i-th brace.
(4) The total number of integers equal to zero or greater than i to the left of the i-th brace is even.

By Definition 2.1, a *coloring* for $SF(k_1, \dots, k_h; n)$ is a sequence of k_h integers $c_1, \dots, c_{k_h}$ such that $1 \leq c_j \leq h$ and the number of integers in the sequence equal to i is $k_i - k_{i-1}$ for $1 \leq i \leq h$. We then have the following definition of a colored sequence of brackets and braces.

Definition 3.3. A *colored sequence of brackets and braces* $(D, c_\bullet)$ of type s for $SF(k_1, \dots, k_h; n)$ is a sequence of brackets and braces of type s for $SG(k_h, n)$ together with a coloring $c_\bullet$ for $SF(k_1, \dots, k_h; n)$.

Notation 3.4. $11]_1 200\}_2 0\}_2 00$ and $300\}_2 20\}_1 10\}_2 0$ are typical examples of sequences of brackets and braces for $SF(1, 3; 8)$. The colorings in these two examples are $1, 2, 2$ and $2, 1, 2$, respectively. The coloring is recorded in the diagram as *subscripts* to the brackets or braces and is read from left to right in order.

Example 3.5. We give some examples that are disallowed by Definition 3.2. Condition (1) disallows $]_1 0000\}_2$ (the first bracket $]_1$ is not in a positive position), $00]_1]_2 00$, $0000\}_1\}_2$, $00]_1\}_2 00$ (two brackets, two braces or a bracket and a brace are in the same position). Condition (2) disallows $00\}_1 0]_2 000$ (a brace is to the left of a bracket). Condition (3) disallows $00\}_1 20\}_2 10$ (there is a 2 in the sequence to the right of the second brace and a 1 in the sequence to the right of the first brace). Condition (4) disallows $2000\}_1 00\}_2$ (the number of zeros to the left of the second brace is odd).

Notation 3.6. • By convention, the brackets are counted from left to right and the braces are counted from right to left. (Note the unusual ordering of the braces!) We write $]^i$ and $\}^i$ to denote the i-th bracket and i-th brace, respectively. The index of a bracket or a brace, which is denoted by a *superscript*, should not be confused with its color, which is denoted by a subscript.

• The positions of the brackets and braces are denoted by $p(]^i)$ and $p(\}^i)$. The position of a bracket or a brace is equal to the number of integers to its left.

• We denote the place of an integer i in the sequence (counted from left to right) by $p(i)$. Since an integer may occur more than once in the sequence, in case of ambiguity, we indicate its place by a subscript. Hence, i_j is the integer i that occurs in the j-th place in the sequence so that $p(i_j) = j$. Whether we are referring to the place of an integer in a sequence or the position of a bracket or brace is always clear from the context. Using the same notation for both should be less taxing on the reader without resulting in confusion.

• We say a bracket or brace at position p is to the left (resp., right) of a bracket or brace at position q if $p < q$ (resp, $p > q$). We say that an integer i is immediately to the left (right) of a bracket or brace at position p if $p(i) = p$ ($p(i) = p + 1$). Similarly, a bracket or brace at position p is immediately to the left (right) of an integer i if $p(i) = p + 1$ ($p(i) = p$).

• Let $l(i)$ denote the number of integers in the sequence that are equal to i.

• Let r_i denote the number of positive integers less than or equal to i to the left of $\}^i$.

• For $1 \leq i \leq k_h - s$, let $\rho(i, 0) = n - p(\}^i)$ and, for $1 \leq i < j \leq k_h - s$, let $\rho(j, i) = p(\}^i) - p(\}^j)$. Equivalently, $\rho(j, i)$ denotes the number of integers to the left of the i-th brace and to the right of the j-th brace and $\rho(i, 0)$ denotes the number of integers to the right of the i-th brace.

• When discussing several sequences simultaneously, to avoid confusion, we indicate each sequence with a subscript.

For the sequence of brackets and braces $1]_1 22]_3 00]_3 00\}_2 00\}_1 0$ for $SF(2, 3, 5; 10)$, the positions are $p(]^1) = 1, p(]^2) = 3, p(]^3) = 5, p(\}^2) = 7, p(\}^1) = 9$. The coloring is $1, 3, 3, 2, 1$. The numbers of integers in the sequence are $l(1) = 1$ and $l(2) = 2$. We have $r_1 = 1$, $r_2 = 3$. Finally, the number of integers between the braces are $\rho(2, 1) = 2$ and $\rho(1, 0) = 1$.

Remark 3.7. To give some context to the combinatorial objects, we remark that a sequence of brackets and braces records a partial flag, where brackets represent isotropic subspaces and braces represent non-isotropic subspaces. The sequence of integers corresponds to an ordering of the basis elements $e_1, \dots, f_m$ (see §4.2 for the precise correspondence). Each bracket or brace represents the vector space spanned by the basis elements associated to the integers to its left. Hence, the dimension of a vector space is equal to the position of the corresponding bracket or brace. The positive integers less than or equal to i to the left of $\}^i$ record the kernel of the restriction of Q to the corresponding non-isotropic space. The quantity r_i is the dimension of this kernel. The codimension of the subspace corresponding to $\}^i$ is $\rho(i, 0)$. The codimension of the subspace corresponding to $\}^j$ in the subspace corresponding to $\}^i$ is $\rho(j, i)$.

The conditions in Definition 3.2 reflect basic properties of skew-symmetric forms. Condition (1) requires every flag element to have positive dimension and the different flag elements to be distinct. Condition (2) says that a subspace of an isotropic space is isotropic. Condition (3) says that the kernel of the restriction of Q to the vector space represented by $\}^i$ is contained in the vector space represented by $\}^i$. Finally, condition (4) reflects the fact that the ranks of skew-symmetric forms are even (see also Remark 4.9).

Definition 3.8. Two colored sequences of brackets and braces $(D_1, c^1_\bullet)$ and $(D_2, c^2_\bullet)$ are *equivalent* if $c^1_\bullet = c^2_\bullet$, the length of sequences of integers in D_1 and D_2 are equal, the brackets and braces occur in the same positions and the integers occurring between any two consecutive brackets and/or braces are the same up to reordering.

Example 3.9. $3300\}_2 200\}_1 10\}_2 0$ and $3030\}_2 002\}_1 10\}_2 0$ are equivalent.

Note that each equivalence class can be represented by a *canonical representative*, where between any two brackets and/or braces the positive integers are listed in non-decreasing order followed by the zeros. Denote the canonical representative of a sequence D by $\mathrm{Can}(D)$. In the example, the first sequence is the canonical representative corresponding to the second sequence. We always represent sequences by their canonical representatives and often blur the distinction between the equivalence class and the canonical representative.

Definition 3.10. A colored sequence of brackets and braces of type s for $SF(k_1, \dots, k_h; n)$ is *saturated* if $\rho(i, i-1) = l(i)$ for every $1 \leq i \leq k_h - s$.

Remark 3.11. Note that shuffling the basis elements as in Definition 3.8 does not change any of the vector spaces recorded by the brackets or braces. We will choose our ordering to make the combinatorial statements simpler. A sequence is saturated if the dimension of the kernel of Q restricted to each non-isotropic subspace is as large as possible (as dictated by the corank bound; see §1).

Definition 3.12. A colored sequence of brackets and braces of type s for $SF(k_1, \dots, k_h; n)$ is *in order* if the sequence of numbers consists of a sequence of non-decreasing positive integers followed by zeros except possibly for one i at the place $p(\}^{i+1}) + 1$ (i.e., immediately to the right of $\}^{i+1}$) for $1 \leq i < k_h - s$. Otherwise, we say that the sequence is *not in order.* A sequence is *in perfect order* if the sequence of numbers consists of non-decreasing positive integers followed by zeros.

Example 3.13. The sequence $1]_1 333]_2 3]_2 00\}_1 0000\}_1 100\}_2 00$ is in order, but it is not saturated. The sequence $11]_1 22]_2 00\}_1 00\}_2 00$ is in perfect order and saturated. The sequences

$$11]_1 00]_2 100\}_1 000 \quad \text{and} \quad 24000000\}_1 0\}_2 1\}_3 0\}_2 0$$

are not in order.

Definition 3.14. A *marked sequence* $(D, c_\bullet, *)$ is a sequence $(D, c_\bullet)$ where either two equal integers or an integer and the place 0 are marked by a $*$. We denote by δ and $p(\delta)$ the rightmost marked integer and its place in the sequence. Let δ', if it exists, denote the leftmost marked integer. Let $p(\delta')$ denote the place in the sequence of the leftmost marking.

Example 3.15. $*00\}_1 1^*\}_2 000$ and $1^*2]_1 00]_2 00\}_3 1^*0\}_4 00$ are typical examples of marked sequences. In the first one, we have $p(\delta') = 0$, $p(\delta) = 3$. In the second, we have $p(\delta') = 1$ and $p(\delta) = 7$.

Let the *canonical representative* $\mathrm{Can}(D, c_\bullet, *)$ of a marked sequence $(D, c_\bullet, *)$ be $\mathrm{Can}(D, c_\bullet)$, where δ and δ' (resp., if δ' does not exist, the place 0) are preserved as the marked integers (resp., the marked integer and the marked place).

Example 3.16. The canonical representative of $*22]_1 022^*20\}_2 00000\}_1$ is $*22]_1 22^*200\}_2 00000\}_1$. The canonical representative of $21^*12]_1 001^*100\}_2 00\}_1 0000$ is $1^*122]_1 1^*10000\}_2 00\}_1 0000$.

Next, we recall the definition of a symplectic diagram for $SG(k_h, n)$ from [C3]. This definition is a technical definition. Every sequence of brackets and braces that occurs in the algorithm is automatically a (possibly-marked) symplectic diagram. Hence, the reader does not need to remember the conditions in this definition to run the algorithm. The properties are needed to ensure that colored symplectic diagrams correspond to subvarieties of $SF(k_1,\dots,k_h;n)$. They also play a role in the dimension counts. The marked diagrams make a brief appearance while running the loop in Steps 3 and 4 of Algorithm FixA2 (3.52) and their definition is included for precision with the necessary changes indicated in parentheses.

Remark 3.17. Before stating the definition, giving some geometric motivation might help the reader. We have not imposed enough conditions on sequences for them to correspond to geometrically meaningful objects. Currently, we have not ruled out sequences such as $222200\}_1 00\}_2$ or $000]_1 110\}_2 00$. In the first case, the restriction of Q to a codimension-two subspace (corresponding to $\}_1^2$) is supposed to have a four-dimensional kernel (depicted by the sequence 2222). This is impossible by the corank bound (see §1). In the second case, a six-dimensional non-isotropic subspace (corresponding to $\}_2^1$) is supposed to contain a five-dimensional isotropic subspace (given by the span of the three-dimensional isotropic subspace $]_1^1$ and the two-dimensional kernel of Q depicted by 11). This violates the linear space bound (see §1). The conditions (S1) and (S2) in Definition 3.18 will impose the corank bound and the linear space bound on sequences to avoid such pathologies (see also Remark 4.9).

More importantly, we have not imposed any conditions on the complexity of the sequence of integers. We do not wish to allow arbitrarily scrambled sequences but rather very closely controlled sequences. Conditions (S3) and (S4) in Definition 3.18 are consequences of the order of specialization. Swapping basis elements increases the dimension of the kernel of the restriction of Q to non-isotropic subspaces. It is very important to increase the dimension of the kernel for the smallest dimensional non-isotropic subspaces (as much as allowed by the corank bound) first. Otherwise, one loses control of the geometry. When one follows such an order, the sequence has a very particular shape as described by conditions (S3) and (S4).

Definition 3.18. A (marked) sequence of brackets and braces D (respectively, $(D,*)$) of type s for $SG(k_h,n)$ is a (*marked*) *symplectic diagram* if it satisfies the following conditions:

- (S1) $l(i) \leq \rho(i,i-1)$ for $1 \leq i \leq k_h - s$.
- (S2) Let τ_i be the sum of $p(]^s)$ and the number of positive integers between $]^s$ and $\}^i$. Then
$$2\tau_i \leq p(\}^i) + r_i, \quad \text{for} \quad 1 \leq i \leq k_h - s.$$
- (S3) Either the sequence is in order or there exists at most one (unmarked) integer $1 \leq \eta \leq k_h - s$ such that the sequence of integers is non-decreasing followed by a sequence of zeros except for at most one occurrence of η between $]^s$ and $\}^{\eta+1}$ and at most one occurrence of i at the place $p(\}^{i+1})+1$ for $1 \leq i < \eta$ (resp., $1 \leq i < \left\{ \begin{array}{ccc} \eta & \text{if} & p(\delta')>0 \\ k_h - s & \text{if} & p(\delta')=0 \end{array} \right.$).
- (S4) Let ξ_j denote the number of positive integers between $\}^j$ and $\}^{j-1}$. If a (unmarked) positive integer i occurs to the left of all the zeros, then either $i=1$ and there is a bracket in the position following it, or there exists at most one index j_0 such that $\rho(j,j-1) = l(j)$ for $j_0 \neq j > \min(i,\eta)$ and $\rho(j_0, j_0-1) \leq l(j_0) + 2 - \xi_{j_0}$. Moreover, in the latter case, any (unmarked) integer η violating order occurs to the right of $\}^{j_0}$.

Definition 3.19. A (marked) colored sequence of brackets and braces $(D, c_\bullet)$ (resp., $(D, c_\bullet, *)$) for $SF(k_1,\dots,k_h;n)$ is a *(marked) colored symplectic diagram* if D (resp., $(D,*)$) is a (marked) symplectic diagram for $SG(k_h,n)$.

Example 3.20. Condition (S3) allows sequences such as $2344]_1 300\}_2 00\}_1 00\}_2 10\}_1 0$ (the second 3 from the left is the only integer violating order), but disallows $22]_1 1100\}_2 22\}_1 0000\}_2 00$. Condition (S4) allows sequences such as $1]_1 33]_2 00\}_2 00\}_2 00\}_1 0$ and $11]_2 3300\}_2 00\}_1 1\}_1 000$, but disallows $144]_1 00\}_2 00\}_3 00\}_3 00\}_1 0$ (1 occurs in the initial non-decreasing part of the sequence and is not followed by a bracket, but 2 and 3 do not occur. On the other hand, $l(3) = 0 \neq \rho(3,2) = 2$, $l(2) = 0 \neq \rho(2,1) = 2$).

Next, we define admissible symplectic diagrams. The conditions in this definition are crucial for running the algorithm and the reader needs to remember them.

Remark 3.21. So far we have not imposed the kernel bound and the second part of the linear space bound (see §1) on our diagrams. The conditions (A1) and (A2) impose these bounds, respectively (see Remark 4.14).

Definition 3.22. A colored symplectic diagram $(D, c_\bullet)$ for $SF(k_1, \dots, k_h; n)$ is called *admissible* if D satisfies the following two conditions:

(A1) For every bracket $]^j$ with $p(]^j) > 1$, the integers in places $p(]^j)$ and $p(]^j) - 1$ are equal. If $p(]^1) = 1$ and $s < k_h$, then the integer in the first place is a 1.

(A2) Let x_i be the number of brackets $]^l$ such that every integer in the sequence to the left of $]^l$ is positive and less than or equal to i. Then, for $1 \leq i \leq k_h - s$,
$$x_i \geq k_h - i + 1 - \frac{p(\}^i) - r_i}{2}.$$

Example 3.23. $11]_1 22]_2 00\}_2 00\}_1 00$ and $22]_3 30000\}_2 0\}_1 100\}_3 0$ are admissible symplectic diagrams in $F(2,4;10)$ and $F(1,2,4;12)$, respectively. $2]_1 0000\}_2 10\}_1 0$ and $12]_2 0000\}_2 10\}_1 00$ are not admissible because they violate condition (A1). $22\}_2 00\}_1 00$ and $200\}_1 2\}_2 00$ are not admissible because they violate condition (A2). See §3 of [C3] for more examples.

We need to slightly modify condition (A1) for marked diagrams to allow for the two integers preceding a bracket not to be equal when one of them is marked.

Definition 3.24. A bracket $]^j$ with $p(]^j) \neq p(\delta)$ in a marked diagram $(D, c_\bullet, *)$ satisfies Condition (A1*) if $]^j$ satisfies condition (A1) after deleting δ and any brackets at position $p(\delta)$ from D. A marked diagram $(D, c_\bullet, *)$ satisfies Condition (A1*) if every bracket $]^j$ with $p(]^j) \neq p(\delta)$ satisfies condition (A1*).

Example 3.25. $1^*1]_1 00]_2 1^*]_3 0]_3 0]_3 0000\}_4 000$ and $2^*2]_1 3455]_1 2^*]_1 00\}_2 00\}_2 0\}_1 0\}_2 000\}_1$ satisfy condition (A1*). In both examples, the two integers in the sequence to the left of $]^3$ are not equal, so these diagrams do not satisfy condition (A1). $1^*1]_1 22]_2 1^*]_4 0]_3 000\}_1 00\}_2 000$ does not satisfy condition (A1*) since the two integers preceding $]^4_3$ omitting 1^* are $2, 0$ and not equal.

Definition 3.26. The dimension of a colored symplectic diagram $(D, c_\bullet)$ is defined by
$$\dim(D, c_\bullet) = \sum_{i=1}^{s} (p(]^i) - i) + \sum_{j=1}^{k_h - s} (p(\}^j) - 1 - 2k_h + 2j + x_j) + \dim(c_\bullet).$$
The dimension of a marked diagram $(D, c_\bullet, *)$ is defined to be the dimension of $(D, c_\bullet)$.

Example 3.27. $11]_1 22]_3 00\}_2 00\}_3 00$ has dimension 9. $00]_3 0000\}_2 2\}_3 000\}_1 00$ has dimension 13.

