BIRATIONAL GEOMETRY OF MODULI SPACES

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ABSTRACT. These are lecture notes and exercises for the VIGRE summer school Birational Geometry and Moduli Spaces at the University of Utah, June 1-12, 2010. In these notes, we discuss the cones of ample and effective divisors on various moduli spaces such as the Kontsevich moduli spaces of stable maps and the moduli space of curves. We describe the stable base locus decomposition of the effective cone in explicit examples and determine the corresponding birational models.

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1. Preliminaries on the Cones of Ample and Effective divisors

In this section, we review the basic terminology and facts about the cones of ample and effective divisors. The two volumes Positivity in Algebraic Geometry I [L1] and II [L2] by Rob Lazarsfeld are great references for the material in this section and for further reading.

A variety X is called $\mathbb{Q}$-*factorial* if every Weil divisor on X is $\mathbb{Q}$-Cartier. Throughout this section, let X be a $\mathbb{Q}$-factorial, normal, projective variety over the complex numbers. The moduli spaces discussed in these notes will be $\mathbb{Q}$-factorial, normal, projective varieties.

Definition 1.1. Two divisors D_1, D_2 are called *numerically equivalent* if the intersection numbers $C \cdot D_1 = C \cdot D_2$ are equal for every irreducible curve $C \subset X$. Numerical equivalence naturally extends to $\mathbb{Q}$ or $\mathbb{R}$ divisors. The *Neron-Severi space*, $N^1(X)$ is the vector space of numerical equivalence classes of $\mathbb{R}$-divisors. The intersection pairing gives a duality between curves and divisors. Two curves C_1, C_2 are called *numerically equivalent* if $C_1 \cdot D = C_2 \cdot D$ for every codimension one subvariety $D \subset X$. Let $N_1(X)$ denote the vector space of curves up to numerical equivalence. The vector spaces $N^1(X)$ and $N_1(X)$ are dual under the intersection pairing.

The vector spaces $N^1(X)$ and $N_1(X)$ contain several natural cones that control the birational geometry of X.

Definition 1.2. Let X be a normal, irreducible, projective variety. A line bundle L on X is called *very ample* if $L = \phi^*\mathcal{O}_{\mathbb{P}^n}(1)$ for some embedding $\phi : X \hookrightarrow \mathbb{P}^n$. A line bundle L is called *ample* if a positive multiple of L is very ample. A divisor D on X is ample if the line bundle associated to a sufficiently divisible, positive multiple is ample.

The Nakai-Moishezon criterion says that a divisor on a projective variety is ample if and only if $D^{\dim(V)} \cdot V > 0$ for every irreducible, positive dimensional subvariety V of X. In particular, being ample is a numerical property: if D_1 and D_2 are numerically equivalent divisors on X, then D_1 is ample if and only if D_2 is ample.

Exercise 1.3. Prove the Nakai-Moishezon criterion by carrying out the following steps (see Theorem 1.2.23 of [L1]).

Step 1. Show that if D is ample, then $D^{\dim(V)} \cdot V > 0$ for every irreducible, positive dimensional subvariety V of X (including X).

Step 2. For the remainder of this exercise assume that $D^{\dim(V)} \cdot V > 0$ for every irreducible, positive dimensional subvariety V of X. Show that if X is a curve, then D is ample. We will use this as the base case of an induction on dimension.

Step 3. Use asymptotic Riemann-Roch and induction on dimension to deduce that $H^0(X, \mathcal{O}_X(mD)) \neq 0$ for $m >> 0$. (Hint: Express $D = A - B$, where A and B are very ample. Use the two standard exact sequences

$$0 \to \mathcal{O}_X(mD - B) \to \mathcal{O}_X(mD) \to \mathcal{O}_B(mD) \to 0$$
$$0 \to \mathcal{O}_X(mD - B) \to \mathcal{O}_X((m+1)D) \to \mathcal{O}_A((m+1)D) \to 0$$

and induction.)

Step 4. Show that $\mathcal{O}_X(mD)$ is generated by global sections. Hence, it gives a morphism to projective space.

Step 5. Check that D is ample.

The tensor product of two ample line bundles is again ample. Moreover, the tensor product of any line bundle with a sufficiently high multiple of an ample line bundle is ample. Consequently, the classes of ample divisors form an open, convex cone called the *ample cone* in the Neron-Severi space.

Definition 1.4. A divisor D is called *NEF* if $D \cdot C \geq 0$ for every irreducible curve $C \subset X$.

Clearly, the property of being NEF is a numerical property. Since the sum of two NEF divisors is NEF, the set of NEF divisors on X forms a closed, convex cone in $N^1(X)$ called the *NEF cone* of X. The NEF cone contains the ample cone. In fact, Kleiman's Theorem characterizes the ample cone as the interior of the NEF cone and the NEF cone as the closure of the ample cone.

Theorem 1.5 (Kleiman's Theorem). *Let D be a NEF $\mathbb{R}$-divisor on a projective variety X. Then $D^k \cdot V \geq 0$ for every irreducible subvariety $V \subseteq X$ of dimension k.*

Exercise 1.6. Prove Kleiman's Theorem by carrying out the following steps (see Theorem 1.4.9 of [L1]).

Step 1. Using the fact that the classes of rational divisors are dense in the Neron-Severi space, reduce to the case of rational divisors by approximation.

Step 2. Observe that the theorem is true for curves. By using the Nakai-Moishezon Criterion and induction on dimension, reduce to proving that $D^n \geq 0$.

Step 3. Fix a very ample divisor H. Consider the polynomial $P(t) = (D+tH)^n$. Notice that the previous statement is implied by $P(0) \geq 0$.

Step 4. If $P(0) < 0$, show that $P(t)$ has a single real root $t_0 > 0$. Show that for $t > t_0$, $D + tH$ is ample.

Step 5. Write $P(t) = Q(t) + R(t)$, where $Q(t) = D(D + tH)^{n-1}$ and $R(t) = tH(D + tH)^{n-1}$. Show $Q(t_0) \geq 0$ and $R(t_0) > 0$. Note that this contradicts the assumption $P(t_0) = 0$. Thus conclude the theorem.

Step 6. Reinterpret the theorem as saying that the NEF cone is the closure of the ample cone and the ample cone is the interior of the NEF cone.

The *cone of curves* in $N_1(X)$ is the closure of the cone of classes that can be represented by non-negative linear combinations of classes of effective curves. Under the intersection pairing, the cone of curves is the dual to the NEF cone. There is a well-developed structure theory for this cone, thanks to the work of many mathematicians including Kawamata, Kollár, Mori, Reid and Shokurov. Birational Geometry of Algebraic Varieties by Kollár and Mori [KM] is a great reference for the cone and contraction theorems about the structure of the cone of curves.

Theorem 1.7 (Cone theorem). *Let X be a smooth projective variety (more generally, a variety with mild (klt) singularities) of dimension n. Then there are at most countably* *many rational curves $C_i \subset X$ with*

$$0 \leq -C_i \cdot K_X \leq n+1$$

such that the cone of curves is generated by C_i and the classes in the cone of curves that have non-negative intersection with K_X. If H is an ample divisor and $\epsilon > 0$, then only for finitely many of the curves C_i, $C_i \cdot (K_X + \epsilon H)$ can be negative.

The importance of the extremal rays in the K_X-negative part of the cone of curves is that they can be contracted. If r is an extremal ray of the cone of curves satisfying $(K_X + \epsilon H) \cdot r < 0$, then there exists a morphism $cont_r : X \to Y$ such that any curve whose class lies in the ray r is contracted. Furthermore, the class of any curve contracted by $cont_r$ lies in the ray r. The Contraction Theorem provides a very important way of constructing new birational models of X.

Unfortunately, we do not understand the K_X-positive part of the cone of curves. Even the K_X-negative part of the cone can be very complicated. The next exercise gives a relatively simple surface for which the K_X-negative part of its cone of curves is not finite polyhedral.

Exercise 1.8. Let X be the blow-up of $\mathbb{P}^2$ at the base points of a general pencil of cubic curves. The nine base points of the pencil give nine sections $E_1, \dots, E_9$ of the elliptic fibration. Fiber-wise translation by differences of two of the sections is a well-defined automorphism of X. Using the action of the automorphism, show that the K_X-negative part of the cone of curves of X has infinitely many extremal rays.

Let L be a line bundle on a normal, irreducible, projective variety X. The semi-group $N(X, L)$ of L is defined to be the non-negative powers of L that have a non-zero section:

$$N(X, L) := \{\ m \geq 0\ :\ h^0(X, L^{\otimes m}) > 0\ \}.$$

Given $m \in N(X, L)$, we can consider the rational map ϕ_m associated to $L^{\otimes m}$.

Definition 1.9. The *Iitaka dimension* of a line bundle L is defined to be the maximum dimension of the image of ϕ_m for $m \in N(X, L)$ provided $N(X, L) \neq 0$. If $N(X, L) = 0$, then the Iitaka dimension of L is defined to be $-\infty$.

Remark 1.10. By definition, the Iitaka dimension of a line bundle L on X is an integer between 0 and $\dim(X)$ or it is $-\infty$.

Definition 1.11. A line bundle L on X is called *big* if its Iitaka dimension is equal to the dimension of X. A smooth, projective variety is called of *general type* if its canonical bundle is big. A singular variety is called of general type if a desingularization is of general type.

Remark 1.12. The same definitions can be made for $\mathbb{Q}$-Cartier divisors instead of line bundles. We use the language of Cartier divisors and line bundles interchangably.

Using Iitaka fibrations, one can prove that the Iitaka dimension of a line bundle L can be characterized as the growth rate of the dimension of the spaces of global sections of L (see Corollary 2.1.38 in [L1] for the proof).

Lemma 1.13. *A line bundle L on a normal, projective variety X of dimension n has Iitaka dimension κ if and only if there exists constants $C_1, C_2 > 0$ such that*

$$C_1 m^\kappa \leq h^0(X, L^{\otimes m}) \leq C_2 m^\kappa$$

for all sufficiently large $m \in N(X, L)$. In particular, L is big if and only if there exists $C > 0$ such that

$$h^0(X, L^{\otimes m}) > Cm^n$$

for all sufficiently large $m \in N(X, L)$.

Kodaira's Lemma allows us to obtain other useful characterizations of big divisors.

Lemma 1.14 (Kodaira's Lemma)**.** *Let D be a big Cartier divisor and E be an arbitrary effective Cartier divisor on a normal, projective variety X. Then*

$$H^0(X, \mathcal{O}_X(mD - E)) \neq 0$$

for all sufficiently large $m \in N(X, D)$.

Proof. Consider the exact sequence

$$(*) \quad 0 \to \mathcal{O}_X(mD - E) \to \mathcal{O}_X(mD) \to \mathcal{O}_E(mD) \to 0.$$

Since D is big by assumption, the dimension of global sections of $\mathcal{O}_X(mD)$ grows like $m^{\dim(X)}$. On the other hand, $\dim(E) < \dim(X)$, hence the dimension of global sections of $\mathcal{O}_E(mD)$ grows at most like $m^{\dim(X)-1}$. It follows that

$$h^0(X, \mathcal{O}_X(mD)) > h^0(E, \mathcal{O}_E(mD)$$

for large enough $m \in N(X, D)$. The lemma follows by the long exact sequence of cohomology associated to the exact sequence (*). □

A corollary of Kodaira's Lemma is the characterization of big divisors as those divisors that are numerically equivalent to the sum of an ample and an effective divisor. We will use this characterization in determining the Kodaira dimension of the moduli space of curves.