Remark 3.28. The admissible colored symplectic diagrams are the main combinatorial objects of this paper. They represent symplectic restriction varieties in $SF(k_1, \dots, k_h; n)$. As we will explain in §4.2, the diagram records a partial flag with respect to a distinguished basis. The coloring dictates the rank conditions imposed on the flags parameterized by $SF(k_1, \dots, k_h; n)$. Informally, the symplectic restriction variety associated to an admissible symplectic diagram $(D, c_\bullet)$ parameterizes partial flags $(W_1, \dots, W_h)$ such that the dimension of intersection of W_j with a subspace recorded by a brace $\}^i$ or a bracket $]^i$ in $(D, c_\bullet)$ is equal to the total number of brackets and braces to the left of $\}^i$ or $]^i$ (inclusive) that are assigned a color less than or equal to j, for every bracket and brace in $(D, c_\bullet)$ and every $1 \leq j \leq h$. (See Definition 4.13 for precise details.) The dimension of an admissible, colored symplectic diagram is equal to the dimension of the corresponding restriction variety.

Next, we discuss how to represent Schubert varieties in $SF(k_1, \dots, k_h; n)$ and intersections of Schubert varieties in $F(k_1, \dots, k_h; n)$ with $SF(k_1, \dots, k_h; n)$ by symplectic diagrams.

Definition 3.29. The symplectic diagram $D(\lambda_\bullet; \mu_\bullet; c_\bullet)$ associated to the Schubert variety $\Sigma_{\lambda_\bullet;\mu_\bullet;c_\bullet}$ in $SF(k_1, \dots, k_h; n)$ is the saturated, colored symplectic diagram in perfect order with brackets at positions λ_i, for $1 \leq i \leq s$, braces at positions $n - \mu_i$, for $s < i \leq k_h$, and coloring $c_\bullet$.

Example 3.30. We have $D(2,4;4,0;1,1,2,2) = 22]_1 22]_1 00\}_2 0000\}_2$ and $D(1,3;3,1;1,2,2,1) = 1]_1 22]_2 0000\}_2 00\}_1 0$ for $SF(2,4;10)$.

Lemma 3.31. *The symplectic diagram $D(\lambda_\bullet; \mu_\bullet; c_\bullet)$ associated to the Schubert variety $\Sigma_{\lambda_\bullet;\mu_\bullet;c_\bullet}$ is an admissible colored symplectic diagram.*

Proof. Definition 3.29 agrees with the definition of a symplectic diagram associated to a Schubert variety in $G(k_h, n)$ given in [C3, Definition 3.20]. By [C3, Lemma 3.22], the diagram underlying $D(\lambda_\bullet; \mu_\bullet; c_\bullet)$ is an admissible symplectic diagram for $G(k_h, n)$. Therefore, $D(\lambda_\bullet; \mu_\bullet; c_\bullet)$ is an admissible colored symplectic diagram. □

Let $\sigma_{a_\bullet, c_\bullet}$ be a Schubert class in $F(k_1, \dots, k_h; n)$. For ease of notation, set $a_0 = 0$. Let i denote the natural inclusion $i : SF(k_1, \dots, k_h; n) \to F(k_1, \dots, k_h; n)$. If $a_j < 2j - 1$ for some $1 \leq j \leq k_h$, then $i^* \sigma_{a_\bullet, c_\bullet} = 0$. Otherwise, the next definition associates a colored symplectic diagram to $\sigma_{a_\bullet, c_\bullet}$.

Definition 3.32. If $a_j < 2j - 1$ for some $1 \leq j \leq k_h$, then we do not associate a symplectic diagram to $\sigma_{a_\bullet, c_\bullet}$. If $a_j \geq 2j - 1$ for $1 \leq j \leq k_h$, then let $P(a_\bullet; c_\bullet)$ be the colored symplectic diagram of type $s = 0$ for $SF(k_1, \dots, k_h; n)$ such that:

- The braces in $P(a_\bullet, c_\bullet)$ occur at positions a_j, for $1 \leq j \leq k_h$.
- The only non-zero integers in the sequence of $P(a_\bullet; c_\bullet)$ consist of a $k_h - j + 1$ at the place $a_{j-1} + 1$ for each odd a_j in $a_\bullet$.
- The colorings in $\sigma_{a_\bullet, c_\bullet}$ and $P(a_\bullet; c_\bullet)$ are equal.

Example 3.33. The diagram $P(3,5,7;1,1,2)$ in $SF(2,3;8)$ is $300\}_1 20\}_1 10\}_2 0$. The diagram

$$P(1,3,6,7,10;1,3,3,2,1) = 5\}_1 40\}_3 000\}_3 2\}_2 000\}_1$$

in $SF(2,3,5;10)$. Notice that the diagram $P(a_\bullet; c_\bullet)$ does not have to be admissible because it fails to satisfy condition (A2) for braces with $a_j = 2j - 1$.

We will later associate a collection of admissible, colored symplectic diagrams to the diagram $P(a_\bullet; c_\bullet)$. For now, we have the following lemma.

Lemma 3.34. *Let $\sigma_{a_\bullet, c_\bullet}$ be a Schubert class in $F(k_1, \dots, k_h; n)$. If $a_j \geq 2j - 1$ for all $1 \leq j \leq k_h$, then diagram $P(a_\bullet; c_\bullet)$ is a colored symplectic diagram. Furthermore, if $a_j > 2j - 1$ for $1 \leq j \leq k_h$, then $P(a_\bullet; c_\bullet)$ is an admissible, colored symplectic diagram.*

Proof. If $a_j > 2j - 1$ for $1 \leq j \leq k_h$, then, by [C3, Definition 3.34], the underlying diagram is the diagram associated to $\sigma_{a_\bullet}$ in $G(k_h, n)$. By [C3, Lemma 3.37], this is an admissible symplectic diagram for $G(k_h, n)$. Therefore, $P(a_\bullet; c_\bullet)$ is an admissible, colored symplectic diagram.

Even when $a_j = 2j - 1$ for some j, then it is easy to see that $P(a_\bullet; c_\bullet)$ is a symplectic diagram. Briefly, since all the integers in $a_\bullet$ are positive and distinct, the braces are in distinct positive positions. Hence, condition (1) is satisfied. Since there are no brackets, condition (2) is automatic. Conditions (3) and (4) hold by construction: There is an integer equal to i in the sequence and it occurs between the braces $\}^{i+1}$ and $\}^i$ if the position of $\}^i$ is odd. Otherwise, there are no integers equal to i. Therefore, every integer equal to i occurs to the left of $\}^i$ and the number of integers equal to 0 or greater than i to the left of $\}^i$ is even. For each $1 \leq j \leq k_h$, there is at most one integer equal to j in the sequence and the diagram is in order. Therefore, conditions (S1), (S3) and (S4) evidently hold. The number of positive integers to the left of $\}^j$ is at most $k_h - j + 1$. Furthermore, $p(\}^j) \geq 2(k_h - j) + 1$ and if equality holds, then $r_j = 1$. We conclude that $2\tau_j \leq 2(k_h - j + 1) \leq p(\}^j) + r_j$. Hence, condition (S2) also holds and $P(a_\bullet; c_\bullet)$ is a colored symplectic diagram. Condition (A1) is also automatic since the diagram does not have any brackets, but condition (A2) may fail. □

After these preliminaries, we are ready to explain the algorithm for computing symplectic restriction coefficients. Given an admissible symplectic diagram, we associate several new symplectic diagrams. These might not be admissible. We will then give an algorithm for replacing them with admissible symplectic diagrams. The goal is to transform every admissible symplectic diagram into a collection of saturated symplectic diagrams in perfect order (which correspond to Schubert classes). The new diagrams will be obtained by swapping some integers in the sequence and moving some of the brackets and braces. We will need to specify the coloring of these new diagrams.

Definition 3.35. If a new sequence of brackets and braces is obtained from a diagram $(D, c_\bullet)$ by moving a set of brackets or braces to new positions or changing a brace to a bracket, the *associated coloring* is the coloring obtained by keeping the colors of all the brackets and braces unchanged and assigning the same color to the bracket as the brace.

For example, if $]^j$ is moved to a position between $]^{i-1}$ and $]^i$ with $i < j$, then the associated coloring is obtained by applying the cycle $(i, i+1, \dots, j-1, j)$ to the coloring $c_\bullet$. Similarly, if $\}^i$ is moved to a position between $\}^j$ and $\}^{j+1}$ with $i < j$, the associated coloring is the coloring obtained by applying the cycle $(k_h - j + 1, k_h - j + 2, \dots, k_h - i + 1)$ to $c_\bullet$.

Remark 3.36. The next definition combinatorially encodes the specialization corresponding to swapping two basis elements. Ideally, we would like to increase the dimension of the kernel of the restriction of Q to the smallest dimensional non-isotropic subspace which is not saturated. Furthermore, while we do this, we would like to keep the sequence of integers in the corresponding diagram in order. Since the rank of a skew-symmetric form is even, we cannot do this arbitrarily. Consequently, we are forced to have 5 cases depending on whether the sequence is in order or not and whether we have room to increase the corank by two or not. In the beginning of §5, we will explicitly say which pair of basis elements need to be swapped to get the corresponding diagram. The diagrams defined in Definitions 3.37 and 3.39 encode the support of the limits of the specialization applied to a symplectic restriction variety. The descriptions of these diagrams emerge from the dimension counts (see Remark 3.65 and the proof of Theorem 5.1).

In the next definition, for ease of notation, read any mention of $k_h - s + 1$ as 0 and any mention of $\}^{k_h - s + 1}$ as $]^s$; and set $p(]^0) = 0$.

Definition 3.37. Given an admissible symplectic diagram $(D, c_\bullet)$ of type s for $SF(k_1, \dots, k_h; n)$ which is not a saturated admissible diagram in perfect order, define a (not-necessarily-admissible) symplectic diagram $(D^a, c_\bullet)$ according to the following exhaustive and mutually-exclusive cases.
Case (1) If D is not in order, let η be the integer violating the order (according to condition (S3), η is unique).

(1)(i) If every integer $\eta < i \leq k_h - s$ occurs to the left of η, let ν be the leftmost integer equal to $\eta + 1$. Let D^a be the canonical representative of the sequence obtained by interchanging η and ν.

(1)(ii) If an integer $\eta < i \leq k_h - s$ does not occur to the left of η (according to condition (S4), i is unique), let ν be the leftmost integer equal to $i+1$. Let ζ be the leftmost 0 to the right of $\}^{i+1}$ which is not equal to ν. Let D^a be the canonical representative of the sequence obtained by swapping η with ζ and changing ν to i.

Case (2) If D is in order, let κ be the largest index for which $l(i) < \rho(i, i-1)$.

(2)(i) If $l(\kappa) < \rho(\kappa, \kappa - 1) - 1$, let ν be the leftmost integer equal to $\kappa + 1$. Let ζ be the leftmost 0 to the right of $\}^{\kappa+1}$ which is not equal to ν. Let D^a be the canonical representative of the sequence obtained by changing ν and ζ to κ.

(2)(ii) If $l(\kappa) = \rho(\kappa, \kappa - 1) - 1$, let η be the integer equal to $\kappa - 1$ at the place $p(\}^{\kappa}) + 1$.

(2)(ii)(a) If κ occurs to the left of η, let ν be the leftmost integer equal to κ. Let D^a be the canonical representative of the sequence obtained by changing ν to $\kappa - 1$ and η to zero.

(2)(ii)(b) If κ does not occur to the left of η, let ν be the leftmost integer equal to $\kappa + 1$. Let ζ be the leftmost 0 to the right of $\}^{\kappa+1}$ which is not equal to ν. Let D^a be the canonical representative of the sequence obtained by swapping η with ζ and changing ν to κ.

In every case, the coloring associated to D^a is the same coloring $c_\bullet$ as in the original diagram $(D, c_\bullet)$.

Notation 3.38. If in Definition 3.37 $\nu \neq 0$, set $\nu' = \nu$. Otherwise, set $\nu' = k_h - s + 1$. If there is a zero in the sequence to the left of a bracket $]^i$, set $y_i = k_h - s + 1$. Otherwise, let y_i denote the largest integer in the sequence to the left of $]^i$. Recall from Definition 3.22 that x_i is the number of brackets $]^l$ such that every integer in the sequence to the left of $]^l$ is positive and less than or equal to i.

Definition 3.39. We preserve the notation from Definition 3.37. Given a diagram $(D, c_\bullet)$ as in Definition 3.37, for every bracket $]^i$ in $(D, c_\bullet)$ with $p(]^i) > p(\nu)$ satisfying the two properties

(1) the equality $p(]^i) - p(\nu) = y_i - \nu' + i - x_{\nu'-1}$ holds,
(2) all the brackets in positions p with $p(\nu) < p < p(]^i)$ are assigned a color less than c_i (the color of $]^i$),

define $(D^b(]^i), c^i_\bullet)$ to be the diagram obtained from $(D^a, c_\bullet)$ by moving the bracket $]^i$ to position $p(\nu)$. The coloring $c^i_\bullet$ is the associated coloring (see Definition 3.35).

The *set of diagrams of type D^b associated to* $(D, c_\bullet)$ is the (possibly-empty) set consisting of diagrams $(D^b(]^i), c^i_\bullet)$ as $]^i$ varies over all brackets $]^i$ with $p(]^i) > p(\nu)$ satisfying the two properties.

Remark 3.40. Conditions (S3), (S4) and (A1) imply that, in an admissible diagram D, a bracket $]^i$ satisfies the equality $p(]^i) - p(\nu) = y_i - \nu' + i - x_{\nu'-1}$ if the number of brackets in positions following an integer j is one less than the number of integers equal to j for every j in the part of the sequence between ν and $]^i$. For example, in $...33]_1 3]_2 4566]_3 6]_2 6]_1 700]_1 0]_2...$ if $\nu = 3$ is the leftmost 3 in the sequence, then every bracket shown here satisfies the equality. Whereas, assuming ν is the leftmost 3 in $...33]_1 33]_2 44]_1 500]_2...$, the brackets other than the first bracket do not satisfy the equality.

We now give several examples to illustrate these definitions.

Example 3.41. Let $D = 2300\}_1 10\}_2 0\}_3 0$, then $\eta = 1$ violates the order and $\nu = 2$ and 3 occur to the left of it. Hence, we are in case (1)(i) and $D^a = 1300\}_1 20\}_2 0\}_3 0$ is obtained by swapping 1 and 2. Similarly, let $D = 200]_1 200\}_2 00\}_3$, then $\eta = 2_4$, $\nu = 0_2$ and $y_1 = \nu' = 3$. Hence, we have $D^a = 220]_1 000\}_2 00\}_3$, $D^b(]^1) = 22]_1 0000\}_2 00\}_3$.

Let $D = 124400\}_1 00\}_2 1\}_2 0\}_1 00$, the 1 in the ninth place violates the order and 3 does not occur to its left, so we are in case (1)(ii). We have $\eta = 1_9$, $\nu = 4_3$ and $\zeta = 0_7$. Hence, $D^a = 123400\}_1 10\}_2 0\}_2 0\}_1 00$.

Let $D = 22]_1 00\}_2 00\}_3 00$, then D is in order and $\kappa = 1$. Since $l(1) = 0 < \rho(1,0) - 1$, we are in case (2)(i). We have $\nu = 2_1$, $\zeta = 0_5$. Hence, $D^a = 12]_1 00\}_2 10\}_3 00$ and $D^b(]^1) = 1]_1 200\}_2 10\}_3 00$.

Let $D = 3300\}_1 200\}_2 0\}_3$, then D is in order and $\kappa = 3$. Since $l(3) = 2 = \rho(3,2) - 1$, we are in case (2)(ii)(a). We have $\eta = 2_5$, $\nu = 3_1$. Hence, $D^a = 2300\}_1 000\}_2 0\}_3$.

Let $D = 330000\}_1 00\}_2 1\}_3 0$, then D is in order and $\kappa = 2$. Since $l(2) = 0 = \rho(2,1) - 1$ and 2 does not occur in the sequence, we are in case (2)(ii)(b). We have $\eta = 1_9$, $\nu = 3_1$ and $\zeta = 0_7$. Hence, $D^a = 230000\}_1 10\}_2 0\}_3 0$.

Let $D = 22]_1 2]_2 344]_3 00\}_1 0000\}_2 0\}_3 000\}_4 00$. Then $D^a = 12]_1 2]_2 344]_3 00\}_1 0000\}_2 0\}_3 100\}_4 00$ and $D^b(]^1) = 1]_1 22]_2 344]_3 00\}_1 0000\}_2 0\}_3 100\}_4 00$, $D^b(]^2) = 1]_2 2]_1 2344]_3 00\}_1 0000\}_2 0\}_3 100\}_4 00$ and $D^b(]^3) = 1]_3 2]_1 2]_2 34400\}_1 0000\}_2 0\}_3 100\}_4 00$.

As can be seen from the examples, the diagrams D^a or D^b may fail to be admissible. D^a may fail to satisfy either condition (A1) or condition (A2). D^b satisfies condition (A2), but may fail to satisfy condition (A1). The next two algorithms transform these diagrams into admissible diagrams. It is easy to fix condition (A1) or (A1*). In contrast, it takes some work to fix condition (A2). This is the most significant difference between the rule for $SG(k_h, n)$ and $SF(k_1, \ldots, k_h; n)$. In the case of $SG(k_h, n)$, it is easy to turn a diagram that fails (A2) to one that satisfies it. In the case of flag varieties, this requires a 'Pieri-like' rule.

Remark 3.42. Conditions (A1) and (A1*) reflect the kernel bound. The algorithm implements the geometric fact that if $]^i$ is a bracket violating (A1) or (A1*) and L is the isotropic subspace corresponding to $]^i$, then the linear spaces have to be contained in $L^\perp$. For example, in the diagram $110]_1 000\}_2 00$ the bracket $]^1_1$ violates (A1). The isotropic spaces that intersect the corresponding linear space L have to be contained not only in the linear space corresponding to $\}^1_2$ but actually in $L^\perp$. We replace the diagram with $111]_1 00\}_2 000$ to reflect this fact. Algorithm FixA1 (3.43) repeatedly applies this principle to obtain a diagram that satisfies (A1).

Algorithm 3.43 (FixA1). Let $(D, c_\bullet)$ (respectively, $(D, c_\bullet, *)$) be a (marked) symplectic diagram.

Step 1. If the diagram satisfies condition (A1) (resp., (A1*)), output the diagram and stop. Otherwise, let $]^j$ be the maximal index bracket such that $p(]^j) \neq p(\delta)$ and condition (A1) (resp., (A1*)) fails. Let i be the integer immediately to the left of $]^j$. Continue to step 2.

Step 2. For the rest of the algorithm, if $i = 0$, read $i-1$ as $k_h - s$. Replace i with $i-1$. If the position $p(\}^{i-1}) - 1$ is unoccupied, move $\}^{i-1}$ to position $p(\}^{i-1}) - 1$ (i.e., one to the left). Output the canonical representative of the resulting sequence with the associated coloring (see Definition 3.35) to Step 1. Otherwise, proceed to Step 3.

Step 3. If the position $p(\}^{i-1}) - 1$ is occupied, let p be the first position to the left of $\}^{i-1}$ which is not occupied and assume that $\}^{i+l}, \ldots, \}^i$ are in positions $p+1, \ldots, p(\}^{i-1}) - 1$. If the color of any of $\}^{i+l}, \ldots, \}^i$ is greater than or equal to the color of $\}^{i-1}$, discard the diagram and stop. The algorithm does not output any diagrams. Otherwise, continue to Step 4.

Step 4. If the color of every $\}^{i+l}, \ldots, \}^i$ is less than the color of $\}^{i-1}$, move $\}^{i-1}$ to position p, subtract one from all the integers in the sequence equal to $i, i+1, \ldots, i+l$ and replace the left most integer equal to $i+l+1$ (or 0 if $i+l = k_h - s$) with $i+l$. The new coloring is the associated coloring. Return the canonical representative of the resulting sequence to Step 1.

Given a symplectic diagram $(D, c_\bullet)$, we write $\mathrm{FixA1}(D, c_\bullet)$ for the (possibly-empty) set of diagrams output by Algorithm FixA1 starting with $(D, c_\bullet)$.

Example 3.44. Let $D = 33]_1 00]_2 0000\}_1 00\}_2 00\}_3 00$. Then $D^a = 23]_1 00]_2 0000\}_1 20\}_2 00\}_3 00$,

$$D^b(]^1) = 2]_1 300]_2 0000\}_1 20\}_2 00\}_3 00 \quad \text{and} \quad D^b(]^2) = 2]_2 3]_1 000000\}_1 20\}_2 00\}_3 00$$

do not satisfy condition (A1). Algorithm FixA1 replaces D^a with $22]_1 00]_2 0000\}_1 2\}_2 000\}_3 00$, which is admissible. Algorithm FixA1, replaces $D^b(]^1)$ with $1]_1 300]_2 0000\}_1 20\}_2 0\}_3 000$, which is admissible. Algorithm FixA1 replaces $D^b(]^2)$ first with $2]_2 2]_1 000000\}_1 2\}_2 000\}_3 000$, which still violates condition (A1). Then with $1]_2 2]_1 000000\}_1 2\}_2 00\}_3 000$ and finally with

$$1]_2 1]_1 000000\}_1 2\}_2 0\}_3 0000,$$

which is admissible.

Let $D = 00]_1 0000\}_2 2\}_3 0\}_4 00$. Then both the diagrams $D^a = 30]_1 2000\}_2 0\}_3 0\}_4 00$ and $D^b(]^1) = 3]_1 20000\}_2 0\}_3 0\}_4 00$ fail condition (A1). Algorithm FixA1 replaces D^a with the admissible diagram $33]_1 200\}_2 00\}_3 0\}_4 00$. Algorithm FixA1 first replaces $D^b(]^1)$ with $2]_1 2300\}_3 0\}_2 00\}_4 00$, which still fails condition (A1), and then with the admissible diagram $1]_1 2300\}_3 0\}_2 0\}_4 000$.

The reader can look ahead to Example 3.51. If we run Algorithm FixA1 on the diagrams in the third example that do not satisfy (A1*), we obtain

$$1^*1]_1 1]_4 233]_2 1^*]_3 00\}_3 00\}_1 0\}_2 0000, \quad 1^*1]_1 22]_4 33]_2 1^*]_3 00\}_2 00\}_1 00\}_3 000$$

and $1^*1]_1 233]_4 3]_2 1^*]_3 00\}_1 000\}_2 0\}_3 000$, respectively.

Next, we give the algorithm for replacing a diagram that fails condition (A2) with an admissible diagram.

Remark 3.45. Condition (A2) reflects the linear space bound. When a brace $\}^i$ in D fails (A2), the flag elements encoded by D are forced to intersect the kernel of the restriction of Q to the linear space represented by $\}^i$ in one-larger dimension. For example, $1^*1]_1 00\}_2 1^*\}_3 000$ is a typical diagram that fails (A2) for $\}_3^1$. By the linear space bound, the linear spaces W_3 parameterized by the variety encoded by this diagram have to intersect the kernel of the linear space corresponding to $\}_3^1$ (which is spanned by the basis elements corresponding to the 1's in the first, second and fifth places) in a two-dimensional subspace. The problem is that the basis elements representing the kernel are not adjacent. In order to get a partial flag, we need to move the basis element in the fifth place to the third place so that it is adjacent to the other basis elements spanning the kernel. Algorithm FixA2 (3.52) records the transformations that happen when we move that basis element past the brackets and braces in between one at a time. In §5.1, we will explicitly define the corresponding specialization.

Before stating the algorithm, first we define the tightening of a diagram. Geometrically, when we move the basis element e past a bracket $\}^i$ (or brace $]^i$), there has to be a $k_h - i + 1$ (respectively, i) dimensional subspace of W_h in the span of e and the subspace corresponding to $\}^{i+1}$ (respectively, $]^{i-1}$). The tightening of the diagram expresses this fact. We then define the diagrams D^α and D^β. These are the analogues of the diagrams D^a and D^b and they encode the varieties that form the support of the flat limit of the specialization. Their description emerges from dimension counts (see Remark 3.65 and the proof of Theorem 5.3).

Definition 3.46. Let $(D, c_\bullet, *)$ be a marked sequence of brackets and braces of type s. The *tightening* of $(D, c_\bullet, *)$ is the (possibly-unmarked) sequence defined by the following steps:

Step 1. • If the bracket or brace at the minimal position $p \geq p(\delta)$ is the bracket $]^j$, move $]^j$ to position $p(\delta)$.

• If the bracket or brace at the minimal position $p \geq p(\delta)$ is the brace $\}^j$, move $\}^j$ to the position $p(\delta)$. If $j = k_h - s$, replace $\}^j$ with $]^{s+1}$ and replace the integers in the sequence equal to $k_h - s$ with 0. If $j < k_h - s$, replace the integers in the sequence equal to $j+1$ with j.

Step 2. Replace any positive integer $1 \leq l \leq k_h - s$ in the sequence that occurs to the right of $\}^l$ by 0. In the resulting marked sequence, if there are no brackets or braces between the markings or if every position $p(\delta') < p < p(\delta)$ is occupied by a bracket, remove the markings.

The coloring is still $c_\bullet$. We denote the tightening of $(D, c_\bullet, *)$ by $\mathrm{Tight}(D, c_\bullet, *)$.

Example 3.47. The tightening of $1^*1]_1 1^*0]_2 0]_3 00\}_4 000$ is $11]_1 1]_2 00]_3 00\}_4 000$.

The tightening of $1^*2]_1 00]_2 00\}_3 1^*0\}_4 00$ is $1^*1]_1 00]_2 00\}_3 1^*\}_4 000$.

The tightening of $2^*2^*\}_1 00\}_2 00$ is $00]_1 00\}_2 00$.

Finally, the tightening of $1^*1]_1 00]_2 0]_2 1^*00\}_1 0\}_1 000$ is $1^*1]_1 00]_2 0]_2 1^*]_1 000\}_1 000$.

We now define analogues of diagrams D^a and D^b for marked diagrams. Recall from Definition 3.14 that $p(\delta') < p(\delta)$ denote the places of the markings in the marked diagram. If $(D, c_\bullet, *)$ is a tightened marked symplectic diagram, then the position $p(\delta)$ is occupied by a bracket or a brace. The definitions depend on whether $p(\delta)$ is occupied by a bracket or a brace. As usual, for ease of notation set $\}^{k_h - s+1} =]^s$ and $p(]^0) = p(\}^{k_h+1}) = 0$.

Definition 3.48. Let $(D, c_\bullet, *)$ be a tightened marked symplectic diagram. Define $(D^\alpha, c_\bullet, *)$ according to the following exhaustive and mutually-exclusive cases.

Case (1) If position $p(\delta)$ is occupied by the bracket $]^{j+1}$, let ϵ be the integer immediately to the left of $]^j$ (i.e., $p(\epsilon) = p(]^j)$). Let $\tilde{D}$ be the canonical representative of the sequence obtained by interchanging ϵ and δ, keeping δ as the marked integer. Define $(D^\alpha, c_\bullet, *)$ to be $\mathrm{Tight}(\tilde{D}, c_\bullet, *)$.

Case (2) If position $p(\delta)$ is occupied by the brace $\}^j$, then let ϵ be the leftmost largest integer in D between $\}^{j+1}$ and $\}^{j+2}$ (or $]^s$ if $j+1 = k_h - s$). Define $(D^\alpha, c_\bullet, *)$ by the following steps:
Step 1. Let D' be the sequence obtained from D by cyclically permuting by one unit to left the integers between ϵ and δ inclusive (so that δ is immediately to the left of $\}^{j+1}$ and ϵ occupies the place previously occupied by δ). Keep δ as the marked integer. Proceed to Step 2.
Step 2. • If $\epsilon = 0$ and there is an integer equal to $j+2$ in the sequence, replace the leftmost integer equal to $j+2$ in D' with $j+1$.

• If $\epsilon = 0$, there is no integer equal to $j+2$ in the sequence and $p(\delta') > 0$, replace the leftmost zero in D' with $j+1$.

• If $\epsilon = 0$, there is no integer equal to $j+2$ in the sequence and $p(\delta') = 0$, replace the leftmost zero between $\}^{j+1}$ and $\}^{j+2}$ (or $]^s$ if $j+1 = k_h - s$) in D' with $j+1$.

Denote the resulting sequence by $\tilde{D}$. Define $(D^\alpha, c_\bullet, *)$ to be $\mathrm{Tight}(\mathrm{Can}(\tilde{D}), c_\bullet, *)$.

In both Cases (1) and (2), the coloring is the original coloring $c_\bullet$.

Definition 3.49. We preserve the notation and the cases from Definition 3.48. Let $(D, c_\bullet, *)$ be a tightened marked symplectic diagram. We associate a set of diagrams of type D^β to $(D, c_\bullet, *)$ as follows. In every case, the coloring $c^\beta_\bullet(t)$ or $c^\beta_\bullet(\})$ is the associated coloring (see Definition 3.35).
Case(1) If position $p(\delta)$ is occupied by the bracket $]^{j+1}$, let p be the minimum of $p_{\tilde{D}}(\delta)$ in $\tilde{D}$ and the first position to the left of $p(]^j)$ not occupied by a bracket in $\tilde{D}$. Define $(D^\beta(t), c^\beta_\bullet(t), *)$ as follows depending on whether there is a bracket between p and $]^j$ or not.

Case(1)(i) If there are no brackets between p and $]^j$, and the colors satisfy $c_j < c_{j+1}$, then for every integer $t > 0$ such that $p + t \leq p(]^j)$, let $(D^\beta(t), c^\beta_\bullet(t), *)$ be the tightening of the sequence obtained from $\tilde{D}$ by moving $]^{j+1}$ to position p and $]^j$ to position $p+t$. If $c_j \geq c_{j+1}$, let the set of diagrams of type D^β be empty.

Case(1)(ii) If there is a bracket between p and $]^j$ in $\tilde{D}$, let q be the position of the rightmost bracket to the left of p. If the colors of all the brackets $]^l$ with $p < p(]^l) < p(]^{j+1})$ are less than c_{j+1}, for each $1 \leq t \leq p-q$, let $(D^\beta(t), c^\beta_\bullet(t), *)$ be the tightening of the sequence obtained from $\tilde{D}$ by moving $]^{j+1}$ to position $q+t$. If $c_l \geq c_{j+1}$ for some $]^l$ with $p < p(]^l) < p(]^{j+1})$, let the set of diagrams of type D^β be empty.

Case (2) If position $p(\delta)$ is occupied by the brace $\}^j$, let p be the first unoccupied position to the left of $p(\}^{j+1})$ in $\tilde{D}$. Let $\}^{j+l}, \ldots, \}^{j+1}$ be the braces between p and $\}^j$. Let γ be the leftmost integer in $\tilde{D}$ equal to $j+1$. Let $]^u,]^{u+1}, \cdots,]^{u+v}$ be the brackets to the right of γ. Define the diagrams of type D^β by the following steps.

Step 1. If $c_{k_h-j+1} > c_i$ for every $k_h - j - l + 1 \leq i < k_h - j + 1$, then let $\overline{D}$ be obtained from $\tilde{D}$ by moving $\}^j$ to position p and subtracting one from the integers $j+l, \ldots, j+1$ in the sequence. Proceed to the next step. Otherwise, let the set of diagrams of type D^β be empty and stop.

Step 2. For every $0 \leq t \leq v$ such that $c_\iota < c_{u+t}$ for all $u \leq \iota < u+t$, let $\overline{D}(t)$ be obtained from $\overline{D}$ by moving $]^{u+t}$ to position $p(\gamma)$. Define $(D^\beta(]^{u+t}), c^\beta_\bullet(t), *) := (\mathrm{Tight}(\mathrm{Can}(\overline{D}(t))), c^\beta_\bullet(t), *)$. Proceed to the next step.

Step 3. • If $p(\delta') = 0$, let D' be the marked sequence obtained from $\overline{D}$ by cyclically permuting the string of integers starting at place $p(\}^{j+l+1}) + 1$ and ending at place $p+1$ one step to the left. Define $(D^\beta(\}^j), c^\beta_\bullet(\}^j), *) := (\mathrm{Tight}(\mathrm{Can}(D')), c^\beta_\bullet(\}^j), *)$.

• If $p(\delta') > 0$ and the color of $\}^j$ (c_{k_h-j+1}) is greater than the color of every bracket and brace in positions $p(\gamma)$ through $p(\}^j) - 1$, let D' be the sequence obtained from $\tilde{D}$ by replacing $\}^j$ in $\tilde{D}$ with a bracket at position $p(\gamma)$ and subtracting one from all the integers greater than j in the sequence of $\tilde{D}$. Define $(D^\beta(\}^j), c^\beta_\bullet(\}^j), *) := (\mathrm{Tight}(\mathrm{Can}(D')), c^\beta_\bullet(\}^j), *)$. Otherwise, there are no diagrams of the form $(D^\beta(\}^j), c^\beta_\bullet(\}^j), *)$.

Given a tightened marked symplectic diagram $(D, c_\bullet, *)$, the *set of diagrams of type* D^β is the (possibly-empty) set of diagrams $(D^\beta(t), c^\beta_\bullet(t), *)$ as t varies over all allowed possibilities in Case (1) and the set of diagrams $(D^\beta(]^{u+t}), c^\beta_\bullet(t), *)$ and $(D^\beta(\}^j), c^\beta_\bullet(\}^j), *)$ as the indices vary over all allowed possibilities in Case (2).

Remark 3.50. In particular, if $c_j \geq c_{j+1}$ in Case (1)(i); or if one of the brackets between position p and $]^{j+1}$ has color greater than or equal to c_{j+1} in Case (1)(ii), the set of diagrams of type D^β is empty. In Case (2) if $c_i \geq c_{k_h-j+1}$ for some $k_h - j - l + 1 \leq i < k_h - j + 1$, then the set of diagrams of type D^β is empty. Similarly, if $p(\delta') > 0$ and the color of a bracket or a brace between $p(\gamma)$ and $p(\}^j) - 1$ is at least c_{k_h-j+1}, then $(D^\beta(\}^j), c^\beta_\bullet(\}^j), *)$ does not exist.

Example 3.51. Let $D = 1^*1]_1 00]_2 00\}_3 1^*\}_4 000$, then we are in Case (2) since δ is followed by $\}^1_4$ and $\epsilon = 0_5$. Hence, $\tilde{D} = 1^*1]_1 20]_2 01^*\}_3 0\}_4 000$. By tightening the canonical representative, we obtain $D^\alpha = 1^*1]_1 00]_2 1^*]_3 00\}_4 000$. We have

$$D^\beta(]^2_2) = 1^*1]_1 1]_2 00\}_4 1^*\}_3 0000 \quad \text{and} \quad D^\beta(\}^1_4) = 00]_1 0]_4 0]_2 0]_3 00000.$$

Let $D = 1^*1]_1 2300]_2 1^*]_3 00\}_1 0\}_2 0\}_3 000$, then we are in Case (1) since δ is followed by $]^3_3$. Hence, $\tilde{D} = 1^*1]_1 1^*230]_2 0]_3 00\}_1 0\}_2 0\}_3 000$. The tightening of the diagram gives

$$D^\alpha = 11]_1 1]_2 2300]_3 00\}_1 0\}_1 0\}_1 000.$$

In Definition 3.49, we are in Case (1)(i), hence, we have

$$D^\beta(1) = 11]_1 1]_3 2]_2 30000\}_1 0\}_2 0\}_3 000, \quad D^\beta(2) = 11]_1 1]_3 23]_2 0000\}_1 0\}_2 0\}_3 000$$

and $D^\beta(3) = 11]_1 1]_3 230]_2 000\}_1 0\}_2 0\}_3 000$. Note that the diagrams of the form D^β do not necessarily satisfy condition (A1) or (A1*).

Let $D = 1^*1]_1 2300]_2 0]_3 1^*]_4 000\}_1 0\}_2 0\}_3 000$, then

$$\tilde{D} = D^\alpha = 1^*1]_1 2300]_2 1^*]_3 0]_4 000\}_1 0\}_2 0\}_3 000.$$

In Definition 3.49, we are in Case (1)(ii) and the diagrams of type D^β are

$$D(1) = 1^*1]_1 2]_4 300]_2 1^*]_3 0000\}_1 0\}_2 0\}_3 000, \quad D(2) = 1^*1]_1 23]_4 00]_2 1^*]_3 0000\}_1 0\}_2 0\}_3 000$$

and $D(3) = 1^*1]_1 230]_4 0]_2 1^*]_3 0000\}_1 0\}_2 0\}_3 000$. (See Example 3.44 for the result of running Algorithm FixA1 on $D(1), D(2)$, and $D(3)$.)