Proposition 1.15. *Let D be a divisor on a normal, irreducible projective variety X. Then the following are equivalent:*

(1) *D is big.*

(2) *For any ample divisor A, there exists an integer $m > 0$ and an effective divisor E such that mD is linearly equivalent to $A + E$.*

(3) *There exists an ample divisor A, an integer $m > 0$ and an effective divisor E such that mD is linearly equivalent to $A + E$.*

(4) *There exists an ample divisor A, an integer $m > 0$ and an effective divisor E such that mD is numerically equivalent to $A + E$.*

Proof. To prove that (1) implies (2) given any ample divisor A, take a large enough positive number r such that both rA and $(r+1)A$ are effective. By Kodaira's Lemma, there is a positive integer m such that $mD-(r+1)A$ is effective, say linearly equivalent to an effective divisor E. We thus get that mD is linearly equivalent to $A+(rA+E)$ proving (2). Clearly (2) implies (3) and (3) implies (4). To see that (4) implies (1), since mD is numerically equivalent to $A+E$, $mD-E$ is numerically equivalent to an ample divisor. Since ampleness is numerical, $mD-E$ is ample. Since ample divisors are big and

$$h^0(X,mD) \geq h^0(X,mD-E),$$

D is big. □

We conclude that the property of being big is a numerical property. Since the sum of two big divisors is again big, the set of big divisors forms an open, convex cone called the *big cone* in the Neron-Severi space. The closure of the big cone consists of all the divisor classes that are limits of divisor classes that are effective. This closed, convex cone is called the *pseudo-effective cone*.

Recent work of Boucksom, Demailly, Paun and Peternell [BDPP] has identified the dual of the pseudo-effective cone in $N_1(X)$ as the cone of mobile curves. Let X be an irreducible, projective variety of dimension n. A curve class $\gamma \in N_1(X)$ is called *mobile* if there exists a projective, birational map $f: X' \to X$ and ample classes $a_1, \dots, a_{n-1}$ in $N^1(X')$ such that $\gamma = f_*(a_1 \cdot \cdots \cdot a_{n-1})$. The *mobile cone* in $N_1(X)$ is the closed convex cone generated by all mobile classes. The mobile cone is the dual of the pseudo-effective cone.

It is possible to define a finer chamber decomposition of the pseudo-effective cone of X.

Definition 1.16. The *stable base locus* of an integral divisor D is the algebraic set

$$\mathbf{B}(D) = \bigcap_{m\geq 1} \mathrm{Bs}(|mD|)$$

obtained by intersecting the base loci of all positive multiples of the complete linear systems $|mD|$.

Exercise 1.17. Show that there exists a positive integer m_0 such that $\mathbf{B}(D) = \mathrm{Bs}(|km_0D|)$ for all $k >> 0$.

The stable base locus is the locus where the rational map associated to a sufficiently high and divisible multiple of D will not be defined. Hence, stable base loci of divisors play an essential role in birational geometry. The pseudo-effective cone may be divided into chambers according to the stable base loci of the divisors (see [ELMNP1] and [ELMNP2] for details). In these notes, we will discuss this decomposition for some moduli spaces.

In general, it is difficult to determine the ample and/or the effective cone of a variety. In these lectures, we will be concerned with some moduli spaces where we can determine these cones. As warm up, determine the NEF and pseudo-effective cones of the following varieties.

Exercise 1.18. Let X be a homogeneous variety. Show that the NEF cone is equal to the effective cone. In particular, determine the NEF cone of a flag variety.

Exercise 1.19. Let $F_n = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n))$. Determine the NEF cone and effective cone of F_n.

Exercise 1.20. Let X be a Del Pezzo surface. Determine the NEF cone and effective cone of X.

Note that for surfaces, the NEF cone and the pseudo-effective cone are duals under the intersection pairing. However, even for very simple surfaces these cones are hard to determine.

Problem 1.21. [1] Let X be the blow-up of $\mathbb{P}^2$ at m general points. Determine the NEF cone of X.

Problem 1.22. Determine the NEF cone of a product of curves $C \times C \times \cdots \times C$.

2. Preliminaries on the Moduli space of curves and the Kontsevich Moduli Space

In this section, we will recall some of the basic facts about the Deligne-Mumford moduli space of stable curves and the Kontsevich moduli space of stable maps. Moduli of Curves by Harris and Morrison [HMo1] and Notes on stable maps and quantum cohomology by Fulton and Pandharipande [FP] are excellent sources for this section and for further reading.

2.1. The moduli space of curves.

Fix non-negative integers g and n such that $2g+n \geq 3$.

Definition 2.1. An *n-pointed, genus g stable curve* $(C, p_1, \ldots, p_n)$ is a reduced, connected, projective, at-worst-nodal curve C of arithmetic genus g together with n distinct, ordered, smooth points $p_i \in C$ such that $\omega_C(\sum_{i=1}^n p_i)$ is ample.

Exercise 2.2. (1) Let C^ν be the normalization of C. A *distinguished point* of C^ν is any point that lies over a marked point p_i or a node of C. Show that the stability condition "$\omega_C(\sum_{i=1}^n p_i)$ is ample" is equivalent to requiring that in the normalization of C, every rational component has at least three distinguished points.

(2) Show that the stability condition is also equivalent to requiring that $(C, p_1, \ldots, p_n)$ have finitely many automorphisms.

Let S be a scheme of finite type over a field. The moduli functor

$$\overline{\mathcal{M}}_{g,n} : \{\text{schemes } /S\} \to \{\text{sets}\}$$

associates to an S-scheme of finite type X, the set of isomorphism classes of families $f : Y \to X$ flat over X with n sections $s_1, \ldots, s_n : X \to Y$ such that for every closed point $x \in X$, $(f^{-1}(x), s_1(x), \ldots, s_n(x))$ is an n-pointed genus g stable curve.

Theorem 2.3 (Deligne, Mumford, [DM], Knudsen, [Kn1], [Kn2])**.** *The functor $\overline{\mathcal{M}}_{g,n}$ is coarsely represented by an irreducible, normal, $\mathbb{Q}$-factorial projective variety $\overline{M}_{g,n}$ of dimension $3g-3+n$ with only finite quotient singularities.*

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[1] In these notes, I will use the label 'exercise' for questions that I believe have been answered in the literature. Some of these exercises are easy and some are challenging and may have taken a research paper to answer. I will reserve the label 'problem' to questions that I believe are open (at least in their full generality).

There is a natural forgetful morphism

$$\pi_{i_1,\ldots,i_{n-k}} : \overline{\mathrm{M}}_{g,n} \to \overline{\mathrm{M}}_{g,k}$$

with $k < n$ that forgets the marked points $p_{i_1}, \ldots, p_{i_{n-k}}$ and then stabilizes the resulting curve. In particular, let $\pi : \overline{\mathrm{M}}_{g,1} \to \overline{\mathrm{M}}_g$ be the forgetful map. Let $\omega = \omega_\pi$ be the relative dualizing sheaf. Then the *Hodge bundle* is the bundle of rank g defined by $\Lambda = \pi_*\omega$. The class λ is defined to be the first Chern class $c_1(\Lambda) = \lambda$ of the Hodge bundle Λ. Similarly, define the classes $\kappa_i = \pi_*(c_1(\omega)^{i+1})$ in $H^{2i}(\overline{\mathrm{M}}_g, \mathbb{Q})$. In particular, λ and κ_1 are divisor classes on $\overline{\mathrm{M}}_g$.

The locus of stable curves that have a node has codimension one in $\overline{\mathrm{M}}_g$. This locus has $\lfloor g/2 \rfloor + 1$ irreducible components, each of codimension one. The locus of curves that have a non-separating node (i.e., a node p such that $C - p$ is connected) forms an irreducible component denoted by Δ_{irr}. The locus of curves that have a separating node p such that $C - p$ has two components one of genus i and one of genus $g - i$ also forms an irreducible components denoted by Δ_i.

In the early 1980s, using the fact that the moduli space is the quotient of Teichmüller space by the action of the mapping class group, Harer was able to compute $H^2(\overline{\mathrm{M}}_g, \mathbb{Q})$. As an important corollary, Harer determined the Neron-Severi space of $\overline{\mathrm{M}}_g$.

Theorem 2.4 (Harer, [Har])**.** *The Picard group $Pic(\overline{M}_g) \otimes \mathbb{Q}$ is generated by λ and the classes of the boundary divisors.*

Most often the divisor class calculations are carried out on the moduli stack $\overline{\mathcal{M}}_g$. Luckily, $\mathrm{Pic}(\overline{\mathcal{M}}_g) \otimes \mathbb{Q} \cong \mathrm{Pic}(\overline{\mathrm{M}}_g) \otimes \mathbb{Q}$. The rational divisor classes corresponding to the boundary in $\mathrm{Pic}(\overline{\mathcal{M}}_g) \otimes \mathbb{Q}$ are denoted by δ_{irr} and δ_i for $1 \leq i \leq \lfloor g/2 \rfloor$. The total boundary class is denoted by δ:

$$\delta = \delta_{irr} + \delta_1 + \cdots + \delta_{\lfloor g/2 \rfloor}.$$

There is one subtle point that the reader should keep in mind. The general point of Δ_1 corresponds to a curve with an automorphism group of order two. Hence, the map from the moduli stack to its coarse moduli scheme is ramified of order two along Δ_1. As long as $g \geq 4$, the map is not ramified along any other divisorial locus. In particular, let $\Delta = \overline{\mathrm{M}}_g - \mathrm{M}_g$. Then the pull back of Δ to the moduli stack is

$$\delta_{irr} + 2\delta_1 + \delta_2 + \cdots + \delta_{\lfloor g/2 \rfloor}$$

and not δ. Sections §3.D and §3.E of [HMo1] explain in detail how to compute divisor classes on $\overline{\mathrm{M}}_g$. Here are some calculations that will become useful later in the discussion.

Exercise 2.5. The case $g = 2$ in Harer's Theorem is special. Show that $\mathrm{Pic}(\mathrm{M}_g) \otimes \mathbb{Q} = 0$. In this case, the class λ can be expressed as a linear combination of boundary divisors. Show that

$$\lambda = \frac{1}{10}\delta_{irr} + \frac{1}{5}\delta_1.$$

Exercise 2.6. More generally, let $\overline{H}_g$ be the closure of the hyperelliptic locus H_g in $\overline{\mathrm{M}}_g$. Show that $\mathrm{Pic}(H_g) \otimes \mathbb{Q} = 0$ (Hint: Realize $\overline{H}_g$ as the quotient of the Hurwitz scheme of degree two covers). Conclude that $\mathrm{Pic}(\overline{H}_g)$ is generated by the boundary divisors.

Show that Δ_{irr} intersects $\overline{H}_g$ in $\lfloor (g-1)/2 \rfloor + 1$ irreducible components. Denote these components by $\Xi_0, \Xi_1, \dots \Xi_{\lfloor (g-1)/2 \rfloor}$. Δ_i restricts to an irreducible divisor on $\overline{H}_g$. Denote the corresponding divisors by Θ_i.

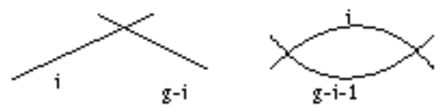

FIGURE 1. The general points of the divisors Θ_i and Ξ_i. The first curve depicts a hyperelliptic curve of genus i and a hyperelliptic curve of genus $g-i$ attached at a Weierstrass point of each. The second curve depicts a hyperelliptic curve of genus i and a hyperelliptic curve of genus $g-i-1$ glued at two points where the two points are interchanged by the hyperelliptic involution.