Finally, let $D = *00\}_1 300\}_2 0\}_3 1^*\}_4 000$. Then we are in Case (2) and we have

$$D^\alpha = *00\}_1 200\}_2 1^*\}_3 0\}_4 000 \quad \text{and} \quad D^\beta(\}_4^1) = 00\}_1 00\}_4 1\}_2 1^*\}_3 0000.$$

The next algorithm fixes symplectic diagrams $(D, c_\bullet)$ that arise as $P(a_\bullet, c_\bullet)$ or $(D^a, c_\bullet)$ so that they satisfy condition (A2).

Algorithm 3.52 (FixA2). Let $(D, c_\bullet)$ be a colored symplectic diagram.

Step 1. If $(D, c_\bullet)$ satisfies condition (A2), output FixA1$(D, c_\bullet)$ and stop. Otherwise, proceed to Step 2.

Step 2. If $(D, c_\bullet)$ does not satisfy condition (A2), let $\}^i$ be the brace with the largest index for which condition (A2) is not satisfied. Let δ be the rightmost positive integer to the left of $\}^i$. If it exists, let δ' be the rightmost positive integer to the left of δ, which is equal to δ but not equal to a 1 that has a bracket to its immediate right. If δ' exists, mark δ and δ'. Otherwise, mark δ and the place 0. Let $(D, c_\bullet, *)$ be the tightening of this marked diagram. If the resulting diagram is not marked, return it to Step 1. Otherwise, proceed to Step 3.

Step 3. Replace $(D, c_\bullet, *)$ with the set of diagrams FixA1$(D^\alpha, c_\bullet, *)$ and FixA1$(D^\beta(t), c_\bullet^\beta(t), *)$, where $(D^\beta(t), c_\bullet^\beta(t), *)$ varies over the set of diagrams of type D^β associated to $(D, c_\bullet, *)$ (see Definitions 3.48 and 3.49), and proceed to Step 4.

Step 4. If a diagram output at Step 3 is not marked, return it to Step 1. Otherwise, return it to Step 3.

Notation 3.53. Let Fix$(D, c_\bullet)$ denote the set of diagrams output by running Algorithm FixA2 on a colored symplectic diagram $(D, c_\bullet)$.

We now give three examples.

Example 3.54. Let $D = 00\}_1 00\}_2 00$, then $D^a = 22\}_1 00\}_2 00$ does not satisfy condition (A2). Step 2 of the algorithm replaces this diagram with first $2^*2^*\}_1 00\}_2 00$ and then with the admissible diagram $00]_1 00\}_2 00$.

Example 3.55. Let $D = 22]_1 00]_2 00\}_3 00\}_4 00$. Then $D^a = 12]_1 00]_2 00\}_3 10\}_4 00$ fails condition (A2). Step 2 marks the two ones in D^a. The tightening of the resulting diagram is $1^*1]_1 00]_2 00\}_3 1^*\}_4 000$. Step 3 of Algorithm FixA2 first replaces this diagram with the three diagrams

$$D^\alpha = 1^*1]_1 00]_2 1^*]_3 00\}_4 000, \quad D^\beta(]^2) = 1^*1]_1 1]_2 00\}_4 1^*\}_3 0000 \text{ and } D^\beta(\}^1) = 00]_1 0]_4 0]_2 0]_3 00000.$$

The third diagram $D^{\beta}(\}^1)$ is admissible. Running the loop in Steps 3 and 4 of Algorithm FixA2 replaces D^{α} with the two diagrams

$$11]_1 1]_2 00]_3 00\}_4 000 \text{ and } 11]_1 1]_3 1]_2 00\}_4 0000,$$

both of which are admissible. Algorithm FixA2 replaces $D^{\beta}(]^2)$ with $11]_1 1]_2 1]_4 00\}_3 0000$ which is also admissible.

Example 3.56. Given $P = 00\}_1 00\}_2 1\}_3 000$, Step 2 forms the marked diagram $*00\}_1 00\}_2 1^*\}_3 000$. Then the loop in Steps 3 and 4 of Algorithm FixA2 first associates the pair of diagrams

$$P_1 = P^{\alpha} = *00\}_1 1^*\}_2 00\}_3 000 \text{ and } P_2 = P^{\beta} = *00\}_1 1^*\}_3 1\}_2 0000.$$

Algorithm FixA2 replaces P_1 with the two admissible diagrams

$$D_1 = 1]_1 00\}_2 00\}_3 000 \text{ and } D_2 = 1]_2 1]_1 00\}_3 0000.$$

Algorithm FixA2 first associates to P_2 the two diagrams

$$P_3 = 1]_1 00\}_3 1\}_2 0000 \quad \text{and} \quad P_4 = 1]_3 1]_1 1\}_2 00000.$$

Then the two admissible diagrams

$$D_3 = 1]_1 1]_3 00\}_2 0000 \text{ and } D_4 = 0]_3 0]_1 0]_2 00000, \text{ respectively.}$$

Hence, Algorithm FixA2 replaces P with the diagrams D_1, D_2, D_3 and D_4.

Before proceeding, we urge the reader to practice Algorithm FixA2 on several examples. We suggest $0000\}_1 3\}_2 0\}_3 1\}_4 0$ and $12]_1 3400]_2 00\}_3 0\}_4 0\}_5 10\}_6 00$ as instructive examples.

Definition 3.57. When $a_j \geq 2j - 1$ for $1 \leq j \leq k_h$, let the diagrams associated to a Schubert class $\sigma_{a_\bullet, c_\bullet}$ in $F(k_1, \dots, k_h; n)$ be the collection of diagrams $\mathrm{Fix}(P(a_\bullet; c_\bullet))$ (see Definition 3.32 and Algorithm FixA2 (3.52)).

We are now ready to state the main algorithm that computes the symplectic restriction coefficients.

Algorithm 3.58. (Main Algorithm) Let $(D, c_\bullet)$ be an admissible, colored symplectic diagram.

Step 1. If $(D, c_\bullet)$ is saturated and in perfect order, then return $(D, c_\bullet)$ and stop. Otherwise, proceed to Step 2.

Step 2. Replace $(D, c_\bullet)$ with the set of diagrams $\mathrm{Fix}(D^a, c_\bullet)$ and $\mathrm{Fix}(D^b(]^i), c^i_\bullet)$, where $(D^b(]^i), c^i_\bullet)$ varies over the set of diagrams of type D^b associated to $(D, c_\bullet)$. Output these diagrams and stop.

Definition 3.59. A *degeneration path* is a sequence of admissible colored symplectic diagrams

$$(D_1, c^1_\bullet) \to (D_2, c^2_\bullet) \to \cdots \to (D_r, c^r_\bullet)$$

such that $(D_{i+1}, c^{i+1}_\bullet)$ is one of the admissible colored symplectic diagrams output by running the main Algorithm 3.58 on $(D_i, c^i_\bullet)$ for $1 \leq i < r$.

The main combinatorial theorem of this paper is the following.

Theorem 3.60. *Let $(D, c_\bullet)$ be an admissible colored symplectic diagram for $SF(k_1, \dots, k_h; n)$. Let $V(D, c_\bullet)$ be the symplectic restriction variety associated to $(D, c_\bullet)$. Then, in terms of the Schubert basis of $SF(k_1, \dots . k_h; n)$, the cohomology class $[V(D, c_\bullet)]$ can be expressed as*

$$[V(D, c_\bullet)] = \sum \alpha_{\lambda_\bullet; \mu_\bullet; c_\bullet} \sigma_{\lambda_\bullet; \mu_\bullet; c_\bullet},$$

where $\alpha_{\lambda_\bullet; \mu_\bullet; c_\bullet}$ is the number of degeneration paths starting with $(D, c_\bullet)$ and ending with the symplectic diagram $D(\lambda_\bullet; \mu_\bullet; c_\bullet)$.

A more precise version of Theorem 1.1 is the following corollary of Theorem 3.60.

Corollary 3.61. *Let $\sigma_{a_\bullet,c_\bullet}$ be a Schubert class in $SF(k_1,\dots,k_h;n)$. If $a_j < 2j-1$ for some $1 \le j \le k_h$, then $i^*\sigma_{a_\bullet,c_\bullet} = 0$. Otherwise, let D_i be the diagrams associated to the Schubert class $\sigma_{a_\bullet,c_\bullet}$ (obtained as $\operatorname{Fix} P(a_\bullet,c_\bullet)$ as in Definition 3.57). Express*

$$i^*\sigma_{a_\bullet,c_\bullet} = \sum \gamma_{\lambda_\bullet;\mu_\bullet;c_\bullet}\sigma_{\lambda_\bullet;\mu_\bullet;c_\bullet}$$

in terms of the Schubert basis of $SF(k_1,\dots,k_h;n)$. Then $\gamma_{\lambda_\bullet;\mu_\bullet,c_\bullet}$ is the number of degeneration paths starting with one of the diagrams D_i and ending with the symplectic diagram $D(\lambda_\bullet;\mu_\bullet;c_\bullet)$.

Proof. In the next two sections, in Lemma 4.19 and Theorem 5.3, we will prove that the intersection of a general Schubert variety $\Sigma_{a_\bullet,c_\bullet}$ with $SF(k_1,\dots,k_h;n)$ has class equal to the sum of the classes of restriction varieties corresponding to admissible symplectic diagram $\operatorname{Fix}(P(a_\bullet,c_\bullet))$. The corollary is then an immediate consequence of Theorem 3.60. $\square$

We now give three examples of the main Algorithm 3.58. We urge the reader to carry out similar calculations for themselves. We note that the algorithm is very efficient and it is no trouble at all to carry out calculations for n as large as 20 easily by hand.

Example 3.62. The first example is a computation in $SF(1,2,3;8)$.

$$300\}_1 20\}_2 10\}_3 0 \to 200\}_1 00\}_2 10\}_3 0 \to 200]_1 00\}_2 10\}_3 0 \to 1]_1 0000\}_2 00\}_3 0 \to 1]_1 2200\}_2 00\}_3 0$$

$$\downarrow$$

$$100]_1 00\}_2 00\}_3 0 \to 11]_1 200\}_2 0\}_3 00$$

$$\downarrow$$

$$100]_1 0]_2 000\}_3 0$$

The calculation shows that

$$i^*\sigma^{1,2,3}_{3,5,7} = \sigma^{1,2,3}_{1;3,1} + \sigma^{1,2,3}_{2;3,2} + \sigma^{1,2,3}_{3,4;1}.$$

Example 3.63. The second example calculates $i^*\sigma^{2,1,2,1}_{2,3,5,8}$ in $SF(2,4;10)$.

$$*00\}_2 3^*\}_1 20\}_2 000\}_1 00 \to *1]_2 00\}_1 2^*\}_2 000\}_1 000 \to 1]_2 1]_2 1]_1 00\}_1 00000 \to 0]_2 0]_2 0]_1 00]_1 00000$$

$$\downarrow$$

$$1]_2 1]_1 00\}_2 00\}_1 0000 \to 1]_2 1]_1 00]_2 00\}_1 0000 \to 0]_2 0]_1 00]_2 0]_1 00000$$

$$\downarrow$$

$$1]_2 1]_1 1]_2 100\}_1 0000$$

We conclude that $i^*\sigma^{2,1,2,1}_{2,3,5,8} = \sigma^{2,2,1,1}_{1,2,3,5;} + \sigma^{2,1,2,1}_{1,2,4,5;} + \sigma^{2,1,2,1}_{1,2,3;4}$.

Example 3.64. As a final more complicated example, we compute $i^*\sigma^{1,2,3,2}_{2,3,6,9}$ in $SF(1,3,4;10)$.

$$*00\}_1 3^*\}_2 000\}_3 100\}_2 0 \to 1]_2 1]_1 00\}_3 100\}_2 000 \to 1]_2 1]_1 00]_3 100\}_2 000 \to 1]_2 1]_1 11]_3 00\}_2 0000$$

$$\downarrow \qquad\qquad\qquad\qquad\qquad\qquad \downarrow$$

$$1]_1 00\}_2 00\}_3 100\}_2 00 \qquad\qquad 1]_2 1]_1 1]_3 0000\}_2 000$$

$$\downarrow$$

$$1]_1 00]_2 00\}_3 100\}_2 00 \to 1]_1 1]_2 200\}_3 10\}_2 000 \to 1]_1 1]_2 100\}_3 00\}_2 000 \to 1]_1 1]_2 100]_3 00\}_2 000$$

$$\downarrow$$

$$1]_1 00]_2 0]_3 1000\}_2 00 \to 1]_1 11]_2 1]_3 00\}_2 0000$$

$$\swarrow \qquad \searrow$$

$1]_1 1]_2 00]_3 0000\}_2 00 \quad 1]_1 1]_3 1]_2 0000\}_2 000$

We conclude that $i^*\sigma^{1,2,3,2}_{2,3,6,9} = \sigma^{2,1,3,2}_{1,2,4;4} + \sigma^{2,1,3,2}_{1,2,3;3} + \sigma^{1,2,3,2}_{1,2,5;3} + \sigma^{1,2,3,2}_{1,3,4;4} + \sigma^{1,3,2,2}_{1,2,3;3} + \sigma^{1,2,3,2}_{1,2,4;2}$.

Remark 3.65. The algorithm given for computing the symplectic restriction coefficients may seem mysterious or complicated at first sight to the reader. However, the reader would have naturally discovered the descriptions of the diagrams D^a, D^b (and D^α, D^β) by reflecting on the geometry of the specialization. We will now explain this in some detail.

Definition 3.66. We say that a sequence of brackets and braces D' of type s' is obtained from a colored sequence $(D, c_\bullet)$ of type s if $s' \geq s$ and the positions p_i, p_i' occupied by brackets or braces in D and D' satisfy $p_i' \leq p_i$ for $1 \leq i \leq k_h$.

The associated coloring (see Definition 3.35) is not the geometrically correct way of assigning a coloring to a new diagram. We are lucky that in every case that occurs in the algorithm, the correct way of assigning a new coloring corresponds to Definition 3.35. The next algorithm gives the geometrically accurate way of assigning a coloring $c_\bullet'$ to a sequence D' obtained from a colored sequence $(D, c_\bullet)$. The coloring $c_\bullet'$ is the least restrictive way of assigning a color to D' under the condition that the number of colors less than or equal to i occurring before position p in D' should be greater than or equal to the number of colors less than or equal to i occurring before position p in D for $1 \leq i \leq h$. The next algorithm makes this precise.

Algorithm 3.67. (Coloring algorithm) Let $p_{u_1} < \cdots < p_{u_{k_1}}$ be the positions in D occupied by brackets and braces to which the coloring $c_\bullet$ assigns the color 1. Let $p_{u_1'}$ be the largest position such that $p_{u_1'} \leq p_{u_1}$ and $p_{u_1'}$ is occupied by a bracket or a brace in D'. Let $c_\bullet'$ assign the color 1 to the bracket or brace in position $p_{u_1'}$. Continuing this way suppose $c_\bullet'$ assigns the color 1 to brackets and braces in positions $p_{u_1'}, \dots, p_{u_j'}$. Let $p_{u_{j+1}'} \leq p_{u_{j+1}}$ be the largest position occupied by a bracket or brace of D' that has not been assigned a color. Let $c_\bullet'$ assign the color 1 to the bracket or brace in position $p_{u_{j+1}'}$. Inductively, suppose that $c_\bullet'$ has assigned the colors $1 \leq i \leq l$ and has assigned the color $l+1$ to $j < k_{l+1} - k_l$ brackets and braces of D'. Let p be the position of the bracket or brace in D that is assigned the $(j+1)$-st color equal to $l+1$. Let $p' \leq p$ be the largest position in D' that has a bracket or a brace that has not been assigned a color. Let $c_\bullet'$ assign the color $l+1$ to the bracket or brace in position p'. Continuing inductively, we obtain a coloring $c_\bullet'$ for D'.

The reader can easily check that the coloring Algorithm 3.67 is well-defined and associates a coloring to every diagram obtained from D. We next give three applications of Algorithm 3.67.

Example 3.68. Let $D = 00]_1 0]_2 000\}_3 00$ and let $D' = 0]0]0]00000$ be a diagram obtained from D. The coloring associated to D' is $3, 1, 2$, so that $D' = 0]_3 0]_1 0]_2 00000$.

Let $D = 0000\}_2 3\}_1 0\}_4 00\}_3 00$ and let $D' = 1]1]00\}00\}0000$ be a diagram obtained from D. The coloring associated to D' is $4, 2, 1, 3$, so that $D' = 1]_4 1]_2 00\}_1 00\}_3 0000$.

Let $D = 00]_1 0]_3 0]_3 0]_1 0]_2 0]_1 0]_2 00000000$ and let $D' = 0]0]0]0]0]0]0]000000000$. Then the coloring associated to D' is $3, 1, 3, 2, 1, 2, 1$ so that $D' = 0]_3 0]_1 0]_3 0]_2 0]_1 0]_2 0]_1 000000000$.

The simple geometric observation is that when we specialize the variety corresponding to a diagram $(D, c_\bullet)$, the linear spaces parameterized by the limit have to satisfy rank conditions with respect to a diagram obtained from it. If one makes the specialization described in Definition 3.37 and lists all diagrams that are obtained from $(D, c_\bullet)$ that look identical to the left of ν and have the same dimension as $(D, c_\bullet)$, one would recover the diagrams D^a and D^b. Similarly, if one started with a marked diagram $(D, c_\bullet, *)$ and make the change described by D^α and list all diagrams obtained from $(D, c_\bullet, *)$ that have the same dimension as $(D, c_\bullet, *)$, one would recover the diagrams of type D^α and D^β. The way we formulated the rule is hard to learn at first because one needs to learn which brackets and braces to move. However, it is much easier to implement in practice because one does not need to use the coloring algorithm or compute dimensions while running the algorithm.

The proof of the rule has three steps. The first step is to check that the combinatorics is consistent in the sense that the algorithm replaces an admissible symplectic diagram with other symplectic diagrams. This is straightforward and will be taken up in the rest of this section. The second step is to interpret the algorithm geometrically as a specialization and argue by a dimension count that the varieties corresponding to the diagrams produced by the algorithm are the only possible varieties that could support the limit of the specialization. Finally, the third and relatively easy step is to say that each of these varieties occur with multiplicity one.