Show that these divisors are independent and give a basis for $\mathrm{Pic}(\overline{H}_g)$. Consider the map $i^* : \mathrm{Pic}(\overline{\mathrm{M}}_g) \to \mathrm{Pic}(\overline{H}_g)$ induced by the inclusion. Show that

$$i^*(\Delta_{irr}) = \Xi_0 + 2 \sum_{i=1}^{\lfloor (g-1)/2 \rfloor} \Xi_i \quad \text{and} \quad i^*(\Delta_i) = \Theta_i/2.$$

Show that the pull-back of λ is given by

$$i^*(\lambda) = \sum_{i=0}^{\lfloor (g-1)/2 \rfloor} \frac{i(g+1-i)}{4g+2} \Xi_i + \sum_{i=1}^{\lfloor g/2 \rfloor} \frac{i(g-i)}{4g+2} \Theta_i.$$

(Hint: See [CoH].)

Exercise 2.7. When $g \geq 3$, by exhibiting appropriate curves, show that there are no linear relations among the classes in Harer's Theorem.

Exercise 2.8. Using Grothendieck-Riemann-Roch, prove the relation

$$12\lambda = \kappa_1 + \delta.$$

Show that this relation is equivalent to the following relation

$$12\lambda - \kappa_1 = [\Delta_{irr}] + \frac{1}{2}[\Delta_1] + [\Delta_2] + \cdots + [\Delta_{\lfloor g/2 \rfloor}]$$

in $\mathrm{Pic}(\overline{\mathrm{M}}_g) \otimes \mathbb{Q}$.

2.2. **The Kontsevich Moduli Space.**

Definition 2.9. Let X be a smooth projective variety. Let $\beta \in H_2(X, \mathbb{Z})$ denote the class of a curve. The Kontsevich moduli space $\overline{\mathrm{M}}_{g,n}(X, \beta)$ parameterizes isomorphism classes of the data $(C, p_1, \dots, p_n, f)$ satisfying the following properties.

(1) C is a reduced, connected, projective, at-worst-nodal curve C of arithmetic genus g.
(2) $p_1, \dots, p_n$ are n ordered, distinct, smooth points on C.

(3) $f : C \to X$ is a morphism with $f_*[C] = \beta$. The map f is stable, i.e., it has only finitely many automorphisms.

Exercise 2.10. Distinguished points of C are points on the normalization of C that lie above the marked points p_i and the nodes of C. Show that a map f is stable if and only if every genus zero component of C on which f is constant has at least three distinguished points and every genus one component of C on which f is constant has at least one distinguished point.

We have already encountered some examples of Kontsevich moduli spaces.

Exercise 2.11. Show that the moduli space of stable maps to a point coincides with the moduli space of curves:

$$\overline{\mathrm{M}}_{g,n}(\mathbb{P}^0, 0) \cong \overline{\mathrm{M}}_{g,n}.$$

Exercise 2.12. Show that the moduli space of degree zero stable maps is isomorphic to

$$\overline{\mathrm{M}}_{g,n}(X, 0) = \overline{\mathrm{M}}_{g,n} \times X.$$

Exercise 2.13. Show that the moduli space of degree one maps to $\mathbb{P}^r$ is isomorphic to the Grassmannian:

$$\overline{\mathrm{M}}_{0,0}(\mathbb{P}^n, 1) = G(2, n+1) = \mathbb{G}(1, n).$$

A generalization of this example is the moduli space of degree one maps to a smooth quadric hypersurface Q in $\mathbb{P}^n$ for $n > 3$. In this case, the Kontsevich moduli space is isomorphic to the orthogonal Grassmannian $OG(2, n+1) = O\mathbb{G}(1, n)$.

Exercise 2.14. Show that the Kontsevich moduli space $\overline{\mathrm{M}}_{0,0}(\mathbb{P}^2, 2)$ is isomorphic to the space of complete conics or, equivalently, to the blow up of the Hilbert scheme of conics in $\mathbb{P}^2$ along the Veronese surface of double lines. (Hint: Exhibit a map (using the universal property of complete conics) from $\overline{\mathrm{M}}_{0,0}(\mathbb{P}^2, 2)$ to the space of complete conics. Check that this is a bijection on points. The claim then follows by Zariski's Main Theorem since the space of complete conics is smooth.)

We will now summarize the main existence theorems for Kontsevich moduli spaces. We refer you to [FP] for their proofs.

Theorem 2.15. *If X is a complex, projective variety, then there exists a projective coarse moduli scheme $\overline{M}_{g,n}(X, \beta)$.*

Note that even when X is a nice, simple variety (such as $\mathbb{P}^2$), $\overline{\mathrm{M}}_{g,n}(X, \beta)$ may have many components of different dimensions.

Exercise 2.16. Let $\overline{\mathrm{M}}_{1,0}(\mathbb{P}^2, 3)$ be the Kontsevich moduli space of genus-one degree-three stable maps to $\mathbb{P}^2$. Show that $\overline{\mathrm{M}}_{1,0}(\mathbb{P}^2, 3)$ has three irreducible components: two of dimension 9 and one of dimension 10. Show that smooth, elliptic cubic curves form an open set in one of the nine dimensional irreducible components of $\overline{\mathrm{M}}_{1,0}(\mathbb{P}^2, 3)$. This component is often referred to as the *main component*. Show that the locus of maps to $\mathbb{P}^2$ from a reducible curve with one genus zero and one genus one component that is constant on the genus one component and has degree three on the genus zero component is an irreducible component of $\overline{\mathrm{M}}_{1,0}(\mathbb{P}^2, 3)$ of dimension 10. Show that there is a third component of dimension 9 by considering maps from elliptic curves with two rational tails which contract the elliptic curve and map the rational tails as a line and a conic.

Exercise 2.17. Even if we restrict ourselves to genus zero stable maps the Kontsevich moduli spaces may have many components of different dimensions. Consider degree two genus zero stable maps to a smooth degree seven hypersurface X in $\mathbb{P}^7$. Assume that X contains a $\mathbb{P}^3$ (write down the equation of such a hypersurface!). Show that $\overline{\mathrm{M}}_{0,0}(X, 2)$ contains at least two components. One component covers X and has dimension 5. The conics in the $\mathbb{P}^3$ give a different component of dimension 8. Expanding on this idea, show that $\overline{\mathrm{M}}_{0,0}(X, 2)$ can have a second component of dimension arbitrarily larger than the dimension of the main component even when X is a Fano hypersurface in $\mathbb{P}^n$.

In fact, the dimension and irreducibility of the Kontsevich moduli spaces of genus-zero stable maps are not known in general even when the target is a general Fano hypersurface in $\mathbb{P}^n$.

Problem 2.18. Prove (or disprove) that if X is a general hypersurface in $\mathbb{P}^n$ of degree $d \leq n-2$, then $\overline{\mathrm{M}}_{0,0}(X, e)$ is irreducible.

Exercise 2.19. Solve the previous problem affirmatively for $e < n$.

In order to obtain an irreducible moduli space with mild singularities one needs to impose some conditions on X. A variety X is *convex* if for every map

$$f : \mathbb{P}^1 \to X,$$

f^*T_X is generated by global sections. Since every vector bundle on $\mathbb{P}^1$ decomposes as a direct sum of line bundles, a variety is convex if for every map

$$f : \mathbb{P}^1 \to X,$$

the summands appearing in f^*T_X are non-negative. If we consider genus zero stable maps to convex varieties, the Kontsevich moduli space has very nice properties.

Theorem 2.20. *Let X be a smooth, projective, convex variety.*

(1) $\overline{M}_{0,n}(X, \beta)$ *is a normal, projective variety of pure dimension*

$$\dim(X) + c_1(X) \cdot \beta + n - 3.$$

(2) $\overline{M}_{0,n}(X, \beta)$ *is locally the quotient of a non-singular variety by a finite group. The locus of automorphism free maps is a fine moduli space with a universal family and it is smooth.*

(3) *The boundary is a normal crossings divisor.*

Observe that the previous theorem in particular applies to homogeneous varieties since homogeneous varieties are convex. In fact, if X is a homogeneous variety, then $\overline{\mathrm{M}}_{0,n}(X, \beta)$ is irreducible (see [KiP]).

Remark 2.21. Although when we do not restrict ourselves to the case of genus zero maps to homogeneous varieties Kontsevich moduli spaces may be reducible with components of different dimensions, $\overline{\mathrm{M}}_{g,n}(X,\beta)$ possesses a virtual fundamental class of the expected dimension. The existence of the virtual fundamental class is the key to Gromov-Witten Theory.

Requiring a variety to be convex is a strong requirement on uniruled varieties. For instance, the blow-up of a convex variety ceases to be convex. In fact, I do not know any examples of rationally connected, projective convex varieties that are not homogeneous.

Problem 2.22. Is every smooth, rationally connected, convex projective variety a homogeneous space? Prove or give a counterexample.

Kontsevich moduli spaces admit some natural maps. As usual there are the forgetful maps

$$\pi_{i_1,\dots,i_k}:\overline{\mathrm{M}}_{g,n}(X,\beta)\to\overline{\mathrm{M}}_{g,n-k}(X,\beta)$$

obtained by forgetting the marked points $p_{i_1},\dots,p_{i_k}$ and stabilizing the resulting map. The Kontsevich moduli spaces also come equipped with n evaluation morphisms

$$ev_i:\overline{\mathrm{M}}_{g,n}(X,\beta)\to X,$$

where ev_i maps $(C,p_1,\dots,p_n,f)$ to $f(p_i)$. Finally, if $2g+n\geq 3$, there are also natural moduli maps

$$\rho:\overline{\mathrm{M}}_{g,n}(X,\beta)\to\overline{\mathrm{M}}_{g,n}$$

given by forgetting the map and stabilizing the domain curve.

Next following Rahul Pandharipande [Pa1] we determine the Picard group of the Kontsevich moduli space. We start by giving the definitions of standard divisor classes.

1. (1) $\mathcal{H}$ is class of the divisor of maps whose images intersect a fixed codimension two linear space in $\mathbb{P}^r$. This divisor is defined provided $r>1$ and $d>0$. Whenever we refer to $\mathcal{H}$ we assume these conditions hold.
2. (2) $\mathcal{L}_i=ev_i^*(\mathcal{O}_{\mathbb{P}^r}(1))$, for $1\leq i\leq n$, are the n divisor classes obtained by pulling back $\mathcal{O}_{\mathbb{P}^r}(1)$ by the n evaluation morphisms.
3. (3) $\Delta_{(A,d_A),(B,d_B)}$ are the classes of boundary divisors consisting of maps with reducible domains. Here $A\sqcup B$ is any ordered partition of the marked points. d_A and d_B are non-negative integers satisfying $d=d_A+d_B$. If $d_A=0$ (or $d_B=0$), we require that $\#A\geq 2$ ($\#B\geq 2$, respectively). When $n=0$, we denote the boundary divisors simply by $\Delta_{k,d-k}$.

Theorem 2.23 (Pandharipande)**.** *Let $r\geq 2$ and $d>0$. The divisor class $\mathcal{H}$, the divisor classes $\mathcal{L}_i$ and the classes of boundary divisors $\Delta_{(A,d_A),(B,d_B)}$ generate the group of $\mathbb{Q}$-Cartier divisors of $\overline{M}_{0,n}(\mathbb{P}^r,d)$.*

Proof. We will prove a more precise version of the theorem and determine the relations between the divisors in the process. For simplicity let

$$P=\mathrm{Pic}(\overline{\mathrm{M}}_{0,n}(\mathbb{P}^r,d))\otimes\mathbb{Q}.$$

Claim 2.24. *If the number of marked points $n \geq 3$, then $\mathcal{H}$ and the boundary divisors generate P.*

Consider the product of $n-3$ copies of $\mathbb{P}^1$. Let W be the complement of diagonals and the locus where one of the factors is $0, 1$ or ∞. Let U be the open subset

$$U \subset \mathbb{P} \oplus_0^r H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d))$$

parameterizing base-point free degree d maps from $\mathbb{P}^1$ to $\mathbb{P}^r$.

Exercise 2.25. Show that the complement of U has codimension at least 2. Show that the product $W \times U$ embeds as an open subset of $\overline{\mathrm{M}}_{0,n}(\mathbb{P}^r, d)$ whose complement is the union of the boundary divisors. Noting that the group of codimension one cycles of $W \times U$ is generated by a multiple of $\mathcal{H}$, deduce the claim.

Claim 2.26. *If the number of marked points $n = 2$, then the boundary, $\mathcal{L}_1$ and $\mathcal{L}_2$ generate P.*

Fix a hyperplane Λ. Consider the inverse image U of Λ under the third evaluation morphism from $\overline{\mathrm{M}}_{0,3}(\mathbb{P}^r, d)$. Away from the inverse image of the locus where the domain of the map is reducible and the images of the marked points lie in Λ, the forgetful map that forgets the third point is finite and projective. Hence, it suffices to show that the divisor class group of this latter space is zero. This is clear.

Claim 2.27. *If the number of marked points $n = 1$, then the boundary, $\mathcal{L}_1$ and $\mathcal{H}$ generate P.*

Exercise 2.28. Prove this claim. (Hint: Fix two general hyperplanes Λ_1, Λ_2 and carry out an argument similar to the proofs of the previous two claims.)

Claim 2.29. *If the number of marked points $n = 0$, then $\mathcal{H}$ and the boundary divisors generate P.*

Fix three hyperplanes H_1, H_2, H_3. Consider the complement Z in $\overline{\mathrm{M}}_{0,0}(\mathbb{P}^r, d)$ of the boundary and the three hypersurfaces of maps intersecting $H_i \cap H_j$, $i \neq j$.

Exercise 2.30. Prove that the divisor class group of Z tensor with $\mathbb{Q}$ is trivial. Use this fact to deduce the claim.

Note that the previous four claims suffice to complete the proof of the theorem. □

As the proof has made clear, the divisors in Theorem 2.23 satisfy certain relations. Using [Ke], these relations can be completely worked out.

Exercise 2.31. Let

$$\pi_{i_1,\dots,i_{n-4}} : \overline{\mathrm{M}}_{0,n} \to \overline{\mathrm{M}}_{0,4}$$

be a forgetful map that forgets all but four of the marked points. Since $\overline{\mathrm{M}}_{0,4} = \mathbb{P}^1$, the three boundary divisors on $\overline{\mathrm{M}}_{0,4}$ are linearly equivalent. Pulling back the boundary divisors via $\pi_{i_1,\dots,i_{n-4}}$ obtain the linear relations

$$\sum_{i,j \in T; k,l \notin T} D_T = \sum_{k,l \in T, i,j \notin T} D_T \quad \text{for} \quad i,j,k,l \in \{1,\dots,n\} \quad \text{distinct}$$

among the boundary divisors of $\overline{\mathrm{M}}_{0,n}$. Show that these (and, of course, $D_T = D_{T^c}$) are the only linear relations among the boundary divisors of $\overline{\mathrm{M}}_{0,n}$ (Hint: See [Ke]).

Exercise 2.32. Using the previous exercise and the map

$$\rho : \overline{\mathrm{M}}_{0,n}(\mathbb{P}^r, d) \to \overline{\mathrm{M}}_{0,n},$$

obtain linear relations among the boundary divisors of $\overline{\mathrm{M}}_{0,n}(\mathbb{P}^r, d)$.

Exercise 2.33. By exhibiting one parameter families that have different intersection numbers show that

(1) $\mathcal{H}$ is not in the span of boundary divisors. (Hint: Consider the Veronese image of a pencil of lines in $\mathbb{P}^2$)

(2) If the number of marked points is one, then $\mathcal{H}$ and $\mathcal{L}_1$ are independent modulo the boundary.

(3) If the number of marked points is two, then $\mathcal{L}_1$ and $\mathcal{L}_2$ are independent modulo the boundary.

Exercise 2.34. Fix a hyperplane Λ in $\mathbb{P}^r$. Show that the locus of stable maps in $\overline{\mathrm{M}}_{0,0}(\mathbb{P}^r, d)$ where $f^{-1}(\Lambda)$ is not d distinct, smooth points is a divisor $\mathcal{T}$ in $\overline{\mathrm{M}}_{0,0}(\mathbb{P}^r, d)$. Calculate the class of this divisor in terms of $\mathcal{H}$ and the boundary divisors. (Hint:

$$\mathcal{T} = \frac{d-1}{d}\mathcal{H} + \sum_{i=1}^{\lfloor d/2 \rfloor} \frac{i(d-i)}{d}\Delta_i.)$$

Exercise 2.35. Generalize the results of this section to the case when the target is a Grassmannian $G(k,n)$. Let $2 \leq k < k+2 \leq n$. Let λ be a partition with k parts $n-k \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 0$. Fix a flag $F_\bullet$. A Schubert class σ_λ is the class of the variety

$$\Sigma_\lambda(F_\bullet) = \{[W] \in G(k,n) \mid \dim(W \cap F_{n-k+i-\lambda_i}) \geq i\}.$$

Let

$$\pi : \overline{\mathrm{M}}_{0,1}(G(k,n), d) \to \overline{\mathrm{M}}_{0,0}(G(k,n), d)$$

be the forgetful morphism and let

$$ev : \overline{\mathrm{M}}_{0,1}(G(k,n), d) \to X$$

be the evaluation morphism. Let $\mathcal{H}_{\sigma_\lambda}$ be the class in $\overline{\mathrm{M}}_{0,0}(G(k,n), d)$ defined by $\pi_*(ev^*\sigma_\lambda)$. Show that $Pic(\overline{\mathrm{M}}_{0,0}(G(k,n), d)) \otimes \mathbb{Q}$ is spanned by $\mathcal{H}_{\sigma_{1,1}}$, $\mathcal{H}_{\sigma_2}$ and the classes of the boundary divisors. Show that these classes are independent.

Exercise 2.36. Generalize the results of this section to the case when the target X is a homogeneous variety G/P such as a flag variety or an orthogonal Grassmannian.

Exercise 2.37. The divisors $\mathcal{H}$ and $\mathcal{T}$ in $\overline{\mathrm{M}}_{0,0}(\mathbb{P}^r, d)$ play an important role in the enumerative geometry of rational curves in projective space. Prove that $\mathcal{H}$ and $\mathcal{T}$ are base-point-free. Calculate the intersection numbers

$$\mathcal{H}^5, \mathcal{H}^4\mathcal{T}, \mathcal{H}^3\mathcal{T}^2, \mathcal{H}^2\mathcal{T}^3, \mathcal{H}\mathcal{T}^4, \mathcal{T}^5$$

on $\overline{\mathrm{M}}_{0,0}(\mathbb{P}^2, 2)$. Interpret these numbers in terms of the enumerative geometry of conics in the plane.

3. The effective cone of the Kontsevich moduli space

In this section, the main problem we would like to address is the following:

Problem 3.1. Describe the cone of effective divisor classes on $\overline{\mathrm{M}}_{0,0}(\mathbb{P}^r, d)$ in terms of the standard generators of the Picard group.

Denote by P_d the $\mathbb{Q}$-vector space of dimension $\lfloor d/2 \rfloor + 1$ with basis labeled $\mathcal{H}$ and $\Delta_{k,d-k}$ for $k = 1, \dots, \lfloor d/2 \rfloor$. For each $r \geq 2$, there is a $\mathbb{Q}$-linear map

$$u_{d,r} : P_d \to \mathrm{Pic}(\overline{\mathrm{M}}_{0,0}(\mathbb{P}^r, d)) \otimes \mathbb{Q}$$

that is an isomorphism of $\mathbb{Q}$-vector spaces.

Definition 3.2. For every integer $r \geq 2$, denote by $\mathrm{Eff}_{d,r} \subset P_d$ the inverse image under $u_{d,r}$ of the effective cone of $\overline{\mathrm{M}}_{0,0}(\mathbb{P}^r, d)$.

Proposition 3.3. *For every integer $r \geq 2$, $\mathit{Eff}_{d,r}$ is contained in $\mathit{Eff}_{d,r+1}$. For every integer $r \geq d$, $\mathit{Eff}_{d,r}$ equals $\mathit{Eff}_{d,d}$.*

Proof of Proposition 3.3. Let $p \in \mathbb{P}^{r+1}$ be a point, denote $U = \mathbb{P}^{r+1} - \{p\}$, and let $\pi : U \to \mathbb{P}^r$ be a linear projection from p. This induces a smooth morphism

$$\overline{\mathrm{M}}_{0,0}(\pi, d) : \overline{\mathrm{M}}_{0,0}(U, d) \to \overline{\mathrm{M}}_{0,0}(\mathbb{P}^r, d).$$

Let $i : U \to \mathbb{P}^{r+1}$ be the open immersion. This induces a morphism

$$\overline{\mathrm{M}}_{0,0}(i, d) : \overline{\mathrm{M}}_{0,0}(U, d) \to \overline{\mathrm{M}}_{0,0}(\mathbb{P}^{r+1}, d)$$

relatively representable by open immersions. The complement of the image of $\overline{\mathrm{M}}_{0,0}(i, d)$ has codimension r, which is greater than 2. Therefore, the pull-back morphism

$$\overline{\mathrm{M}}_{0,0}(i, d)^* : \mathrm{Pic}(\overline{\mathrm{M}}_{0,0}(\mathbb{P}^{r+1}, d)) \to \mathrm{Pic}(\overline{\mathrm{M}}_{0,0}(U, d))$$

is an isomorphism. So there is a unique homomorphism

$$h : \mathrm{Pic}(\overline{\mathrm{M}}_{0,0}(\mathbb{P}^r, d)) \to \mathrm{Pic}(\overline{\mathrm{M}}_{0,0}(\mathbb{P}^{r+1}, d))$$

such that

$$\overline{\mathrm{M}}_{0,0}(\pi, d)^* = \overline{\mathrm{M}}_{0,0}(i, d)^* \circ h.$$

Recalling that $u(r, d)$ identifies the Picard group of $\overline{\mathrm{M}}_{0,0}(\mathbb{P}^r, d))$ with the vector space spanned by $\mathcal{H}$ and the boundary divisors $\Delta_{k,d-k}$, we see that $h \circ u_{d,r}$ equals $u_{d,r+1}$. So to prove $\mathrm{Eff}_{d,r}$ is contained in $\mathrm{Eff}_{d,r+1}$, it suffices to prove that $\overline{\mathrm{M}}_{0,0}(\pi, d)$ pulls back effective divisors to effective divisors classes, which follows since $\overline{\mathrm{M}}_{0,0}(\pi, d)$ is smooth.

Next assume $r \geq d$. Let D be any effective divisor in $\overline{\mathrm{M}}_{0,0}(\mathbb{P}^r, d)$. A general point in the complement of D parameterizes a stable map $f : C \to \mathbb{P}^r$ such that $f(C)$ spans a d-plane. Denote by $j : \mathbb{P}^d \to \mathbb{P}^r$ a linear embedding whose image is this d-plane. There is an induced morphism

$$\overline{\mathrm{M}}_{0,0}(j, d) : \overline{\mathrm{M}}_{0,0}(\mathbb{P}^d, d) \to \overline{\mathrm{M}}_{0,0}(\mathbb{P}^r, d).$$

The map $\overline{\mathrm{M}}_{0,0}(j,d)^* \circ u_{d,r}$ equals $u_{d,d}$. By construction, $\overline{\mathrm{M}}_{0,0}(j,d)^*([D])$ is the class of the effective divisor $\overline{\mathrm{M}}_{0,0}(j,d)^{-1}(D)$, i.e., $[D]$ is in $\mathrm{Eff}_{d,d}$. Thus $\mathrm{Eff}_{d,d}$ contains $\mathrm{Eff}_{d,r}$, which in turn contains $\mathrm{Eff}_{d,d}$ by the last paragraph. Therefore $\mathrm{Eff}_{d,r}$ equals $\mathrm{Eff}_{d,d}$. □

In view of Proposition 3.3, it is especially interesting to understand $\mathrm{Eff}_{d,d}$. We will concentrate on this case.