We conclude this section by proving that the main Algorithm 3.58 is well-defined and terminates. The proof of Theorem 3.60 is geometric and will be taken up in the next two sections.

Proposition 3.69. *Let $(D, c_\bullet)$ be an admissible, colored, symplectic diagram for the symplectic flag variety $SF(k_1, \dots, k_h; n)$. Algorithm 3.58 replaces $(D, c_\bullet)$ with admissible, colored, symplectic diagrams. Furthermore, the algorithm terminates after finitely many steps.*

Proof. The formation of D^a from D is defined exactly as in [C3]. Therefore, by [C3, Proposition 3.39], D^a is a (not-necessarily-admissible) symplectic diagram. Since D satisfies condition (A1) and $\nu \neq 1$, there cannot be a bracket at the position $p(\nu)$ in D. The diagrams of the type D^b are all formed by moving a bracket to the right of position $p(\nu)$ to position $p(\nu)$. Moving a bracket to the left does not effect conditions (3) and (4) and can only improve condition (2). Since the position to which we are moving the bracket is not occupied, condition (1) holds. Conditions (S1), (S3) and (S4) are unaffected and the inequality can only improve in condition (S2) when we move a bracket to the left. We conclude that both D^a and diagrams of type D^b are (not-necessarily-admissible) symplectic diagrams.

Since Algorithms FixA1 (3.43) and FixA2 (3.52) differ from the corresponding algorithms in [C3], we need to check that the outputs of these algorithms are still symplectic diagrams. We first check that Algorithm FixA1 (3.43) preserves condition (A2) and outputs (marked) symplectic diagrams. This has been checked in the proof of [C3, Proposition 3.39] except in the case when position $p(\}^{i-1}) - 1$ is occupied by a brace. (Notice that the procedure described in Steps 2-4 of Algorithm FixA1 is the same whether the diagram is marked or not.) In that case, we move $\}^{i-1}$ to the first unoccupied position to the left of $p(\}^{i-1})$ and change the integers in the sequence as specified by Algorithm FixA1. The conditions (1)-(4) do not change for any of the braces or brackets except for the brace that we move. Since we are moving the brace to an unoccupied position, condition (1) holds. By condition (4), in the initial diagram there must be at least one zero to the left of $\}^{i-1}$. Condition (S2) then implies that there is an unoccupied position between the brackets and $\}^{i-1}$, so condition (2) holds. Conditions (3) and (4) hold by construction. Observe that by changing one of the integers from $i + l + 1$ to $i + l$, we guarantee condition (4). Similarly, (S1) holds for $\}^{i+l}$ and $\}^{i+l+1}$ by construction. If the diagram is in order, applying Algorithm FixA1 preserves order. If η violates order, then η remains the only integer violating the order and conditions (S3) and (S4) are preserved. Finally, condition (S2) must hold for the new $\}^{i+l}$ (the only brace for which the quantities change) since τ_u does not change unless $u = i + l = k_h - s$, $p(\}^{i+l})$ decreases by one and r_{i+l} increases by one. Hence, the inequality remains the same. If $i + l = k_h - s$, then τ_{i+l} increases by one provided that the leftmost zero is to right of all the brackets. However, in that case, condition (S2) holds automatically. We conclude that Algorithm FixA1 preserves symplectic diagrams. Similarly, the operation preserves the inequality in condition (A2). If $p(\}^{i-1})-1$ is not occupied, then x_{i-1} and r_{i-1} both increase by one and $p(\}^{i-1})$ decreases by one, preserving the inequality in condition (A2). If $p(\}^{i-1})-1$ is occupied and after moving $\}^{i-1}$ to p condition (A2) is violated for a brace, then it is immediate to check that (A2) would be violated for $\}^{i-1}$ in the input. We conclude that Algorithm FixA1 preserves condition (A2).

Next we check that Algorithm FixA2 outputs admissible symplectic diagrams. The diagram D^a may fail to satisfy condition (A2). The formation of D^a does not change the positions of the brackets or braces and the position $p(\nu)$ in D is not occupied by a bracket. Hence, the quantities $p(\}^j)$ and x_j remain constant. In cases (1)(i) and (2)(ii)(a), the quantities r_j also remain constant. Hence, in these cases D^a satisfies condition (A2). In cases (1)(ii), (2)(i) and (2)(ii)(b), these quantities remain unchanged except r_i increases by two in (1)(ii) and r_κ increases by two in cases (2)(i) and (2)(ii)(b). We conclude that D^a can violate condition (A2) only by one for one index when the equality $x_j = k-j+1-\frac{p(\}^j)-r_j}{2}$ holds for $j=i$ in case (1)(ii) and $j=\kappa$ in cases (2)(i) and (2)(ii)(b). Observe, however, if (A2) is violated for the index i in case (1)(ii) or the index κ in case (2)(ii)(b), then condition (A2) would be violated for the indices $i-1$ and $\kappa-1$ in the diagram D, respectively. We conclude that condition (A2) can be violated for D^a only in case (2)(i) and only for the index κ. In particular, when a diagram D^a violates (A2), in Step 2 of Algorithm FixA2 we have $p(\delta')>0$. For future reference, observe that if equality holds in the inequality in condition (A2) for an index j' in an admissible diagram D, then equality holds for every index $j \geq j'$. Notice that since x_i (respectively, x_κ) increases by one in case (1)(ii) (respectively, (2)(i) and (2)(ii)(b)) in the diagrams of the type D^b, we conclude that diagrams of type D^b always satisfy condition (A2). An initial diagram $P(a_\bullet, c_\bullet)$ fails to satisfy condition (A2) for indices j such that $a_j = 2j-1$. In this case, in Step 2 of Algorithm FixA2 we have $p(\delta')=0$.

We remark that whether $p(\delta')=0$ or not distinguishes whether the initial input of Algorithm FixA2 is a diagram of the form D^a or $P(a_\bullet, c_\bullet)$. Notice that when the initial input is D^a, then Algorithm FixA2 preserves condition (A2) for braces with larger indices than the initial brace violating (A2). This is automatic in Case (1) and clear for the diagram D^α. It is true for diagrams of type D^β in Case (2) by construction since the formation of diagrams of type D^β (when $p(\delta')>0$) increases x_{j+1} through x_{j+l} by at least one. The braces with indices smaller than i are not affected by the algorithm and hence satisfy condition (A2). Once we check that Algorithm FixA2 outputs admissible symplectic diagrams, we can conclude that one run of the loop in Steps 3 and 4 of the algorithm fixes the diagram when the initial input is D^a. When the initial input is $P(a_\bullet, c_\bullet)$, some of the diagrams of type D^β may have larger index braces that fail to satisfy condition (A2) (for diagrams that arise as $(D^\beta(\}^j), c_\bullet^\beta(\}^j), *)$ in case (2)). Nevertheless, the only brackets in diagrams produced by the algorithm have the property that $p(]^i)=i$ and each of the brackets are preceded by a 1. Hence, Step 2 of Algorithm FixA2 always has $p(\delta')=0$ through out the algorithm when the initial input is $P(a_\bullet, c_\bullet)$. Note also that after every return to Step 1 in Algorithm FixA2, there is at least one more bracket and one fewer brace. Since the total number of braces is bounded, Algorithm FixA2 terminates when starting with initial input $P(a_\bullet, c_\bullet)$, even though more braces may fail to satisfy (A2) midway through the algorithm.

By construction, it is easy to see that the diagrams D^α and D^β satisfy conditions (1)-(4), (S1) and (S2). When running Algorithm FixA2 on a diagram derived from D^a, the only possibly out of order integer is the marked integer δ. Hence, (S3) holds. Condition (S4) holds by construction. The fact that the resulting diagrams satisfy condition (A1) is built into Algorithm FixA2. When running Algorithm FixA2 with initial input $P(a_\bullet, c_\bullet)$, the diagrams may have one more integer η other than δ out of order. In that case, the integer is $\cdots \}^{j+1} j-1 \}^j \delta^* \}^{j-1} \cdots$ and at the next stage of the algorithm, we swap δ and $j-1$ to obtain a diagram where again only δ (possibly) violates the order. Hence these diagrams also satisfy (S3) and (S4). We conclude that every diagram output by Algorithm FixA2 is an admissible symplectic diagram. Hence, Algorithm 3.58 replaces admissible symplectic diagrams or an initial diagram with a collection of admissible symplectic diagrams.

The termination of the algorithm is clear. In case (2)(i), the formation of D^a from D increases the number of positive integers in the sequence. In cases (1)(i), (1)(ii), (2)(ii)(a) and (2)(ii)(b), the formation of D^a from D either increases the number of positive integers in the initial part of the sequence or decreases at least one of the positive integers in the initial part of the sequence. The formation of D^b shifts one bracket to the left. Similarly, Algorithm FixA1 (3.43) decreases at least one positive integer in the initial part of the sequence or increases the number of positive integers in the sequence and shifts at least one brace to the left. Similarly, Algorithm FixA2 (3.52) decreases the number of braces and increases the number of brackets. Since the total number of brackets and braces is fixed at k_h, no new braces are formed during the algorithm and no bracket or brace ever moves to the right, each of these steps can only be repeated finitely many times until the resulting diagram is saturated and in order.

□

4. The symplectic restriction varieties

In this section, we define symplectic restriction varieties in $SF(k_1, \dots, k_h; n)$ and show that they can be represented by admissible colored symplectic diagrams. We develop the basic geometric properties of these varieties. The translation between the combinatorics and the geometry is almost identical to the Grassmannian case discussed in [C3]. For the convenience of the reader, we recall the main definitions from [C3].

Let Q denote a non-degenerate skew-symmetric form on an n-dimensional vector space V. Let L_{n_j} denote an isotropic subspace of Q of dimension n_j. Let $Q_{d_i}^{r_i}$ denote a linear space of dimension d_i such that the restriction of Q to it has corank r_i. Let K_i denote the kernel of the restriction of Q to $Q_{d_i}^{r_i}$.

A sequence $(L_\bullet, Q_\bullet)$ for $SG(k_h, n)$ is a partial flag of linear spaces

$$L_{n_1} \subset \cdots \subset L_{n_s} \subset Q_{d_{k_h-s}}^{r_{k_h-s}} \subset \cdots \subset Q_{d_1}^{r_1} \ \text{ such that}$$

- $\dim(K_i \cap K_l) \geq r_i - 1$ for $l > i$ and $1 \leq i \leq k_h - s$,
- $\dim(L_{n_j} \cap K_i) \geq \min(n_j, \dim(K_i \cap Q_{d_{k_h-s}}^{r_{k_h-s}}) - 1)$ for every $1 \leq j \leq s$ and $1 \leq i \leq k_h - s$.

Definition 4.1. A *colored sequence* $(L_\bullet, Q_\bullet, c_\bullet)$ for $SF(k_1, \dots, k_h; n)$ is a sequence $(L_\bullet, Q_\bullet)$ for $SG(k_h, n)$ together with a coloring $c_\bullet$ for $SF(k_1, \dots, k_h; n)$.

The main geometric objects of this paper are colored sequences satisfying further properties. A colored sequence for $SF(k_1, \dots, k_h; n)$ is *in order* if the underlying sequence $(L_\bullet, Q_\bullet)$ satisfies:

- $K_i \cap K_l = K_i \cap K_{i+1}$, for all $l > i$ and $1 \leq i \leq k_h - s$, and
- $\dim(L_{n_j} \cap K_i) = \min(n_j, \dim(K_i \cap Q_{d_{k_h-s}}^{r_{k_h-s}}))$, for $1 \leq j \leq s$ and $1 \leq i < k_h - s$.

A colored sequence $(L_\bullet, Q_\bullet, c_\bullet)$ is *in perfect order* if

- $K_i \subseteq K_{i+1}$, for $1 \leq i < k_h - s$, and
- $\dim(L_{n_j} \cap K_i) = \min(n_j, r_i)$ for all i and j.

Definition 4.2. A colored sequence $(L_\bullet, Q_\bullet, c_\bullet)$ is called *saturated* if $d_i + r_i = n$, for $1 \leq i \leq k_h - s$.

Definition 4.3. A colored sequence $(L_\bullet, Q_\bullet, c_\bullet)$ is called a *symplectic sequence* if it satisfies the following properties.

(GS1) The sequence $(L_\bullet, Q_\bullet, c_\bullet)$ is either in order or there exists at most one integer $1 \leq \eta \leq k_h - s$ such that
$$K_i \subseteq K_l \text{ for } l > i > \eta \text{ and } K_i \cap K_l = K_i \cap K_{i+1} \text{ for } i < \eta \text{ and } l > i.$$
Furthermore, if $K_\eta \subseteq K_{k_h-s}$, then
$$\dim(L_{n_j} \cap K_i) = \min(n_j, \dim(K_i \cap Q_{d_{k_h-s}}^{r_{k_h-s}})) \text{ for } i < \eta \text{ and}$$
$$\dim(L_{n_j} \cap K_i) = \min(n_j, \dim(K_i \cap Q_{d_{k_h-s}}^{r_{k_h-s}}) - 1) \text{ for } i \geq \eta.$$
If $K_\eta \not\subseteq K_{k_h-s}$, then $\dim(L_{n_j} \cap K_i) = \min(n_j, \dim(K_i \cap Q_{d_{k_h-s}}^{r_{k_h-s}}))$ for all i.

(GS2) If $\alpha = \dim(K_i \cap Q_{d_{k_h-s}}^{r_{k_h-s}}) > 0$, then either $i = 1$ and $n_\alpha = \alpha$ or there exists at most one j_0 such that, for $j_0 \neq j > \min(i, \eta)$, $r_j - r_{j-1} = d_{j-1} - d_j$. Furthermore,
$$d_{j_0-1} - d_{j_0} \leq r_{j_0} - r_{j_0-1} + 2 - \dim(K_{j_0-1}) + \dim(K_{j_0-1} \cap Q_{d_{j_0}}^{r_{j_0}})$$
and $K_\eta \not\subset Q_{d_{j_0}}^{r_{j_0}}$.

Remark 4.4. Given a sequence $(L_\bullet, Q_\bullet, c_\bullet)$, the basic principles about skew-symmetric forms imply inequalities among the invariants of a sequence. The evenness of rank implies that $d_i - r_i$ is even for every $1 \leq i \leq k_h - s$. The corank bound implies that $r_i - \dim(Q_{d_i}^{r_i} \cap K_{i-1}) \leq d_{i-1} - d_i$. The linear space bound implies that $2(n_s + r_i - \dim(K_i \cap L_{n_s})) \leq r_i + d_i$ for every $1 \leq i \leq k_h - s$. These inequalities are implicit in the sequence $(L_\bullet, Q_\bullet, c_\bullet)$.

Remark 4.5. For a symplectic sequence $(L_\bullet, Q_\bullet, c_\bullet)$, the coloring $c_\bullet$, the invariants n_j, r_i, d_i and the dimensions $\dim(L_{n_j}, K_i)$ and $\dim(Q_{d_h}^{r_h} \cap K_i)$ determine the sequence $(L_\bullet, Q_\bullet, c_\bullet)$ up to the action of the symplectic group. This will become clear when we construct these sequences by choosing bases.

Definition 4.6. A colored symplectic sequence $(L_\bullet, Q_\bullet, c_\bullet)$ is *admissible* if it satisfies the following additional conditions:

(GA1) $n_j \neq \dim(L_{n_j} \cap K_i) + 1$ for any $1 \leq j \leq s$ and $1 \leq i \leq k_h - s$.

(GA2) Let x_i denote the number of isotropic subspaces L_{n_j} that are contained in K_i. Then
$$x_i \geq k_h - i + 1 - \frac{d_i - r_i}{2}, \quad \text{for } 1 \leq i \leq k_h - s.$$

The symplectic restriction varieties will be defined in terms of colored admissible sequences.

4.1. **Associating a symplectic diagram to a geometric sequence.** Colored symplectic sequences can be represented by colored symplectic diagrams introduced in §3. An isotropic linear space L_{n_j} is represented by a bracket] in position n_j. A linear space $Q_{d_i}^{r_i}$ is represented by a brace } in position d_i such that there are exactly r_i positive integers less than or equal to i to the left of the i-th brace. Finally, $\dim(L_{n_j} \cap K_i)$ and $\dim(Q_{d_l}^{r_l} \cap K_i)$, $l > i$, are recorded by the number of positive integers less than or equal to i to the left of $]^j$ and $\}^l$, respectively (see Remark 3.7). The colorings associated to the sequence and the diagram are the same.

More explicitly, given a colored symplectic sequence $(L_\bullet, Q_\bullet, c_\bullet)$, the corresponding symplectic diagram $D(L_\bullet, Q_\bullet, c_\bullet)$ is determined as follows: The sequence of integers begins with $\dim(L_{n_1} \cap K_1)$ integers equal to 1, followed by $\dim(L_{n_1} \cap K_i) - \dim(L_{n_1} \cap K_{i-1})$ integers equal to i, for $2 \leq$

$i \leq k_h - s$, in increasing order, followed by $n_1 - \dim(L_{n_1} \cap K_{k_h-s})$ integers equal to 0. The sequence then continues with $\dim(L_{n_j} \cap K_1) - \dim(L_{n_{j-1}} \cap K_1)$ integers equal to 1, followed by $\dim(L_{n_j} \cap K_i) - \max(\dim(L_{n_{j-1}} \cap K_i), \dim(L_{n_j} \cap K_{i-1}))$ integers equal to i in increasing order, followed by $n_j - \max(n_{j-1}, \dim(L_{n_j} \cap K_{k_h-s}))$ zeros for $j = 2, \ldots, s$ in increasing order. The sequence then continues with $\dim(Q^{r_{k_h-s}}_{d_{k_h-s}} \cap K_1) - \dim(L_{n_s} \cap K_1)$ integers equal to 1, followed by $\dim(Q^{r_{k_h-s}}_{d_{k_h-s}} \cap K_i) - \max(\dim(Q^{r_{k_h-s}}_{d_{k_h-s}} \cap K_{i-1}), \dim(L_{n_s} \cap K_i))$ integers equal to i in increasing order, followed by zeros until position d_{k_h-s}. Between positions d_i and d_{i-1} ($i > k_h - s$), the sequence has $\dim(Q^{r_{i-1}}_{d_{i-1}} \cap K_1) - \dim(Q^{r_i}_{d_i} \cap K_1)$ integers equal to 1, followed by $\dim(Q^{r_{i-1}}_{d_{i-1}} \cap K_l) - \max(\dim(Q^{r_i}_{d_i} \cap K_l), \dim(Q^{r_{i-1}}_{d_{i-1}} \cap K_{l-1}))$ integers equal to l in increasing order, for $l \leq i-1$, followed by zeros until position d_{i-1}. Finally, the sequence ends with $n - d_1$ zeros. The brackets occur at positions n_j and the braces occur at positions d_i. The colorings are the same.