When $r = d$, the locus parameterizing stable maps $f : C \to \mathbb{P}^d$ of degree d whose set theoretic image does not span $\mathbb{P}^d$. We will denote its class by D_{deg}. The class is easily calculated in terms of the standard divisors.

Lemma 3.4. *The class D_{deg} equals*

$$D_{deg} = \frac{1}{2d}\left[(d+1)\mathcal{H} - \sum_{k=1}^{\lfloor d/2 \rfloor} k(d-k)\Delta_{k,d-k}\right]. \qquad (1)$$

Proof. We will prove the equality (1) by intersecting D_{deg} by test curves. Fix a general rational normal scroll of degree i and a general rational normal curve of degree $d-i-1$ intersecting the scroll in one point p. Consider the one-parameter family C_i of degree d curves consisting of the fixed degree $d-i-1$ rational normal curve union curves in a general pencil (that has p as a base-point) of degree $i+1$ rational normal curves on the scroll. When $2 \leq i \leq \lfloor d/2 \rfloor$, C_i has the following intersection numbers with $\mathcal{H}$ and D_{deg}.

$$C_i \cdot \mathcal{H} = i, \quad C_i \cdot D_{\mathrm{deg}} = 0.$$

The curve C_i is contained in the boundary divisor $\Delta_{i+1,d-i-1}$ and has intersection number

$$C_i \cdot \Delta_{i+1,d-i-1} = -1.$$

The intersection number of C_i with the boundary divisors $\Delta_{i,d-i}$ and $\Delta_{1,d-1}$ is non-zero and given as follows:

$$C_i \cdot \Delta_{i,d-i} = 1, \quad C_i \cdot \Delta_{1,d-1} = i+1.$$

Finally, the intersection number of C_i with all the other boundary divisors is zero. When $i = 1$, we have to modify the intersection number of C_1 with $\Delta_{1,d-1}$ to read $C_1 \cdot \Delta_{1,d-1} = 3$. Next consider the one-parameter family B_1 of rational curves of degree d that contain $d+2$ general points and intersect a general line. The intersection number of B_1 with all the boundary divisors but $\Delta_{1,d-1}$ is zero. Clearly $B_1 \cdot D_{\mathrm{deg}} = 0$. By the algorithm for counting rational curves in projective space given in [V1], it follows that

$$B_1 \cdot \mathcal{H} = \frac{d^2+d-2}{2}, \quad B_1 \cdot \Delta_{1,d-1} = \frac{(d+2)(d+1)}{2}.$$

This determines the class of D_{deg} up to a constant multiple. In order to determine the multiple, consider the curve C that consists of a fixed degree $d-1$ curve and a pencil of lines in a general plane intersecting the curve in one point. The curve C has intersection number zero with all the boundary divisors but $\Delta_{1,d-1}$ and has the following intersection numbers:

$$C \cdot \mathcal{H} = 1, \quad C \cdot D_{\mathrm{deg}} = 1, \quad C \cdot \Delta_{1,d-1} = -1.$$

The lemma follows from these intersection numbers. □

D_{deg} plays a crucial role in describing the effective cone of $\overline{\mathrm{M}}_{0,0}(\mathbb{P}^d, d)$. The following theorem completely describes the effective cone of $\overline{\mathrm{M}}_{0,0}(\mathbb{P}^d, d)$.

Theorem 3.5. *The class of a divisor lies in the effective cone of $\overline{M}_{0,0}(\mathbb{P}^d, d)$ if and only if it is a non-negative linear combination of the class of D_{deg} and the classes of the boundary divisors $\Delta_{k,d-k}$ for $1 \leq k \leq \lfloor d/2 \rfloor$.*

Proof. Since D_{deg} and the boundary divisors are effective, any non-negative rational linear combination of these divisors lies in the effective cone. The main content of the theorem is to show that there are no other effective divisor classes.

Definition 3.6. A reduced, irreducible curve C on a scheme X is a *moving curve* if the deformations of C cover a Zariski open subset of X. More precisely, a curve C is a moving curve if there exists a flat family of curves $\pi : \mathcal{C} \to T$ on X such that $\pi^{-1}(t_0) = C$ for $t_0 \in T$ and for a Zariski open subset $U \subset X$ every point $x \in U$ is contained in $\pi^{-1}(t)$ for some $t \in T$. We call the class of a moving curve a *moving curve class.*

An obvious observation is that the intersection pairing between the class of an effective divisor and a moving curve class is always non-negative. Intersecting divisors with a moving curve class gives an inequality for the coefficients of an effective divisor class. The strategy for the proof of Theorem 3.5 is to produce enough moving curves to force the effective divisor classes to be a non-negative linear combination of D_{deg} and the boundary classes.

Lemma 3.7. *If $C \subset \overline{M}_{0,0}(\mathbb{P}^d, d)$ is a reduced, irreducible curve that intersects the complement in $\overline{M}_{0,0}(\mathbb{P}^d, d)$ of the boundary divisors and the divisor of maps whose image is degenerate, then C is a moving curve.*

Proof. The automorphism group of $\mathbb{P}^d$ acts transitively on rational normal curves. An irreducible curve of degree d that spans $\mathbb{P}^d$ is a rational normal curve. Hence, a curve $C \subset \overline{\mathrm{M}}_{0,0}(\mathbb{P}^d, d)$ that intersects the complement in $\overline{\mathrm{M}}_{0,0}(\mathbb{P}^d, d)$ of the boundary divisors and the divisor of maps whose image is degenerate, contains a point that represents a map that is an embedding of $\mathbb{P}^1$ as a rational normal curve. The translations of C by $\mathbb{P}GL(d+1)$ cover a Zariski open set of $\overline{\mathrm{M}}_{0,0}(\mathbb{P}^d, d)$. □

First, observe that if D is an effective divisor on $\overline{\mathrm{M}}_{0,0}(\mathbb{P}^d, d)$ and D has the class

$$a\mathcal{H} + \sum_{k=1}^{\lfloor d/2 \rfloor} b_{k,d-k}\Delta_{k,d-k},$$

then $a \geq 0$. Furthermore, if $a = 0$, then $b_{k,d-k} \geq 0$. Consider a general projection of the d-th Veronese embedding of $\mathbb{P}^2$ to $\mathbb{P}^d$. Consider the image of a pencil of lines in $\mathbb{P}^2$. By Lemma 3.7, this is a moving one-parameter family C of degree d rational curves that has intersection number zero with the boundary divisors. It follows from the inequality $C \cdot D \geq 0$ that $a \geq 0$.

Furthermore, suppose that $a = 0$. Consider a general pencil of $(1,1)$ curves on $\mathbb{P}^1 \times \mathbb{P}^1$. Take a general projection to $\mathbb{P}^d$ of the embedding of $\mathbb{P}^1 \times \mathbb{P}^1$ by the linear system

$\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(i, d-i)$. By Lemma 3.7, the image of the pencil gives a moving one-parameter family C of degree d curves whose intersection with $\Delta_{k,d-k}$ is zero unless $k = i$. The relation $C \cdot D \geq 0$ implies that if $a = 0$, then $b_{i,d-i} \geq 0$. We conclude that Theorem 3.5 is true if $a = 0$. We can, therefore, assume that $a > 0$.

Suppose that for every $1 \leq i \leq \lfloor d/2 \rfloor$, we could construct a moving curve C_i in $\overline{\mathrm{M}}_{0,0}(\mathbb{P}^d, d)$ with the property that $C_i \cdot \Delta_{k,d-k} = 0$ for $k \neq i$ and that the ratio of $C_i \cdot \Delta_{i,d-i}$ to $C_i \cdot \mathcal{H}$ is given by

$$\frac{C_i \cdot \Delta_{i,d-i}}{C_i \cdot \mathcal{H}} = \frac{d+1}{i(d-i)}. \tag{2}$$

Observe that given these intersection numbers, Lemma 3.4 implies that $C_i \cdot D_{\mathrm{deg}} = 0$. Theorem 3.5 follows from the inequalities $C_i \cdot D \geq 0$.

We now construct approximations to these curves.

Proposition 3.8. *Let k, j and d be positive integers subject to the condition that $2k \leq d$. There exists an integer $n(k,d)$ depending only on k and d such that the linear system*

$$L'(j) = d\ F_1 + \left(\frac{jk(k+1)}{2} - 1\right)\ F_2 - \sum_{i=1}^{j(d+1)-n(k,d)} k\ E_i - \sum_{i=j(d+1)-n(k,d)+1}^{j(d+1)+n(k,d)\frac{(k-1)(k+2)}{2}} E_i$$

on the blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$ at $j(d+1) + n(k,d)\frac{(k-1)(k+2)}{2}$ general points is non-special (i.e., has no higher cohomology) for every $j >> 0$. The integer $n(k,d)$ may be taken to be

$$n(k,d) = \lceil 2(d+1)/k \rceil.$$

Proposition 3.8 implies Theorem 3.5. Consider the blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$ in

$$j(d+1) + \frac{n(k,d)(k-1)(k+2)}{2}$$

general points. The proper transform of the fibers F_2 under the linear system

$$d\ F_1 + \frac{jk(k+1)}{2}\ F_2 - \sum_{i=1}^{j(d+1)-n(k,d)} k\ E_i - \sum_{i=j(d+1)-n(k,d)+1}^{j(d+1)+n(k,d)\frac{(k-1)(k+2)}{2}} E_i$$

gives a one-parameter family $C_k(j)$ of rational curves of degree d that has intersection number zero with D_{deg}. Letting j tend to infinity we obtain a sequence of moving curves $C_k(j)$ in $\overline{\mathrm{M}}_{0,0}(\mathbb{P}^d, d)$ that has intersection zero with all the boundary divisors but $\Delta_{1,d-1}$ and $\Delta_{k,d-k}$. Unfortunately, the intersection of $C_k(j)$ with $\Delta_{1,d-1}$ is not zero and the ratio of $C_k(j) \cdot \mathcal{H}$ to $C_k(j) \cdot \Delta_{k,d-k}$ is not the one required by Equation (2). However, as j tends to infinity, the ratio of the intersection numbers $C_k(j) \cdot \Delta_{1,d-1}$ to $C_k(j) \cdot \mathcal{H}$ tends to zero and the ratio of $C_k(j) \cdot \Delta_{k,d-k}$ to $C_k(j) \cdot \mathcal{H}$ tends to the desired ratio $\frac{d+1}{k(d-k)}$. Theorem 3.5 follows.

Exercise 3.9. Let S be the blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$ described in Proposition 3.8. Using the exact sequence,

$$0 \to \mathcal{O}_S(L'(j)) \to \mathcal{O}_S(L'(j) + F_2) \to \mathcal{O}_{F_2}(L'(j) + F_2) \to 0,$$

Proposition 3.8 and Lemma 3.7, show that $C_k(j)$ is a moving curve in $\overline{\mathrm{M}}_{0,0}(\mathbb{P}^d, d)$.

Let S be the blow-up of $\mathbb{P}^2$ at general points. Let $|M|$ be a complete linear system on S. The Harbourne-Hirschowitz Conjecture asserts that if $E \cdot M$ is non-negative for every (-1)-curve on S, then M is non-special.

Exercise 3.10. Show that using the Harbourne-Hirschowitz Conjecture, we can construct the cone of moving curves dual to the effective cone of $\overline{\mathrm{M}}_{0,0}(\mathbb{P}^d, d)$ without the need for approximation. Work out these moving curves explicitly for $\overline{\mathrm{M}}_{0,0}(\mathbb{P}^6, 6)$ and show that they exist by verifying the Harbourne-Hirshowitz Conjecture directly in the required cases.

Proof of Proposition 3.8. The specialization technique in §2 of [Y] yields the proof of the proposition. We will specialize the points of multiplicity k one by one onto a point q. At each stage the k-fold point that we specialize will be in general position. We will first slide the point along a fiber f_1 in the class F_1 onto the fiber f_2 in the fiber class F_2 containing the point q. We then slide the point onto q along f_2. We will record the flat limit of this degeneration.

There is a simple checker game that describes the limits of these degenerations. This checker game for $\mathbb{P}^2$ is described in §2 of [Y]. The details for $\mathbb{P}^1 \times \mathbb{P}^1$ are identical. The global sections of the linear system $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(a, b)$ are bi-homogeneous polynomials of bi-degree a and b in the variables x, y and z, w, respectively. A basis for the space of global sections is given by $x^i y^{a-i} z^j w^{b-j}$, where $0 \leq i \leq a$ and $0 \leq j \leq b$. We can record these monomials in a rectangular $(a+1) \times (b+1)$ grid. In this grid the box in the i-th row and the j-th column corresponds to the monomial $x^i y^{a-i} z^j w^{b-j}$.

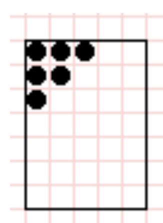

FIGURE 2. Imposing a triple point on $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(4, 6)$.

If we impose an ordinary k-fold point on the linear system at $([x : y], [z : w]) = ([0 : 1], [0 : 1])$, then the coefficients of the monomials

$$y^a w^b, xy^{a-1} w^b, \ldots, x^{k-1} y^{a-k+1} z^{k-1} w^{b-k+1}$$

must vanish. We depict this by filling in a $k \times k$ triangle of checkers into the boxes at the upper left hand corner as in Figure 2. The coefficients of the monomials represented by boxes that have checkers in them must vanish.

We first slide the k-fold point along the fiber f_1 onto the point $([x : y], [z : w]) = ([1 : 0], [0 : 1])$. This correspond to the degeneration

$$([x : y], [z : w]) \mapsto ([x : ty], [z : w]).$$

The flat limit of this degeneration is described by the vanishing of the coefficients of certain monomials (assuming none of the checkers fall out of the rectangle). The monomials whose coefficients must vanish are those that correspond to boxes with checkers in them when we let the checkers fall according to the force of gravity. The first two panels in Figure 3 depict the result of applying this procedure to a 4-fold point when there is an aligned ideal condition at the point $([x:y],[z:w]) = ([1:0],[1:0])$.

We then follow this degeneration with a degeneration that specializes the k-fold point to q by sliding along the fiber f_2. This degeneration is explicitly given by

$$([x:y],[z:w]) \mapsto ([x:y],[z:tw]).$$

The flat limit is described by the vanishing of the coefficients of the monomials that have checkers in them when we slide all the checkers as far right as possible. The last two panels of Figure 3 depict this degeneration.

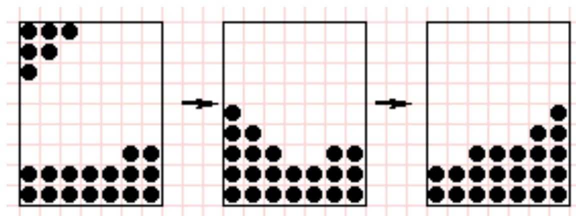

FIGURE 3. Depicting the degenerations by checkers.

Exercise 3.11. Show that, provided none of the checkers fall out of the ambient rectangle during these moves, these checker movements do record the flat limits of the linear systems under the given degenerations.

If one can play this checker game with all the multiple points that one imposes on a linear system so that during the game none of the checkers fall out of the rectangle, one can conclude that the multiple points impose independent conditions on the linear system. The limit linear system has the expected dimension. In particular, it is non-special. By upper semi-continuity, the original linear system must also have the expected dimension and be non-special.

Exercise 3.12. Unfortunately, when one plays this game, occasionally checkers may fall out of the rectangle. In that case, we lose information about the flat limits. This may happen even if the original linear system has the expected dimension. Find examples where the game fails even though the linear system has the expected dimension (see [Y]).

In order to conclude the proposition we need to show that if we impose at most $j(d+1)-n(k,d)$ points of multiplicity k on the linear system $\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(d, jk(k+1)/2)$ where $2k \leq d$, we do not lose any checkers when we specialize all the k-fold points by the degeneration just described. This suffices to conclude the proposition because general simple points always impose independent conditions.

The main observation is that if there is a safety net of empty boxes at the top of the rectangle, then the checkers will not fall out of the box. The proof of the proposition is completed by noting the following simple facts.

(1) At any stage of the degeneration the height of the checkers in the rectangle is at most k larger than the highest row full of checkers.
(2) The left most checker of a row is to the lower left of the left most checker of any row above it.

If there are at least $(k+1)(d+1)$ empty boxes in our rectangle, then by the above two observations when we specialize a k-fold point we do not lose any of the checkers. As long as $n(k,d) \geq \lceil 2(d+1)/k \rceil$, there is always at least $(k+1)(d+1)$ boxes empty. Hence until the stage where we specialize the last k-fold point we cannot lose any checkers. This concludes the proof. □

This also concludes the proof of the theorem. □

Exercise 3.13. Show that the class of D_{deg} is not an effective divisor class in $\overline{\mathrm{M}}_{0,0}(\mathbb{P}^{d-1}, d)$ (Hint: Consider the moving curve obtained by taking a pencil of degree d rational curves on a rational normal surface scroll in $\mathbb{P}^{d-1}$. Show that this curve has negative intersection number with D_{deg}). Conclude that the inclusion in Proposition 3.3 is strict for $r < d$.

Exercise 3.14. Determine that the effective cone of $\overline{\mathrm{M}}_{0,0}(\mathbb{P}^2, 3)$ (Hint: Consider the locus of maps that fail to be an isomorphism over a point contained in a fixed line in $\mathbb{P}^2$. Show that this is a divisor and together with the boundary divisor generates the effective cone of $\overline{\mathrm{M}}_{0,0}(\mathbb{P}^2, 3)$). Determine the effective cone of $\overline{\mathrm{M}}_{0,0}(\mathbb{P}^3, 4)$. (Hint: Consider the locus of maps that fail to be an isomorphism onto their image. Show that this is a divisor and calculate its class. Show that the effective cone is generated by this divisor and the boundary divisors.) If you like a challenge, try to determine the effective cone of $\overline{\mathrm{M}}_{0,0}(\mathbb{P}^2, 4)$.

Problem 3.15. Determine the effective cone of $\overline{\mathrm{M}}_{0,0}(\mathbb{P}^r, d)$ when $r < d$.

Exercise 3.16. Much of the theory discussed in this section can be generalized to other homogeneous varieties. In this exercise, you will work out the case of $\overline{\mathrm{M}}_{0,0}(G(k,n), k)$ with $n \geq 2k$.

(1) Identify the Neron-Severi space of $\overline{\mathrm{M}}_{0,0}(G(k,n), k)$ with the vector space generated by the symbols $\mathcal{H}_{\sigma_{1,1}}, \mathcal{H}_{\sigma_2}$ and Δ_i for $1 \leq i \leq \lfloor k/2 \rfloor$. Let $\mathrm{Eff}(\overline{\mathrm{M}}_{0,0}(G(k,r), k))$ denote the image of the effective cone in this vector space. Show that
$$\mathrm{Eff}(\overline{\mathrm{M}}_{0,0}(G(k,r), k)) \subseteq \mathrm{Eff}(\overline{\mathrm{M}}_{0,0}(G(k,r+1), k))$$
with equality if $r \geq 2k$. We may, therefore, restrict the discussion to the effective cone of $\overline{\mathrm{M}}_{0,0}(G(k,2k), k)$.
(2) Show that in $\overline{\mathrm{M}}_{0,0}(G(k,2k), k)$ the locus of maps such that the span of the vector spaces parameterized by f lie in a proper subspace
$$\{(C,f) \in \overline{\mathrm{M}}_{0,0}(G(k,2k), k) \mid \dim(\text{span of } f(p), p \in C) \leq 2k-1\}$$
is a divisor D_{deg} in $\overline{\mathrm{M}}_{0,0}(G(k,2k), k)$. Calculate the class of this divisor. Informally, this is the locus of degenerate maps.
(3) Let S be the tautological bundle of $G(k,2k)$. Let D_{unb} be the closure of the locus of maps f with irreducible domain such that $H^0(C, f^*S \otimes \mathcal{O}_C(-2)) \neq 0$. Show

that D_{unb} is a divisor and calculate its class. Informally, this is the locus of maps for which the tautological bundle has unbalanced splitting.

(4) Using the moving curves constructed for the case of $\mathbb{P}^n$, construct moving curves in $\overline{\mathrm{M}}_{0,0}(G(k,2k),k)$ to show that that the effective cone of $\overline{\mathrm{M}}_{0,0}(G(k,2k),k)$ is generated by D_{deg}, D_{unb} and the boundary divisors.

Generalize this discussion to the case $\overline{\mathrm{M}}_{0,0}(G(k,n),d)$ for $n \geq d+k$. Show that it suffices to consider the case $n = d+k$. Show that the locus of degenerate maps is still a divisor. When k divides d, there is a natural generalization of the divisor D_{unb}. What happens when k does not divide d? Find another extremal divisor in this case. For more details see [CS].

4. The effective cone of the moduli space of curves

There are very few cases when the effective cone of the moduli space of curves is completely known. In this section, we will see a few examples.

As a first example, we study the effective cone of $\overline{\mathrm{M}}_{0,n}/\mathfrak{S}_n$. The moduli space of n-pointed genus g curves admits a natural action of the symmetric group, where the symmetric group acts by permuting the marked points. The Neron-Severi space of $\overline{\mathrm{M}}_{0,n}/\mathfrak{S}_n$ is generated by the classes of the boundary divisors $\Delta_2, \Delta_3, \dots, \Delta_{\lfloor n/2 \rfloor}$.

Theorem 4.1 (Keel-McKernan, [KeM]). *The effective cone of $\overline{M}_{0,n}/\mathfrak{S}_n$ is the cone spanned by the classes of the boundary divisors Δ_i for $2 \leq i \leq \lfloor n/2 \rfloor$.*

Proof. Let D be an irreducible effective divisor different from a boundary divisor. We would like to show that the class of D is a non-negative linear combination of boundary divisors. Write $D = \sum_{i=2}^{\lfloor n/2 \rfloor} a_i[\Delta_i]$. We show that $a_i \geq 0$ by induction on i. Let C_2 be the curve obtained in $\overline{\mathrm{M}}_{0,n}/\mathfrak{S}_n$ by fixing $n-1$ points on $\mathbb{P}^1$ and varying the n-th point on $\mathbb{P}^1$. $C_2 \cdot \Delta_2 = n-1$ and $C_2 \cdot \Delta_i = 0$ for $i > 2$. Moreover, C_2 is a moving curve. We conclude that $a_2 \geq 0$. Suppose $a_j \geq 0$ for $2 \leq j < i \leq \lfloor n/2 \rfloor$. Fix a $\mathbb{P}^1$ with i distinct fixed points $p_1, \dots, p_{i-1}$ and q_1. Fix another $\mathbb{P}^1$ with $n-i+1$ fixed points $p_i, \dots, p_n$ and one variable point q_2. Glue the two $\mathbb{P}^1$'s along q_1 and q_2. Let C_i be the curve in $\overline{\mathrm{M}}_{0,n}/\mathfrak{S}_n$ obtained by letting q_2 vary. Then $C_i \cdot \Delta_i = n-i+1$, $C_i \cdot \Delta_{i-1} = 2-(n-i+1) = -n+i+1 < 0$ and $C_i \cdot \Delta_j = 0$ for $j \neq i-1, 1$. Curves with the class C_i cover the boundary divisor Δ_{i-1}. Since D is an irreducible divisor different from the boundary divisors, we conclude that a general curve with the class C_i cannot be contained in D. Hence, $C_i \cdot D \geq 0$. It follows that $a_i \geq 0$ concluding the induction step. □

Exercise 4.2. Examine the previous proof to show that Δ_i is in the stable base locus of a divisor with class $D = \sum a_j[\Delta_j]$ unless

$$\frac{a_{i+1}}{a_i} \geq \frac{n-i-2}{n-i}.$$

Exercise 4.3. Using the Theorem of Keel and McKernan, deduce that the effective cone of the locus of hyperelliptic curve $\overline{H}_g$ is spanned by the boundary divisors.