Example 4.7. Given a sequence $L_1(1) \subset L_3(3) \subset L_5(2) \subset L_7(1) \subset Q^6_{10}(2) \subset Q^3_{13}(3) \subset Q^2_{16}(1)$ for $SF(3,5,7;18)$, where the numbers in parentheses denote the coloring, and the relations $K_2 \subset K_3$, $\dim(K_1 \cap Q^3_{13}) = 1$, $\dim(K_1, L_1) = 1$, $L_3 = K_2$ and $L_5 = K_3 \cap L_7$, the corresponding diagram is $1]_1 22]_3 33]_2 00]_1 300\}_2 000\}_3 100\}_1 00$. The dimensions of the isotropic spaces are $1, 3, 5, 7$, hence the brackets occur at these positions. The dimensions of the non-isotropic spaces are $10, 13, 16$, so the braces occur at these positions. The dimensions of the kernels are $6, 3, 2$, so the sequence has 2 integers equal to 1. Since $\dim(K_1 \cap Q^3_{13}) = 1$ and $\dim(K_1, L_1) = 1$, one of these occur at place 1 and one at place 14. The sequence then has 2 integers equal to 2 and since $L_3 = K_2$, these occur at places 2 and 3. Finally, the sequence has 3 integers equal to 3. Since $L_5 = K_3 \cap L_7$, these have to occur at places 4,5 and 8.

Proposition 4.8. *The diagram $D(L_\bullet, Q_\bullet, c_\bullet)$ is a symplectic diagram for $SF(k_1, \ldots, k_h; n)$. Furthermore, if $(L_\bullet, Q_\bullet, c_\bullet)$ is admissible, then $D(L_\bullet, Q_\bullet, c_\bullet)$ is admissible.*

Proof. The construction of $D(L_\bullet, Q_\bullet, c_\bullet)$ from $(L_\bullet, Q_\bullet, c_\bullet)$ for $SF(k_1, \ldots, k_h; n)$ is identical to the construction of $D(L_\bullet, Q_\bullet)$ from $(L_\bullet, Q_\bullet)$ for $G(k_h, n)$ except for the data of the coloring. By [C3, Proposition 4.9], $D(L_\bullet, Q_\bullet)$ is a symplectic diagram for $G(k_h, n)$, which is admissible if $(L_\bullet, Q_\bullet)$ is admissible. We conclude that $D(L_\bullet, Q_\bullet, c_\bullet)$ is a symplectic diagram for $SF(k_1, \ldots, k_h; n)$ which is admissible if $(L_\bullet, Q_\bullet, c_\bullet)$ is admissible. ☐

Remark 4.9. Proposition 4.8 explains the conditions in the definition of a symplectic diagram in geometric terms. Condition (1) holds since every linear space in the flag is a distinct vector space and its dimension corresponds to the position of the corresponding bracket or brace. Condition (2) reflects the fact that isotropic subspaces precede the non-isotropic ones in the sequence. Integers in the sequence equal to i represent vectors in the kernel of the restriction of Q to $Q^{r_i}_{d_i}$. Hence, condition (3) holds. Condition (4) of Definition 3.3 is implied by the evenness of rank and simply states that $d_i - r_i$ is even. Condition (S1) is a translation of the corank bound saying that the codimension of $K_{i-1} \cap K_i$ in K_i is bounded by the codimension of $Q^{r_i}_{d_i}$ in $Q^{r_{i-1}}_{d_{i-1}}$. Condition (S2) is a consequence of the linear space bound since the linear space $Q^{r_i}_{d_i}$ contains a linear space of dimension at least τ_i.

4.2. **Associating a geometric sequence to a colored symplectic diagram.** Conversely, we can associate an admissible sequence to every admissible symplectic diagram $(D, c_\bullet)$ for $SF(k_1, \ldots, k_h; n)$. Let $e_1, \ldots, e_m, f_1, \ldots, f_m$ be a basis that diagonalizes the form Q with $f_i = e_i^*$. Given an admissible symplectic diagram, we associate $e_1, \ldots, e_{p(]^s)}$ to the integers to the left of $]^s$ in order. We then associate $e_{p(]^s)+1}, \ldots, e_{r'}$ to the positive integers to the right of $]^s$ and left of $\}^{k_h-s}$ in order. Let $e_{i_1}, \ldots, e_{i_l}$ be vectors that have so far been associated to zeros. Then associate

$f_{i_1}, \dots, f_{i_l}$ to the remaining zeros to the left of $\}^{k_h-s}$ in order. If there are any zeros to the left of $\}^{k_h-s}$ that have not been assigned a basis vector, assign them $e_{r'+1}, f_{r'+1}, \dots, e_{r''}, f_{r''}$ in pairs in order. Continuing this way, if there is a positive integer between $\}^{i+1}$ and $\}^i$ associate to it the smallest index basis element e_α that has not yet been assigned. Assume that the integers equal to $i+1$ have been assigned the vectors $e_{j_1}, \dots, e_{j_l}$. Assign to the zeros between $\}^{i+1}$ and $\}^i$, the vectors $f_{j_1}, \dots, f_{j_l}$. If there are any zeros between $\}^{i+1}$ and $\}^i$ that have not been assigned a vector, assign them $e_{\alpha+1}, f_{\alpha+1}, \dots, e_\beta, f_\beta$ in pairs until the zeros are exhausted. Let L_{n_j} be the span of the basis elements associated to the integers to the left of $]^j$. Let $Q^{r_i}_{d_i}$ be the span of the basis elements associated to the integers to the left of $\}^i$. We thus obtain a sequence $(L_\bullet, Q_\bullet, c_\bullet)$ whose associated symplectic diagram is $(D, c_\bullet)$. The vectors associated to the integers in a marked diagram $(D, c_\bullet, *)$ is the same as the sequence associated to $(D, c_\bullet)$.

Example 4.10. Let $D = 11]_3 233]_1 00]_2 30000\}_2 200000\}_3 100\}_1 000$ be a diagram for $SF(2,4,6;24)$. To this diagram we associate the vectors

$$e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, f_6, f_7, e_9, f_9, e_{10}, f_4, f_5, f_8, e_{11}, f_{11}, e_{12}, f_3, f_{10}, f_1, f_2, f_{12}$$

in order. Then $L_2(3)$ is the span of e_1 and e_2. $L_5(1)$ is the span of e_i for $1 \leq i \leq 5$. $L_7(2)$ is the span of e_i for $1 \leq i \leq 7$. $Q^6_{12}(3)$ is the span of e_i, $1 \leq i \leq 9$ and f_6, f_7, f_9. $Q^4_{18}(3)$ is the span of $Q^6_{12}(3)$ and the vectors $e_{10}, e_{11}, f_4, f_5, f_8, f_{11}$. Finally, $Q^3_{21}(1)$ is the span of $Q^4_{18}(3)$ and the vectors e_{12}, f_3, f_{10}.

Remark 4.11. As observed in [C3, Remark 4.13], the construction of $(L_\bullet, Q_\bullet, c_\bullet)$ from $(D, c_\bullet)$ is well-defined thanks to conditions (4), (S1), (S2) and (S3). The formation of a sequence $(L_\bullet, Q_\bullet, c_\bullet)$ from an admissible diagram $(D, c_\bullet)$ and the formation of an admissible symplectic diagram $(D, c_\bullet)$ from an admissible sequence are inverse constructions. Further note that equivalent symplectic diagrams (see Definition 3.8) correspond to permutations of the basis elements that do not change the vector spaces in $(L_\bullet, Q_\bullet, c_\bullet)$.

Remark 4.12. For some purposes, it is more convenient to reorder the basis elements between any two consecutive brackets and/or braces so that all the e_i precede all the f_j and are listed in *increasing* order from left to right and all the f_j are listed in *decreasing* order from left to right. Since this operation does not change any of the vector spaces recorded by the diagram, we are free to reorder the basis elements in this way. For example, given a saturated admissible diagram in perfect order, if we so order the basis elements, the ordering becomes $e_1, e_2, \dots, e_m, f_m, \dots, f_2, f_1$.

We are now ready to define symplectic restriction varieties.

Definition 4.13. Let $(L_\bullet, Q_\bullet, c_\bullet)$ be an admissible sequence for $SF(k_1, \dots, k_h; n)$. Then the *symplectic restriction variety* $V(L_\bullet, Q_\bullet, c_\bullet)$ is the Zariski closure of the locus in $SF(k_1, \dots, k_h; n)$ parameterizing

$$\begin{aligned}\{(W_1, \dots, W_h \in SF(k_1, \dots, k_h; n) \mid \ & \dim(W_u \cap L_{n_j}) = \#\{c_l \mid l \leq j, c_l \leq u\} \text{ for } 1 \leq j \leq s,\\ & \dim(W_u \cap Q^{r_i}_{d_i}) = \#\{c_l \mid l \leq k_h - i + 1, c_l \leq u\} \text{ and}\\ & \dim(W_u \cap K_i)) = \#\{c_l \mid l \leq x_i, c_l \leq u\} \text{ for } 1 \leq i \leq k_h - s\}.\end{aligned}$$

Remark 4.14. The geometric reasons for imposing conditions (A1) and (A2) in Definition 3.22 are now clear. Condition (A1) is an immediate consequence of the kernel bound. If $\dim(L_{n_j} \cap K_i) = n_j - 1$ and a linear space of dimension $k_h - i + 1$ intersects n_j in dimension j and K_i in dimension $j-1$, then the linear space is contained in $L^\perp_{n_j}$. Hence, we need to impose condition (A1).

The inequality

$$x_i \geq k_h - i + 1 - \frac{d_i - r_i}{2}$$

is an immediate consequence of the linear space bound. We require the k_h-dimensional isotropic subspaces to intersect $Q_{d_i}^{r_i}$ in a subspace of dimension $k_h - i + 1$ and to intersect the singular locus of $Q_{d_i}^{r_i}$ in a subspace of dimension x_i. By the linear space bound, any linear space of dimension $k_h - i + 1$ has to intersect the singular locus in a subspace of dimension at least $k_h - i + 1 - \frac{d_i - r_i}{2}$, hence the inequality in condition (A2).

Example 4.15. The two most basic examples of symplectic restriction varieties are:

(1) A Schubert variety $\Sigma_{\lambda_\bullet;\mu_\bullet;c_\bullet}$ in $SF(k_1, \dots, k_h; n)$, which is the restriction variety associated to a symplectic diagram $D(\lambda_\bullet; \mu_\bullet; c_\bullet)$, and
(2) The intersection $\Sigma_{a_\bullet, c_\bullet} \cap SF(k_1, \dots, k_h; n)$ of a general Schubert variety in $F(k_1, \dots, k_h; n)$ satisfying $a_j > 2j - 1$ for all $1 \leq j \leq k_h$ with $SF(k_1, \dots, k_h; n)$, which is the restriction variety associated to $D(a_\bullet, c_\bullet)$.

In general, symplectic restriction varieties interpolate between these two examples. Unlike the case of Grassmannians, the intersection $\Sigma_{a_\bullet, c_\bullet} \cap SF(k_1, \dots, k_h; n)$ when $a_j = 2j - 1$ for some j need not be a symplectic restriction variety. However, it can be degenerated into a union of symplectic restriction varieties as we will see in the next section.

Lemma 4.16. *A symplectic restriction variety corresponding to a saturated and perfectly ordered admissible sequence is a Schubert variety in $SF(k_1, \dots, k_h; n)$. Conversely, every Schubert variety in $SF(k_1, \dots, k_h; n)$ can be represented by such a sequence.*

Proof. By Remark 4.12, the partial flag recorded by a saturated and perfectly ordered admissible sequence is an isotropic partial flag. Therefore, the definition of a symplectic restriction variety reduces to the definition of a Schubert variety. □

The next proposition shows that symplectic restriction varieties are irreducible and calculates their dimension. Note that the dimension of a symplectic diagram introduced in Definition 3.26 is equal to the dimension of the corresponding symplectic restriction variety.

Proposition 4.17. *Let $(L_\bullet, Q_\bullet, c_\bullet)$ be an admissible colored sequence. Then the symplectic restriction variety $V(L_\bullet, Q_\bullet, c_\bullet)$ is an irreducible subvariety of $SF(k_1, \dots, k_h; n)$ of dimension*

$$\dim(V(L_\bullet, Q_\bullet, c_\bullet)) = \sum_{j=1}^{s} (n_j - j) + \sum_{i=1}^{k_h - s} (d_i - 1 - 2k_h + 2i + x_i) + \dim(c_\bullet). \tag{1}$$

Proof. The projection $\pi_h(V(L_\bullet, Q_\bullet, c_\bullet))$ is the symplectic restriction variety $V(L_\bullet, Q_\bullet)$ in the symplectic Grassmannian $SG(k_h, n)$. By [C3, Proposition 4.21], the latter variety is irreducible of dimension

$$\sum_{j=1}^{s} (n_j - j) + \sum_{i=1}^{k_h - s} (d_i - 1 - 2k_h + 2i + x_i).$$

The linear spaces $(L_\bullet, Q_\bullet)$ determine a complete flag on W_h, where W_h is a point contained in the image of the Zariski open set defining $V(L_\bullet, Q_\bullet, c_\bullet)$ under π_h. The fiber of π_h over W_h is the Schubert variety with class $c_\bullet$ in $F(k_1, \dots, k_{h-1}; k_h)$ defined with respect to this flag. By the Theorem on the Dimension of Fibers [S, I.6.7], $V(L_\bullet, Q_\bullet, c_\bullet)$ is irreducible of the claimed dimension. □

Next, we show that the intersection of a general Schubert variety $\Sigma_{a_\bullet, c_\bullet}$ with $SF(k_1, \dots, k_h; n)$ is non-empty if and only if $a_j \geq 2j - 1$ for $1 \leq j \leq k_h$. Furthermore, the intersection is a symplectic restriction variety if $a_j > 2j - 1$ for $1 \leq j \leq k_h$. Otherwise, the class of the intersection is the sum of the classes of the restriction varieties in $\mathrm{Fix}(P(a_\bullet, c_\bullet))$. We postpone the proof of the last statement to the next section.

Lemma 4.18. *Let $\Sigma_{a_\bullet,c_\bullet}$ be a Schubert variety defined with respect to a general partial flag $F_{a_1} \subset \cdots \subset F_{a_{k_h}}$. Then $\Sigma_{a_\bullet,c_\bullet} \cap SF(k_1,\dots,k_h;n) \neq \emptyset$ if and only if $a_j \geq 2j-1$ for $1 \leq j \leq k_h$.*

Proof. Suppose $a_j < 2j-1$ for some j. If $[W_1 \subset \cdots \subset W_h] \in \Sigma_{a_\bullet,c_\bullet} \cap SF(k_1,\dots,k_h;n)$, then $W_h \cap F_{a_j}$ is an isotropic subspace of $Q \cap F_{a_j}$ of dimension at least j. Since F_{a_j} is general, the corank of $Q \cap F_{a_j}$ is 0 or 1 depending on whether a_j is even or odd. By the linear space bound, the largest dimensional isotropic subspace of $Q \cap F_{a_j}$ has dimension less than or equal to $j-1$. Therefore, W_h cannot exist and $\Sigma_{a_\bullet,c_\bullet} \cap SF(k_1,\dots,k_h;n) = \emptyset$.