In contrast to $\overline{\mathrm{M}}_{0,n}/\mathfrak{S}_n$, the effective cone of $\overline{\mathrm{M}}_{0,n}$ seems to be very complicated.

Exercise 4.4. Using the fact that $\overline{\mathrm{M}}_{0,5}$ is isomorphic to the Del Pezzo surface D_5, show that the effective cone of $\overline{\mathrm{M}}_{0,5}$ is the cone spanned by the boundary divisors.

Already for $\overline{\mathrm{M}}_{0,6}$ the boundary divisors do not generate the effective cone. There are several ways of generating effective divisors on $\overline{\mathrm{M}}_{0,n}$. First, there are natural gluing maps

$$gl : \overline{\mathrm{M}}_{0,2n} \to \overline{\mathrm{M}}_g$$

obtained by gluing the points marked p_{2i-1}, p_{2i} to obtain an n-nodal genus n curve. One can pull-back effective divisors that do not contain the image of gl to obtain effective divisors on $\overline{\mathrm{M}}_{0,2n}$. There are several other such gluing maps that one may consider. For example, one may glue a fixed one-pointed elliptic curve at each of the marked points to obtain a map

$$gl' : \overline{\mathrm{M}}_{0,g} \to \overline{\mathrm{M}}_g.$$

Pulling back effective divisors not containing the image of gl' produces effective divisors on $\overline{\mathrm{M}}_{0,n}$. Next, given an effective divisor in $\overline{\mathrm{M}}_{0,n-k}$, one may pull-back this divisor via the forgetful maps

$$\pi_{i_1,\dots,i_k} : \overline{\mathrm{M}}_{0,n} \to \overline{\mathrm{M}}_{0,n-k}$$

to obtain effective divisors in $\overline{\mathrm{M}}_{0,n}$. More interestingly, by appropriately choosing the forgetful morphisms, one may construct birational morphisms from $\overline{\mathrm{M}}_{0,n}$ to a product $\overline{\mathrm{M}}_{0,n_i}$. Again by choosing the numerics carefully, one sometimes obtains divisorial contractions (see [CaT1]). The exceptional divisor in that case is an extremal ray of the effective cone.

As already mentioned, the effective cone of $\overline{\mathrm{M}}_{0,6}$ is not generated by the boundary divisors. Keel and Vermeire [Ve] have constructed an effective divisor that is not in the non-negative span of the boundary.

Exercise 4.5. Consider the gluing map

$$gl : \mathrm{M}_{0,6} \to \overline{\mathrm{M}}_3.$$

Show that the locus of hyperelliptic curves Hyp is a divisor in $\overline{\mathrm{M}}_3$ that does not contain the image of gl. Let D_{KV} be the closure of $gl^{-1}(Hyp)$. Show that the class of D_{KV} is not a non-negative linear combination of boundary divisors to conclude that the effective cone of $\overline{\mathrm{M}}_{0,6}$ is not generated by boundary divisors. In fact, show that there are 15 such different divisor classes (obtained by different possible identifications of the pairs of points).

Exercise 4.6. Show that a different way of thinking of D_{KV}, which is probably more convenient for computations, is as the locus of curves that are invariant under the element

$$(i_1, i_2)(i_3, i_4)(i_5, i_6) \in \mathfrak{S}_6.$$

Exercise 4.7. Show that pulling back the previous divisor by the forgetful morphisms produces effective divisors in any $\overline{\mathrm{M}}_{0,n}$ that are not the spans of boundary divisors for any $n \geq 6$.

Hassett and Tschinkel have proved that, in fact, the effective cone of $\overline{\mathrm{M}}_{0,6}$ is generated by the boundary divisors and the 15 Keel-Vermeire divisors constructed in the previous exercises.

Theorem 4.8 (Hassett-Tschinkel, [HT]). *The effective cone of $\overline{M}_{0,6}$ is generated by the boundary divisors and the Keel-Vermeire divisors.*

We do not know the effective cone of $\overline{\mathrm{M}}_{0,n}$ for $n > 6$. However, recently Ana-Maria Castravet and Jenia Tevelev have studied the effective cone of $\overline{\mathrm{M}}_{0,n}$ (see [CaT1] and [CaT2]) in general. They have constructed new effective divisors called *hypertree divisors*.

Definition 4.9. A *hypertree* $\Gamma = \{\Gamma_1, \dots, \Gamma_d\}$ on a set N is a collection of subsets of N such that

- Each subset Γ_j has at least three elements;
- Any element $\alpha \in N$ is contained in at least two of the subsets Γ_j;
- For any subset $S \subset \{1, \dots, d\}$,
$$(*) \qquad |\bigcup_{j \in S} \Gamma_j| - 2 \geq \sum_{j \in S} (|\Gamma_j| - 2)$$
- $$|N| - 2 = \sum_{j=1}^{d} (|\Gamma_j| - 2).$$

A hypertree Γ is *irreducible* if the inequality (*) is strict for $1 < |S| < d$.

Exercise 4.10. Find all the irreducible hypertrees when N has cardinality $6, 7$ or 8.

Definition 4.11. A *planar realization* of a hypertree Γ of $N = \{1, \dots, n\}$ is a configuration of distinct points $p_1, \dots, p_n \in \mathbb{P}^2$ such that for any subset $S \subset N$ of cardinality at least three $\{p_i\}_{i \in S}$ are collinear if and only if $S \subset \Gamma_j$ for some j.

Given a hypertree Γ of $\{1, \dots, n\}$, Castravet and Tevelev construct an effective divisor D_Γ of $\overline{\mathrm{M}}_{0,n}$. D_Γ is the closure of the locus of n-pointed curves that occur as the projection of a planar realization of Γ from a point $p \in \mathbb{P}^2$.

Theorem 4.12 (Castravet-Tevelev, [CaT2]). *For any irreducible hypertree Γ, the locus $D_\Gamma \subset \overline{M}_{0,n}$ is a non-empty, irreducible divisor, which generates an extremal ray of the effective cone of $\overline{M}_{0,n}$. Furthermore, there exists a birational contraction such that the divisor D_Γ is the irreducible component of the exceptional locus that intersects the interior $M_{0,n}$.*

Castravet and Tevelev go as far as conjecturing that the hypertree divisors and boundary divisors generate the effective cone of $\overline{\mathrm{M}}_{0,n}$.

Conjecture 4.13 (Castravet-Tevelev, [CaT2]). *The effective cone of $\overline{M}_{0,n}$ is generated by the boundary divisors and the divisors D_Γ parameterized by irreducible hypertrees.*

Exercise 4.14. Show that the Keel-Vermeire divisor in $\overline{\mathrm{M}}_{0,6}$ is a hypertree divisor. Conclude by the theorem of Hassett and Tschinkel that the Castravet-Tevelev Conjecture holds for $n = 6$.

Problem 4.15. Consider the locus of curves in $\overline{\mathrm{M}}_{10}$ whose canonical images occur as hyperplane sections of $K3$ surfaces. This locus is a divisor D_{K3} discussed below. Does the closure of $gl^{-1}(D_{K3})$ in $\mathrm{M}_{0,20}$ give an effective divisor on $\overline{\mathrm{M}}_{0,20}$ that is in the cone generated by hypertree and boundary divisors?

Problem 4.16. Determine the effective cone of $\overline{\mathrm{M}}_{0,n}$ for $n > 6$.

It is not even known whether the effective cone of $\overline{\mathrm{M}}_{0,n}$ has finitely many extremal rays. A *Mori dream space* is a $\mathbb{Q}$-factorial, projective variety X with $\mathrm{Pic}(X) \otimes \mathbb{R} = N^1(X)$ and whose Cox ring is finitely generated. Mori dream spaces were introduced in [HuK]. They satisfy many nice properties. For example, in Mori dream spaces, one can run Mori's program for every divisor. The NEF cone of a Mori dream space is generated by finitely many semi-ample divisors. The effective cone of a Mori dream space is finite polyhedral. One may be bold and conjecture the following:

Conjecture 4.17. *$\overline{M}_{0,n}$ is a Mori dream space for all n.*

In particular, the conjecture would imply that the cones of ample and effective divisors are finite polyhedral cones. The conjecture is easy for $n \leq 5$ (check it in these cases!) and known for $n = 6$ by Castravet's work [Ca].

Our knowledge of the effective cone of $\overline{\mathrm{M}}_{g,n}$ is even more limited. We can determine the effective cone for some (very) small genus examples. The work of Harris-Mumford [HM], Eisenbud-Harris [EH4], [H], Farkas-Popa [FaP], Farkas [Far1], [Far3], Khosla [Kh] and many many others construct interesting effective divisors. Of course, each time one constructs an effective divisor, one determines part of the effective cone. Recently, there has been some work by Harris-Morrison [HMo2], Chen [Ch2], Fedorchuk [Fe] and Pandharipande [Pa3] for bounding the cone of effective divisors by constructing moving curves. Each time one constructs a moving curve, the effective cone has to lie to one side of the hyperplane in N^1 determined by that moving curve. One thus obtains a cone containing the effective cone. Unfortunately, we will not be able to survey this literature in any detail. None the less, let us turn to a few fun examples.

Exercise 4.18. Show that a general genus 2 curve occurs as a $(2,3)$ curve on $\mathbb{P}^1 \times \mathbb{P}^1$. More generally, show that a general hyperelliptic curve of genus g can be embedded in $\mathbb{P}^1 \times \mathbb{P}^1$ as a $(2, g+1)$ curve.

Proposition 4.19. *The effective cone of $\overline{M}_2$ is generated by the boundary divisors δ_{irr} and δ_1.*

Proof. Since in genus 2, the divisors δ_{irr}, δ_1 and λ satisfy the linear relation

$$10\lambda = \delta_{irr} + 2\delta_1,$$

the Neron-Severi space has dimension two. We need to determine the two rays bounding the effective cone. Write the class of an effective divisor as $D = a\delta_{irr} + b\delta_1$. We would like to show that $a, b \geq 0$. We can assume that D is an irreducible divisor that does not contain any of the boundary divisors. Take a general pencil of $(2,3)$ curves in $\mathbb{P}^1 \times \mathbb{P}^1$. This pencil induces a moving curve C in $\overline{\mathrm{M}}_2$. Since none of the curves in this pencil is reducible and 20 members of the family are singular, we conclude $C \cdot D = 20a \geq 0$. Hence, $a \geq 0$. Let B be the curve in $\overline{\mathrm{M}}_2$ obtained by taking a fixed elliptic curve E with a fixed point $p \in E$ and identifying a variable point $q \in E$ with p to form a genus two nodal curve. Note that B is a moving curve in the boundary divisor Δ_{irr}. Since

$$B \cdot \delta_{irr} = -2, \quad B \cdot \delta_1 = 1,$$

we conclude that $-2a + b \geq 0$. Hence, $b \geq 2a \geq 0$. □

Exercise 4.20. Verify the intersection numbers in the previous proof.