Conversely, let $a_j = 2j-1$ for every j. Then $G_1 = F_1$ is isotropic, $G_2 = F_1^\perp$ in F_3 is the unique two-dimensional isotropic subspace of $Q \cap F_3$ containing G_1. By induction, we see that $G_j = G_{j-1}^\perp$ is the unique subspace of dimension j isotropic with respect to $Q \cap F_{2i-1}$ that contains G_{j-1}. Continuing this way, we construct a unique isotropic subspace W_h of dimension k_h contained in $\Sigma_{a_\bullet,c_\bullet} \cap SG(k_h,n)$. The flag $F_\bullet$ induces a complete flag $G_\bullet$ on W_h and the intersection $\Sigma_{a_\bullet,c_\bullet} \cap SF(k_1,\dots,k_h;n)$ is the Schubert variety Ω (defined with respect to $G_\bullet$) in $F(k_1,\dots,k_{h-1};k_h)$ corresponding to $c_\bullet$. In particular, the intersection is non-empty. If $a_j \geq 2j-1$, Ω is still contained in $\Sigma_{a_\bullet,c_\bullet} \cap SF(k_1,\dots,k_h;n)$, hence this intersection is non-empty. □

Lemma 4.19. *Let $\Sigma_{a_\bullet,c_\bullet}$ be a Schubert variety defined with respect to a general partial flag $F_{a_1} \subset \cdots \subset F_{a_{k_h}}$ such that $a_j \geq 2j-1$ for $1 \leq j \leq k_h$. Then $\Sigma_{a_\bullet,c_\bullet} \cap SF(k_1,\dots,k_h;n) = V(P(a_\bullet,c_\bullet))$ and is irreducible. Furthermore, if $a_j > 2j-1$ for $1 \leq j \leq k_h$, then $\Sigma_{a_\bullet,c_\bullet} \cap SF(k_1,\dots,k_h;n) = V(D(a_\bullet,c_\bullet))$.*

Proof. The Schubert variety $\Sigma_{a_\bullet,c_\bullet}$ is irreducible. The complement of the open cell in $\Sigma_{a_\bullet,c_\bullet}$ is a finite union of lower dimensional Schubert varieties. By Kleiman's Transversality Theorem, their intersection with $SF(k_1,\dots,k_h;n)$ have lower dimension and cannot form components of the intersection. We may, therefore, restrict our attention to the open cell U. The lemma follows by induction on h and k_h. If $h=1$, the irreducibility is clear by induction on k_h and has been observed in [C3, Lemma 4.18]. The fibers of the restriction of the projection $\pi_h : SF(k_1,\dots,k_h;n) \to SG(k_h,n)$ to $U \cap SF(k_1,\dots,k_h;n)$ are open subsets of a Schubert variety in $F(k_1,\dots,k_{h-1};k_h)$. Therefore, the intersection $\Sigma_{a_\bullet,c_\bullet} \cap SF(k_1,\dots,k_h;n)$ is irreducible and when $a_j > 2j-1$ for all j equal to $V(D(a_\bullet,c_\bullet))$. □

Similarly, we can associate a variety to a marked diagram $(D, c_\bullet, *)$ occurring while running Algorithm FixA2 (3.52). Let e_δ be the vector corresponding to the rightmost marking δ and let F' be the span of the vectors up to and including the leftmost marking δ'. Let F be the span of F' and e_δ. Assume that the number of brackets to the left of position $p(\delta')$ inclusive is t. Then the variety corresponding to the marked diagram $(D, c_\bullet, *)$ is defined by

$$\begin{aligned}\{(W_1,\dots,W_h \in SF(k_1,\dots,k_h;n) \mid \ &\dim(W_u \cap L_{n_j}) = \#\{c_l \mid l \leq j, c_l \leq u\} \text{ for } 1 \leq j \leq s,\\ &\dim(W_u \cap Q_{d_i}^{r_i}) = \#\{c_l \mid l \leq k_h - i + 1, c_l \leq u\}\\ \dim(W_u \cap K_i)) = \#\{c_l \mid l \leq x_i, c_l \leq u\} &\text{ for } 1 \leq i \leq k_h - s \text{ and } \dim(W_h \cap F) = t+1\}.\end{aligned}$$

As in the proof of Lemmas 4.17 and 4.19, induction on h and [C3, Lemma 4.18] imply that the variety associated to a marked diagram $(D, c_\bullet, *)$ is irreducible and has dimension equal to the dimension of the marked diagram.

5. The geometric explanation of the rule

In this section, we interpret the combinatorial rule introduced in §3 as a specialization of the corresponding symplectic restriction variety. We will analyze this specialization and show that the diagrams that replace $(D, c_\bullet)$ in the main Algorithm 3.58 parameterize the irreducible components of the flat limit and each of these components are generically reduced.

The main specialization. The specialization we use is identical to the specialization in the Grassmannian case introduced in [C3]. Of course, the flat limits of restriction varieties will typically have more irreducible components in the case of flag varieties. For the convenience of the reader, we recall the specialization. There are several cases depending on whether $(D, c_\bullet)$ is in order and whether $l(\kappa) < \rho(\kappa, \kappa-1) - 1$ or not. These cases correspond to the cases in Definition 3.37.

In the previous section, given an admissible symplectic diagram $(D, c_\bullet)$, we associated an admissible sequence by defining each of the vector spaces $(L_\bullet, Q_\bullet)$ as a union of basis elements that diagonalize the skew-symmetric form Q. All our specializations will replace exactly one of the basis elements $v = e_u$ or $v = f_u$ for some $1 \leq u \leq m$ in a one-parameter family $v(t) = e_u(t)$ or $v(t) = f_u(t)$. For $t \neq 0$, the resulting set of vectors will be a new basis for V, but when $t = 0$ two of the basis elements will become equal. Correspondingly, we get a one-parameter family of vector spaces $(L_\bullet(t), Q_\bullet(t))$ by changing every occurrence of the vector v to $v(t)$. We thus get a flat family of symplectic restriction varieties $V(D(t))$, where the restriction variety at $t \neq 0$ is defined with respect to the linear spaces $(L_\bullet(t), Q_\bullet(t))$. We now explicitly describe the specialization.

In case (1)(i), D is not in order, η is the unique integer violating the order, and ν is the leftmost integer equal to $\eta + 1$. Suppose that under the translation between symplectic diagrams and sequences of vector spaces, e_u is the vector associated to η and e_v is the vector associated to ν. Then consider the one-parameter family obtained by changing e_v to $e_v(t) = te_v + (1-t)e_u$ and keeping every other vector fixed. When the set of basis elements spanning a vector space L_{n_j} or $Q^{r_i}_{d_i}$ contains e_v, $L_{n_j}(t)$ or $Q^{r_i}_{d_i}(t)$ is obtained by replacing e_v with $e_v(t)$. Otherwise, $L_{n_j}(t) = L_{n_j}$ and $Q^{r_i}_{d_i}(t) = Q^{r_i}_{d_i}$.

In case (1)(ii), D is not in order, η is the unique integer violating the order, $i > \eta$ does not occur in the sequence to the left of η and ν is the leftmost integer equal to $i+1$. Let e_u be the vector associated to η and let e_v be the vector associated to ν. Consider the one-parameter family obtained by changing f_v to $f_v(t) = tf_v + (1-t)e_u$. When the set of basis elements spanning a vector space L_{n_j} or $Q^{r_i}_{d_i}$ contains f_v, $L_{n_j}(t)$ or $Q^{r_i}_{d_i}(t)$ is obtained by replacing f_v with $f_v(t)$. Otherwise, $L_{n_j}(t) = L_{n_j}$ and $Q^{r_i}_{d_i}(t) = Q^{r_i}_{d_i}$.

In case (2)(i), D is in order and $l(\kappa) < \rho(\kappa, \kappa-1) - 1$. Suppose that e_v is the vector associated to ν, the leftmost $\kappa + 1$. Let e_u and f_u be two vectors associated to two zeros between $\}^\kappa$ and $\}^{\kappa-1}$. These exist since $l(\kappa) < \rho(\kappa, \kappa-1) - 1$. Consider the one-parameter specialization replacing f_v with $f_v(t) = tf_v + (1-t)e_u$. When the set of basis elements spanning a vector space L_{n_j} or $Q^{r_i}_{d_i}$ contains f_v, $L_{n_j}(t)$ or $Q^{r_i}_{d_i}(t)$ is obtained by replacing f_v with $f_v(t)$. Otherwise, $L_{n_j}(t) = L_{n_j}$ and $Q^{r_i}_{d_i}(t) = Q^{r_i}_{d_i}$.

In case (2)(ii)(a), D is in order and $l(\kappa) = \rho(\kappa, \kappa-1) - 1$. Let ν be the leftmost integer equal to κ and suppose that e_v is the vector associated to ν. Let e_u be the vector associated to the $\kappa - 1$ following $\}^\kappa$. Then let $e_v(t) = te_v + (1-t)e_u$. When the set of basis elements spanning a vector space L_{n_j} or $Q^{r_i}_{d_i}$ contains e_v, $L_{n_j}(t)$ or $Q^{r_i}_{d_i}(t)$ is obtained by replacing e_v with $e_v(t)$. Otherwise, $L_{n_j}(t) = L_{n_j}$ and $Q^{r_i}_{d_i}(t) = Q^{r_i}_{d_i}$.

Finally, in case (2)(ii)(b), D is in order, $l(\kappa) = \rho(\kappa, \kappa-1)-1$ and there does not exist an integer equal to κ to the left of κ. Let e_v be the vector associated to ν, the leftmost integer equal to $\kappa+1$ and let e_u be the vector associated to $\kappa-1$ to the right of $\}^\kappa$. Then let $f_v(t) = tf_v + (1-t)e_u$. When the set of basis elements spanning a vector space L_{n_j} or $Q_{d_i}^{r_i}$ contains f_v, $L_{n_j}(t)$ or $Q_{d_i}^{r_i}(t)$ is obtained by replacing f_v with $f_v(t)$. Otherwise, $L_{n_j}(t) = L_{n_j}$ and $Q_{d_i}^{r_i}(t) = Q_{d_i}^{r_i}$.

The flat limit of the vector spaces is easy to describe. If L_{n_j} or $Q_{d_i}^{r_i}$ does not contain the vector v, then $L_{n_j}(t) = L_{n_j}$ and $Q_{d_i}^{r_i}(t) = Q_{d_i}^{r_i}$ for all t. Similarly, if L_{n_j} or $Q_{d_i}^{r_i}$ contains both of the basis elements spanning $v(t)$, then $L_{n_j}(t) = L_{n_j}$ and $Q_{d_i}^{r_i}(t) = Q_{d_i}^{r_i}$ for all $t \neq 0$. Then in the limit $L_{n_j}(0) = L_{n_j}$ and $Q_{d_i}^{r_i}(0) = Q_{d_i}^{r_i}$. A vector space changes under the specialization only when it contains one and not the other of the two basis elements spanning $v(t)$. In that case, in the limit $t = 0$ one swaps the basis vector with coefficient t with the basis vector with coefficient $(1-t)$.

In each of these cases, the set of limiting vector spaces is depicted by the symplectic diagram D^a up to taking the canonical representative. In case (1)(i), the degeneration swaps e_v and e_u. The effect on symplectic diagrams is simply switching η and ν. In case (1)(ii), the degeneration switches e_u and f_v. As a result, the restriction of Q to the linear space $Q_{d_i}^{r_i}$ has e_v in its kernel. The resulting set of linear spaces is depicted by switching η and the zero corresponding to f_v and replacing the integer corresponding to e_v with i. In case (2)(i), as a result of the specialization, the restriction of Q to $Q_{d_\kappa}^{r_\kappa}$ contains e_u and e_v in its kernel. The resulting set of vector spaces is denoted by switching the zero corresponding to e_u and ν to κ. The cases (2)(ii)(a) and (2)(ii)(b) are analogous to the cases (1)(i) and (1)(ii), respectively.

Correspondingly, we get a family of restriction varieties $V(D(t))$, where $V(D(t))$ is the symplectic restriction variety defined with respect to the linear spaces $L_\bullet(t)$ and $Q_\bullet(t)$. As long as $t \neq 0$, the corresponding varieties are projectively equivalent, hence form a flat family. The first theorem of this section, which is the main geometric theorem of this paper, describes the flat limit when $t = 0$.

Theorem 5.1. *Let $(D, c_\bullet)$ be an admissible colored symplectic diagram. Let $V(D(t))$ be the main specialization described in this section and let $V(D(0))$ be the flat limit at $t = 0$. Then $V(D(0))$ is supported along the union $\bigcup V(D_i)$ of restriction varieties, where D_i range over the admissible symplectic diagrams replacing D in Algorithm 3.58. Furthermore, the flat limit is generically reduced along each $V(D_i)$. In particular, we have the equality of cohomology classes*

$$[V(D)] = \sum [V(D_i)].$$

Proof. The proof has two steps. We first show that the flat limit is supported along the union of the varieties $V(D_i)$, where D_i ranges over the diagrams replacing D in Algorithm 3.58. We then do a local calculation to show that the flat limit is generically reduced along each D_i. The second step is a straightforward local calculation. The first step is the step that requires more care.

We determine the components of the support of the flat limit using a dimension count. The following observation puts very strong restrictions on the support of the flat limit.

Observation 5.2. Let $c_\bullet : c_1, \dots, c_{k_h}$ be the coloring. If $j \leq s$, let $\gamma_l(j)$ denote the number of integers among $c_1, \dots, c_j$ less than or equal to l. Then the linear spaces $W_l(t)$ parameterized by $V(D(t))$ intersect the linear spaces $L_{n_j}(t)$ in a subspace of dimension at least $\gamma_l(j)$. Similarly, let $\gamma_l(k_h - i + 1)$ denote the number of integers less than or equal to l among $c_1, \dots, c_{k_h - i+1}$. Then the linear spaces $W_l(t)$ parameterized by $V(D(t))$ intersect the linear spaces $Q_{d_i}^{r_i}(t)$ in a subspace of dimension at least $\gamma_l(k_h - i + 1)$. Since intersecting a proper variety in a subspace of dimension greater than or equal to a given integer is a closed condition, the flag elements W_l parameterized by the flat limit $V(D(0))$ have to intersect the linear spaces $L_{n_j}(0)$ in a subspace of dimension at least $\gamma_l(j)$ and the linear spaces $Q^{r_i}_{d_i}(0)$ in a subspace of dimension at least $\gamma_l(k_h - i + 1)$.

We use Observation 5.2 repeatedly to obtain strong restrictions on the flat limit. Let Y be an irreducible component of the support of $V(D(0))$. Since the family is flat, $\dim(Y) = \dim(V(D))$. Consider the complete flag (not necessarily isotropic) determined by the vectors ordered from left to right depicted by D^a. We claim that the rank conditions that the generic element of Y satisfies with respect to this flag determines Y.

Build a diagram $D(Y)$ that depicts the flag elements for which the dimension of the intersection of the flag elements W_h parameterized by the generic element in Y jumps. By Observation 5.2, each W_h has to intersect the flag elements $L_{n_j}(0)$ in dimension at least n_j and the flag elements $Q^{r_i}_{d_i}(0)$ in dimension at least $k_h - i + 1$. Consequently, the diagram $D(Y)$ is a diagram obtained from D^a. Assign $D(Y)$ the coloring obtained by Algorithm 3.67. A priori, the flag elements W_l for $l < h$ may intersect the flag elements more specially. However, by Observation 5.2, they have to satisfy the rank conditions imposed by $D(Y)$ with the coloring assigned by Algorithm 3.67.

We now compare the dimension of linear spaces satisfying the rank conditions specified by $D(Y)$ to the dimension of the diagram D and determine under which conditions the two dimensions can be the same. This almost determines the set of diagrams occurring in Algorithm 3.58. The change in dimension is determined by Equation (1) in Proposition 4.17.

(1) If we replace a linear space $L^a_{n_{i+j}}$ of dimension n_{i+j} in the $(i+j)$-th position in $(L^a_\bullet, Q^a_\bullet)$ with a linear space F_{u_i} not contained in $(L^a_\bullet, Q^a_\bullet)$, then according to Equation (1) the dimension changes as follows. Let y^a_{i+j} be the index of the smallest index linear space $Q^{r_l}_{d_l}$ such that $L^a_{n_{i+j}} \subset K_l$. Similarly, let y^u_i be the smallest l such that $F_{u_i} \subset K_l$. The left sum in Equation (1) decreases by $n^a_{i+j} - u_i$. The quantities x_l increase by one for $y^u_i \leq l < y^a_{i+j}$. Hence, the dimension decreases by $n^a_{i+j} - n^u_i + y^a_{i+j} - y^u_i$. By condition (S4) for D^a and condition (A1) for D, in D^a, there is at most one missing integer among the positive integers to the left of the brackets and the two integers preceding all but at most one of the brackets are equal. Therefore, $n^a_{i+j} - n^u_i + y^a_{i+j} - y^u_i \geq j$ with equality only if the index $i = x_{\nu'-1} + 1$ and in D the equality $p(]^i) - p(\nu) - 1 = y_{x_{\nu'-1}+1} - \nu'$ holds. Let the color of $L^a_{n_{i+j}}$ be c_{i+j}. Then the change in $\dim(c_\bullet)$ between the original coloring and the coloring assigned by Algorithm 3.67 can be calculated as follows. Let $i \leq u_1 < u_2 < \cdots < u_z = i + j$ be the indices with $c_{u_1} > c_{u_2} > \cdots > c_{u_z} = c_{i+j}$ and there does not exist and index between u_q and $i + j$ of color larger than or equal to c_{u_q}. Then $\dim(c_\bullet)$ increases by $(u_1 - i) - \#\{i \leq l < u_1 \mid c_l = c_{u_1}\} + \sum_{q=2}^{z}(u_q - u_{q-1} - 1) - \#\{u_{q-1} < l < u_q \mid c_l = c_{u_q}\})$. In particular, the increase is at most j with equality if and only if the color of every linear space in D^a with index $i, \ldots, i + j - 1$ is strictly less than c_{i+j}.

(2) If we replace the linear space $Q^{r_i,a}_{d_i}$ of dimension d^a_i in $(L^a_\bullet, Q^a_\bullet)$ with a non-isotropic linear space $F_{u_{k-i-j+1}}$ of dimension d^u_{i+j}, then, by Equation (1), the dimension changes as follows. Let x^u_{i+j} be the number of linear spaces that are contained in the kernel of the restriction of Q to $F_{u_{k-i-j+1}}$. Then the dimension decreases by $d^a_i - d^u_{i+j} + x^a_i - x^u_{i+j}$. This decrease is at least j, with strict inequality unless Condition (A1) fails for an integer equal to i. As in the previous case, the increase in $\dim(c_\bullet)$ is at most j with equality only if the color of every linear space with index between $i + j$ and i in D^a is strictly less than the color of $Q^{r_i,a}_{d_i}$.

(3) Finally, if we replace a linear space $Q^{r_{k-s}}_{d_{k-s}}$ of dimension d^a_{k-s} in $(L^a_\bullet, Q^a_\bullet)$ with an isotropic linear space $F_{u_{s+1}}$, then the first sum in Equation (1) changes by $u_{s+1} - s - 1$. The second

sum changes by $-d^a_{k-s} + y^u_{s+1} - x^a_{k-s} + (2s+1)$, where y^u_{s+1} denotes the number of non-isotropic subspaces containing $F_{u_{s+1}}$ in the kernel of the restriction of Q. Hence, the total change is

$$-d^a_{k-s} + u_{s+1} - x^a_{k-s} - y^u_{s+1} + s.$$

Since the dimension of an isotropic linear space is bounded by $(d_{k-s}+r_{k-s})/2$, we conclude that this quantity is less than or equal to zero, with strict inequality unless $x_{k-s} = s$, $r_{k-s} = d_{k-s}$ and $d_{k-s} = u_{s+1}$. In this case, since we are not changing the ordering of the linear spaces, the coloring does not change.

We will now conclude from the dimension calculation that $D(Y)$ can only be one of the diagrams D^a or type D^b replacing D in the main Algorithm 3.58. We need one further observation from the proof of Theorem 5.2 in [C3]. If we replace $L^a_{n_{i+j}}$ with F_{u_i}, if u_i is less than $p(\nu)$, by semi-continuity of the dimension of intersection of $W_h^\perp$ with $K_\nu \cap L_{x_{\nu-1}+1}$, not every linear space parameterized by $D(Y)$ can be in the flat limit. Hence, Y yielding such a $D(Y)$ has strictly smaller dimension than $V(D(Y))$ and we cannot obtain a component of the support of the flat limit by moving a bracket to the left of ν.

Now the proof of the first statement of the theorem is immediate. If D^a is admissible and the generic linear space parameterized by Y does not satisfy more special rank conditions, then Y is contained in $V(D^a)$. Since the two varieties are irreducible of the same dimension, they are equal. If D^a does not satisfy condition (A2) for the index i, then, by the linear space bound, the linear spaces parameterized by Y have to intersect the kernel K_i in dimension at least $x_i + 1$. Note that K_i is the span of e_δ (the basis elements corresponding to δ) and the basis elements corresponding to the integers to the left of and including δ', where δ and δ' are the integers marked in Step 2 of Algorithm FixA2. Since the $k_h - i + 1$ dimensional subspace contained in $Q^{r_i}_{d_i}$ has to be contained in the span of e_δ and $Q^{r_{i+1}}_{d_{i+1}}$, we conclude that Y is contained in the variety corresponding to the marked diagram defined in Step 2 of Algorithm FixA2 starting with D^a. Since these are both irreducible of the same dimension, we conclude that they are equal. We will analyze this case in detail in the next theorem.

If a (marked) diagram violates condition (A1) (or (A1*)) for a bracket $]^j$, assume that $i_0 < i_i$ are the integers in places $p(]^j)-1$ and $p(]^j)$ (or the two integers immediately preceding $]^j$ omitting δ). Then, by the kernel bound, the linear spaces of dimension $k_h - i + 1$ contained in $Q^{r_i}_{d_i}$ for $i_0 \leq i < i_1$ have to be contained in $L^\perp_{n_j}$. Notice that $L^\perp_{n_j} \cap Q^{r_i}_{d_i} = Q^{r_i+1}_{d_i-1}$ corresponds to the linear space depicted by moving $\}^i$ one unit to the left. If $d_{i+1} = d_i - 1$, then this expresses the fact that the $k_h - i + 1$ dimensional space contained in $Q^{r_i}_{d_i}$ is actually contained in $Q^{r_{i+1}}_{d_{i+1}}$. Hence, Y has to be contained in the the variety corresponding to the diagram obtained by moving the braces with indices $i_0 \leq i < i_1$ to the left in decreasing order as described in Algorithm FixA1. By our dimension count, such a diagram can have the same dimension as D only in the cases specified in Steps 2 and 4 of the algorithm and has strictly smaller dimension in the case specified in Step 3. We conclude that if D^a fails condition (A1), Y has to be contained in the variety corresponding to FixA1(D^a). Since they are both irreducible varieties of the same dimension, they have to be equal.

If $D(Y)$ is obtained by moving a bracket $]^i$ from D^a to a position p, then by the first dimension count and the requirement that $p \geq p(\nu)$, we can move at most one bracket and $p = p(\nu)$. Furthermore, the bracket has to satisfy the equality $p(]^i) - p(\nu) = y_i - \nu' + i - x_{\nu'-1}$ in D and every bracket between $p(]^i)$ and $p(\nu)$ must have color strictly less than the color of $]^i$. If the resulting diagram is admissible, then by the second and third dimension counts, we cannot move a brace or replace a brace with a bracket without obtaining a strictly smaller dimensional locus. We conclude that Y has to be contained in $V(D^b(]^i))$. Since they have the same dimension, Y is equal to $V(D^b(]^i))$. If the resulting diagram is not admissible, then it fails condition (A1). As in the previous case, by the kernel bound, Y has to be contained in the variety associated to $\mathrm{FixA}(D^b(]^i))$. We thus conclude that the support of the flat limit is contained in the union of the varieties $V(D^a)$ and $V(D^b(]^i))$ described in Algorithm 3.58.

Finally, we need to check that each of the components are generically reduced. This is a routine local calculation, which is almost identical to the calculation in [C2] or [C3] for orthogonal flag varieties or symplectic Grassmannians. As a sample, we check the case (2)(i) and leave the modifications in other cases to the reader. For each D^a and $D^b(]^i)$, we exhibit varieties $Z(D^a)$ and $Z(D^b(]^i))$ with the following intersection numbers:

$$Z(D^a)\cdot V(D) = 1, \quad Z(D^a)\cdot V(D^a) = 1, \quad Z(D^a)\cdot V(D^b(]^i)) = 0 \ \forall i,$$

$$Z(D^b(]^i))\cdot V(D) = 1, \quad Z(D^b(]^i))\cdot V(D^a) = 0, \quad Z(D^b(]^i))\cdot V(D^b(]^j)) = \delta_{i,j}.$$

It follows that each of the components occur and are generically reduced. Since this is a local calculation, we may assume that $\kappa = 1, d_\kappa + r_\kappa = n-2$ and $x_\kappa = 0$. For each $D^b(]^i)$ and D^a, there is a dual Schubert variety obtained as follows. Apply Algorithm 3.58 to $D^b(]^i)$ or D^a taking the branch corresponding to the diagram D^a at each step. (Notice that the coefficient of the resulting Schubert class is one in the class of $V(D)$ according to Algorithm 3.58.) Let $\Sigma(D^b(]^i))$, respectively $\Sigma(D^a)$, denote the opposite Schubert variety. By the same argument as in [C2] or [C3], Kleiman's Transversality Theorem implies that if we set $Z(D^a) = \Sigma(D^a)$ and $Z(D^b(]^i)) = \Sigma(D^b(]^i))$, the desired equalities are satisfied. We conclude that the flat limit is generically reduced along each component. ☐

5.1. **The specialization for Algorithm FixA2.** Now we analyze Algorithm FixA2 (3.52). We can interpret Algorithm FixA2 also as a sequence of specializations. We begin by describing this specialization.

Let $(D, c_\bullet, *)$ be a tightened diagram which is marked. Recall that the vectors associated to a marked diagram are the same as the vectors associated to $(D, c_\bullet)$.

• Let e_δ be the vector corresponding to δ.

• If ϵ is positive or to the left of a bracket, let e_ϵ be the vector corresponding to ϵ. If ϵ is zero and there are no brackets to the right of ϵ, then let f_ϵ be the least index f vector among the f vectors associated to the zeros between the two consecutive braces or the consecutive bracket and brace that bound ϵ. (Recall that we are free to shuffle the basis vectors corresponding to the integers between any consecutive brackets and/or braces without changing any of the vector spaces.) To unify the notation, let v_ϵ denote e_ϵ in the first case and f_ϵ in the second case.

Consider the specialization that replaces v_ϵ by $tv_\epsilon + (1-t)e_\delta$. When $t=1$, we have the original set of vector spaces. When $t=0$, the specialization replaces v_ϵ with e_δ. If a vector space in the sequence contains both v_ϵ and e_δ or if it does not contain v_ϵ, then the specialization leaves the vector space fixed. Otherwise, it replaces the basis vector v_ϵ with e_δ.

The diagram $\tilde{D}$ (see Definition 3.48) encodes the flat limit of the sequence of vector spaces (up to taking an equivalent representative). If ϵ is positive or to the left of a bracket, then the limit of the linear spaces that contain e_ϵ but not e_δ is obtained by swapping e_ϵ and e_δ. If further we are in Case (1) (i.e., $p(\delta)$ is occupied by a bracket), then these vector spaces are represented by swapping ϵ and δ, which is the diagram $\tilde{D}$ in Case (1) (up to taking the canonical representative). If ϵ is positive and $p(\delta)$ is occupied by the brace $\}^j$, then the diagram representing the limiting vector spaces is obtained by swapping ϵ and δ. Note that cyclically permuting one place to the left the string of integers $\epsilon \cdots \delta$ gives an equivalent diagram (and this representative is chosen for $\tilde{D}$ to simplify the combinatorial description of diagrams of type D^β). If $\epsilon = 0$ and $p(\delta)$ is occupied by the brace $\}^j$, the process also swaps ϵ and δ. However, in this case the kernel of the restriction of Q to the non-isotropic subspace $Q^{r_{j+1}}_{d_{j+1}}(0)$ also contains the vector $e_\epsilon = f^*_\epsilon$. The vector e_ϵ corresponds to the leftmost integer equal to $j+2$ if there exists an integer equal to $j+2$ in the sequence. Otherwise, e_δ corresponds to a zero and its location depends on whether the initial input of Algorithm FixA2 is a diagram D^a or a diagram $P(a_\bullet, c_\bullet)$. By the description in 4.2, in the first case, e_ϵ corresponds to the leftmost zero in the diagram. In the second case, it corresponds to a zero between $\}^{j+2}$ (or $]^s$ if $j+1 = k_h - s$) and $\}^{j+1}$. As remarked in the proof of Proposition 3.69, the two cases are combinatorially distinguished by whether $p(\delta') = 0$ or $p(\delta') > 0$. Hence, we conclude that $\tilde{D}$ (up to choosing an equivalent diagram) represents the vector spaces that are the limit of the specialization.

The next theorem determines the flat limit of the symplectic restriction varieties under this specialization and shows that Algorithm FixA2 (3.52) replaces a diagram with a collection of admissible diagrams whose cohomology classes sum to the cohomology class of the original diagram.

Theorem 5.3. *Let $(D, c_\bullet, *)$ be a marked symplectic diagram that arises while running Algorithm* Fix*A2 with input $(D^\alpha, c_\bullet)$ or $P(a_\bullet, c_\bullet)$. Then the flat limit of the specialization applied to $V(D, c_\bullet, *)$ is supported along the varieties associated to D^α and the diagrams of type D^β described by Algorithm* Fix*A2. Moreover, the flat limit is generically reduced along each component. In particular, the class of $V(D, c_\bullet, *)$ is equal to the sum of the classes of restriction varieties corresponding to admissible diagrams replacing $(D, c_\bullet, *)$ in Algorithm* Fix*A2.*

Proof. As in the previous theorem, the proof has two steps. We interpret Algorithm FixA2 (3.52) as the specialization described above. We first show that the flat limit of the specialization is supported along the restriction varieties corresponding to the diagrams $\mathrm{Fix}(D, c_\bullet, *)$ described in Algorithm FixA2. We then check that the flat limit is generically reduced along each of the components. The proofs of both steps are almost identical to the proofs in the previous theorem.

First, using the dimension count in the proof of Theorem 5.1, we determine the support of the flat limit. In a diagram $P(a_\bullet, c_\bullet)$, for each j such that $a_j = 2j-1$, the linear space Q^1_{2j-1} contains a j-dimensional isotropic subspace. Hence, by the linear space bound, this j-dimensional subspace must contain the kernel of the restriction of Q to Q^1_{2j-1}. The kernel for the smallest j for which $a_j = 2j-1$ is e_δ. Similarly, if D^a fails condition (A2) for $\}^i$, then by the linear space bound, the $(k_h - i + 1)$-dimensional subspace contained in $Q^{r_i}_{d_i}$ intersects the kernel K_i in a subspace of dimension at least $x_i + 1$. This kernel is denoted by the span of e_δ and the vectors corresponding to the integers up to and including δ'.

Suppose δ is between $]^j$ and $]^{j+1}$ (or $\}^{i+1}$ and $\}^i$). In the first case, the $(j+1)$-dimensional subspace contained in $L_{n_{j+1}}$ must already be contained in the span of e_δ and L_{n_j}, since this span contains a $(j+1)$-dimensional subspace. Similarly, in the second case, the $(k_h - i + 1)$-dimensional subspace in $Q^{r_i}_{d_i}$ has to be contained in the span of $Q^{r_{i+1}}_{d_{i+1}}$ and e_δ. Moreover, if $i = k_h - s$, then δ is between $]^s$ and $\}^i$ and the span of L_{n_s} and δ is isotropic. We conclude that if linear spaces satisfy the rank conditions imposed by a marked diagram $(D, c_\bullet, *)$, they actually satisfy the rank conditions specified by $\mathrm{Tight}(D, c_\bullet, *)$.

If δ is between $\}^{j+1}$ and $\}^j$, let $F = Q^{r_{j+1}}_{d_{j+1}}$ and if δ is between $]^j$ and $]^{j+1}$, let $F = L_{n_j}$. In the first case, let $m_j = k_h - j$. In the latter case, let $m_j = j$. The flag elements W_h parameterized by $(D, c_\bullet, *)$ intersect F in a subspace of dimension m_j and intersect the span of F with e_δ in a subspace of dimension $m_j + 1$. When we make the specialization, e_δ becomes a vector of the flat limit $F(0)$ of F. There are two possibilities. Either the flag elements W_h parameterized by a component of the flat limit intersect $F(0)$ in a subspace of dimension m_j or greater than m_j. In the former case, the intersection of the linear spaces with $F(0)$ have to be contained in the span of the one smaller flag element in the sequence and e_δ. Hence, this component is the variety corresponding to D^α, the tightening of $\tilde{D}$. Notice that by the dimension count in the proof of Theorem 5.1, there cannot be any components where the linear spaces are more special. Such a component would be contained in a variety corresponding to a diagram where one or more of the brackets or braces are shifted to the left. By the dimension counts in the proof of Theorem 5.1, shifting any of the brackets or braces would produce a locus of strictly smaller dimension.

Else, W_h intersects $F(0)$ in a subspace of dimension $m_j + 1$ (or greater). In this case, we have to determine the possible limits. Let Y be an irreducible component of the support of the flat limit. Associate to it the diagram $D(Y)$ depicting the rank conditions satisfied by a linear space corresponding to a general point of Y. Let D' be the diagram one gets from $\tilde{D}$ by moving the brace $\}^j$ or $]^{j+1}$, depending on the case, to the first unoccupied position to its left. By Observation 5.2, $D(Y)$ is a diagram obtained from D'. We have to check whether it is possible to obtain any components of the same dimension by moving other bracket and/or braces to the left.

Now we can go through the possibilities quickly. If δ is between $\}^{j+1}$ and $\}^j$ and $p(\delta') > 0$, then by the linear space bound, the linear spaces have to intersect the kernel of the restriction of Q to $Q^{r_{j+1}}_{d_{j+1}}$ in dimension $x_{j+1} + 1$. This kernel is spanned by the vectors corresponding to the integers up to and including γ. Hence, we need to increase the number of brackets in positions less than or equal to $p(\gamma)$ by at least one. As observed in the proof of Theorem 5.1, moving a bracket or brace to the left of γ results in smaller dimensional loci. By the first dimension count, we can move at most one of the brackets to $p(\gamma)$ and the bracket we move cannot cross any brackets of equal or larger color. We recover the diagrams $D^\beta(]^i)$. In this case, there is one more possibility. We can move $\}^j$ to $p(\gamma)$ provided that the color of all the brackets and braces in between is strictly less than the color of $\}^j$. We recover the diagram $D^\beta(\}^j)$. If these diagrams satisfy (A1) or (A1*), then by the dimension counts in the proof of Theorem 5.1, moving any other bracket or brace corresponds to a locus with smaller dimension. We also remark that when running Algorithm FixA2, the equality (1) in Definition 3.39 automatically holds for $\nu = \gamma$ and all the brackets at positions $p > p(\gamma)$. Since $p(\delta') > 0$, the initial input is a diagram D^a. As observed in the proof of Proposition 3.69, equality must hold in condition (A2) for all indices $j \geq i$ in D. This implies that all brackets of D to the right of $p(\gamma)$ satisfy the equality in (1) of Definition 3.39. This is the reason we do not need to specify that the brackets satisfy this equality in Algorithm FixA2. When δ is between $\}^{j+1}$ and $\}^j$ and $p(\delta') = 0$, there are no brackets in the diagram that can be moved to the left. Because of the location of γ (between $\}^{j+2}$ (or $]^s$ if $j+1 = k_h - s$) and $\}^{j+1}$) in this case, we only recover the diagram $D^\beta(\}^j)$. If any of the resulting diagrams fail condition (A1) or (A1*), then, by the kernel bound, the component Y is contained in the locus corresponding to the diagram obtained by running Algorithm FixA1. By our dimension counts, moving any other bracket or brace corresponds to a locus of smaller dimension. We conclude that Y is the variety associated to one of the diagrams of type D^β.

If δ is between $]^j$ and $]^{j+1}$ and there are no brackets between position p (see Definition 3.49) and $]^j$, we need to move the bracket $]^{j+1}$ to position p since there is a j-dimensional subspace contained in the span of $L_{n_{j-1}}$ and e_δ and a $(j+1)$-dimensional subspace contained in L_{n_j}. This has the same dimension if $c_j < c_{j+1}$. Otherwise, it has strictly smaller dimension. By the dimension count in the proof of Theorem 5.1, moving $]^j$ to any position $p+t \leq p(]^j)$ preserves the dimension and moving any other bracket or brace to the left strictly decreases the dimension. If any of the resulting diagrams fail condition (A1*), then, by the kernel bound, the component Y is contained in the locus corresponding to the diagram obtained by running Algorithm FixA1. We conclude that Y is the variety associated to one of the diagrams of type D^{β}.

If δ is between $]^j$ and $]^{j+1}$ and there are brackets between position p and $]^j$, we need to move the bracket $]^{j+1}$ to a position less than or equal to p since there is a $(j+1)$-dimensional subspace contained in L_{n_j}. By the first dimension count in the proof of Theorem 5.1, moving $]^{j+1}$ to any of the positions greater than q (see Definition 3.49 Case(1)(ii)) and less than or equal to p preserves the dimension provided that the brackets between p and $]^j$ all have color less than c_{j+1}. Moving any other bracket strictly decreases the dimension. If the resulting diagram is admissible, then moving any brace also decreases the dimension. If any of the resulting diagrams fail condition (A1*), then, by the kernel bound, the component Y is contained in the locus corresponding to the diagram obtained by running Algorithm FixA1. We conclude that Y is one of the diagrams of type D^{β}. We conclude that the flat limit is supported along the union of the varieties associated to the diagrams assigned to $(D, c_{\bullet}, *)$ by Algorithm FixA2.

The multiplicity calculation is identical to the calculation in the proof of the previous theorem, so we leave it to the reader. This concludes the proof of the theorem. □

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