Exercise 4.21. Show that the boundary divisor δ_{irr} is in the stable base locus of $D = a\delta_{irr} + b\delta_1$ if $b < 2a$. (Hint: Use the curve B introduced in the previous proof.) Conversely, show that the stable base locus of $D = \delta_{irr} + 2\delta_1$ is empty.

Exercise 4.22. Show that the boundary divisor δ_1 is in the stable base locus of $D = a\delta_{irr} + b\delta_1$ if $12a < b$. (Hint: Consider the curve in $\overline{\mathrm{M}}_2$ obtained by taking a pencil of cubics in $\mathbb{P}^2$ and attaching a fixed elliptic curve at one of the base points.)

Exercise 4.23. To complement the previous two exercises, show that D is ample if and only if $12a > b > 2a > 0$. Conclude that the effective cone decomposes into three chambers consisting of the ample cone bounded by the rays $\delta_{irr} + 2\delta_1$ and $12\delta_{irr} + \delta_1$, a cone bounded by the rays δ_{irr} and $\delta_{irr} + 2\delta_1$, where the stable base locus is the divisor Δ_{irr} ,and a cone bounded by the rays $12\delta_{irr} + \delta_1$ and δ_1, where the stable base locus is the boundary divisor Δ_1.

Theorem 4.24 (Rulla, [Ru])**.** *The effective cone of $\overline{M}_3$ is generated by the classes of the divisor of hyperelliptic curves D_{hyp} and the boundary divisors δ_{irr} and δ_1.*

Proof. First, the class of D_{hyp} is given by $[D_{hyp}] = 18\lambda - 2\delta_{irr} - 6\delta_1$. As usual, express $D = a[D_{hyp}] + b_0\delta_{irr} + b_1\delta_1$. We may assume that D is the class of an irreducible divisor that does not contain any of the boundary divisors or D_{hyp}. Take a general pencil of quartic curves in $\mathbb{P}^2$. This pencil induces a moving curve C_1 in the moduli space which is disjoint from Δ_1 and D_{hyp} and has intersection number $C_1 \cdot \delta_{irr} = 27$ (note also that $C_1 \cdot \lambda = 3$). It follows that $b_0 \geq 0$. Fix a genus 2 curve A and a pointed genus one curve (E, p). Let C_2 be the curve in moduli space induced by attaching (E, p) to A at a variable point $q \in A$. We have the intersection numbers

$$C_2 \cdot \lambda = 0, \quad C_2 \cdot \delta_{irr} = 0, \quad C_2 \cdot D_{hyp} = 12, \quad C_2 \cdot \delta_1 = -2.$$

Since the class of C_2 is a moving curve class in Δ_1, we conclude that $12a - 2b_1 \geq 0$. Next fix a genus 2 curve A and a point $p \in A$. Let C_3 be the curve induced in the moduli space by the one-parameter family of nodal genus 3 curves obtained by gluing p to a variable point $q \in A$. The intersection numbers of C_3 are

$$C_3 \cdot \lambda = 0, \quad C_3 \cdot D_{hyp} = 2, \quad C_3 \cdot \delta_{irr} = -4, \quad C_3 \cdot \delta_1 = 1.$$

Since C_3 is a moving curve class in Δ_{irr}, we have that $2a - 4b_0 + b_1 \geq 0$. Rewriting, we see that $2a + b_1 \geq 4b_0 \geq 0$. Since $a \geq \frac{1}{6}b_1$, we conclude that a has to be non-negative. Finally, to see that b_1 is non-negative, restrict the class of D to D_{hyp} (In Exercise 2.6 we have calculated this restriction.) D_{hyp} is ample in the Satake compactification of M_g. Hence, D_{hyp} intersects D in an effective divisor. Since the coefficient of Θ_1 in $i^*(D_{hyp})$ to D_{hyp} is negative, we conclude that $b_1 \geq 0$. (In fact, show that if $b_1 < 3/7a$, then D_{hyp} must be in the base locus of D.) □

There are a few other curves that are worth analyzing. Let C_4 be the curve obtained by attaching a general one-pointed genus 2 curve to a pencil of cubic curves at a base point. Then

$$C_4 \cdot \lambda = 1, \quad C_4 \cdot D_{hyp} = 0, \quad C_4 \cdot \delta_{irr} = 12, \quad C_4 \cdot \delta_1 = -1.$$

Conclude that if $b_0 < b_1/12$, then Δ_1 is in the stable base locus of D. Similarly, let C_5 be a general pencil of $(2,4)$ curves on a quadric surface. Then

$$C_5 \cdot \lambda = 3, \quad C_5 \cdot \delta_1 = 0, \quad C_5 \cdot \delta_{irr} = 28, \quad C_5 \cdot D_{hyp} = -2.$$

Conclude that D_{hyp} is in the stable base locus of D if $b_0 < a/14$.

Exercise 4.25. Let C be a general pencil of plane curves of degree d. Show that the normalization of every member of C is irreducible. Similarly, let C be a general pencil of curves of bi-degree (a,b) on $\mathbb{P}^1 \times \mathbb{P}^1$. Show that if a and b are larger than one, then the normalization of every curve parameterized by C is irreducible.

Exercise 4.26. Let C be a plane quartic curve with one node. Let C^v denote the normalization of C and let p and q be the points lying over the node in the normalization. Show that C is in the closure of the locus of hyperelliptic curves if and only if $p+q$ is linearly equivalent to K_{C^v}. Conclude that a general pencil of quartic curves does not intersect the locus of hyperelliptic curves in $\overline{\mathrm{M}}_3$.

Exercise 4.27. Verify the intersection numbers of $C_1, \cdots C_5$ with the standard divisors claimed above.

Exercise 4.28. Using the curve class C_1, C_2 and C_4, verify that $[D_{hyp}] = 18\lambda - 2[\Delta_{irr}] - 3[\Delta_1]$.

Exercise 4.29. Imitate the cases of $g = 2, 3$ to explore the effective cone of $\overline{\mathrm{M}}_g$ when $g = 4$ and $g = 5$.

Problem 4.30. Determine the effective cone of $\overline{\mathrm{M}}_g$.

Currently this problem seems to be out of reach. A simpler problem is to determine the intersection of the effective cone with the plane spanned by λ and δ. Since the Satake compactification is a compactification of the moduli space of smooth curves where the boundary has codimension two, one extremal ray of this cone is generated by δ. The slope of the other boundary ray of the effective cone in the $(\lambda - \delta)$-plane is not known and is called the *slope of* $\overline{M}_g$. The importance of the slope will become clearer below. We will see that the slope determines the g for which the moduli space is of general type. Unfortunately, even the slope of the moduli space of curves is not known. We will discuss this problem in more detail in section §7.

5. The canonical class of $\overline{\mathrm{M}}_g$

The canonical class of the moduli space of curves can be calculated using the Grothendieck -Riemann-Roch formula (see [HM]).

Theorem 5.1. *The canonical class of the coarse moduli scheme $\overline{M}_g$ is given by*

$$K_{\overline{M}_g} = 13\lambda - 2\delta - \delta_1.$$

Proof. Let $\pi : \mathcal{C}_g^0 \to \overline{\mathcal{M}}_g^0$ be the universal family over the moduli space of stable curves without any automorphisms. The cotangent bundle of $\overline{\mathrm{M}}_g$ at a smooth, automorphism-free curve is given by the space of quadratic differentials. More generally, over the automorphism-free locus the canonical bundle will be the first chern class of

$$\pi_*(\Omega_{\overline{\mathrm{M}}_{g,1}/\overline{\mathrm{M}}_g} \otimes \omega_{\overline{\mathrm{M}}_{g,1}/\overline{\mathrm{M}}_g}).$$

We can calculate this class in the Picard group of the moduli stack using Grothendieck -Riemann-Roch:

$$\pi_*\left(\left(1 + c_1(\Omega\otimes\omega) + \frac{c_1^2(\Omega\otimes\omega)}{2} - c_2(\Omega\otimes\omega)\right)\left(1 - \frac{c_1(\Omega)}{2} + \frac{c_1^2(\Omega) + c_2(\Omega)}{12}\right)\right)$$

Note that

$$\Omega = I_{Sing}\cdot\omega,$$

where I_{Sing} denotes the ideal of the singular locus. Hence, expanding (and simplifying using the relations we have already discussed), we see that this expression equals

$$\pi_*\left(2c_1^2(\omega) - [Sing] - c_1^2(\omega) + \frac{c_1^2(\omega) + [Sing]}{12}\right) = 13\lambda - 2\delta.$$

We need to adjust this formula to take into account that every element of the locus of curves with an elliptic tail has an automorphism given by the hyperelliptic involution on the elliptic tail. The effect of this can be calculated in local coordinates to see that it introduces a simple zero along δ_1. □

Exercise 5.2. Check the last statement in the computation of the canonical class. Choose local coordinates $t_1, \dots, t_{3g-3}$ for the deformation space of a curve $C \cup E$ (where $C \cup E$ is a general point of Δ_1) such that the automorphism g of $C \cup E$ acts by $g^*t_1 = -t_1$ and $g^*t_i = t_i$ for $i > 1$. Then one can choose local coordinates for $\overline{\mathrm{M}}_g$ near $C \cup E$ so that $s_1 = t_1^2$ and $s_i = t_i$ for $i > 1$. Finish the computation by comparing

$$ds_1 \wedge \cdots \wedge ds_{3g-3} = 2t_1\ dt_1 \wedge \cdots \wedge dt_{3g-3}.$$

Remark 5.3. In terms of the class of the boundary divisors in $\overline{\mathrm{M}}_g$, the canonical class is

$$13\lambda - 2[\Delta] + \frac{1}{2}[\Delta_1].$$

6. Ample divisors on the moduli space of curves

In order to show that the moduli space is of general type we need to show that the canonical bundle is big (on a desingularization). In view of the discussion in the first section, we can try to express the canonical bundle as a sum of an ample and an effective divisor. The G.I.T. construction gives us a large collection of ample divisors.

For our purposes in the next section, we need only the following fact:

Lemma 6.1. *The divisor class λ is big and NEF.*

Proof. The shortest proof of this result is based on some facts about the Torelli map and the moduli spaces of abelian varieties. We can map the moduli space of curves M_g to the moduli space A_g of principally polarized abelian varieties of dimension g by sending C to the pair $(Jac(C), \Theta)$ consisting of the Jacobian of C and the theta divisor. In characteristic zero, this map extends from $\overline{\mathrm{M}}_g$ to the Satake compactification of A_g. The class λ is a multiple of the pull-back of an ample divisor on A_g. The lemma follows. □

A much more precise theorem due Cornalba and Harris [CoH], which we will need in the last section, determines the restriction of the ample cone of $\overline{\mathrm{M}}_g$ to the plane spanned by λ and δ.

Theorem 6.2. *Let a and b be any positive integers. Then the divisor class $a\lambda - b\delta$ is ample on $\overline{\mathrm{M}}_g$ if and only if $a > 11b$.*

For a nice exposition of the proof see [HMo1] §6.D.

Remark 6.3. Note that λ itself is not ample, but since it is big it is a sum of an ample and an effective divisor. Consequently, to show that the canonical bundle of $\overline{\mathrm{M}}_g$ is big, it suffices to express it as a sum of λ and an effective divisor.

Exercise 6.4. Fix a curve C of genus $g-1$ and a pointed curve (E, p) of genus one. Let B be the curve in $\overline{\mathrm{M}}_g$ obtained by attaching (E, p) to (C, q) at a variable point $q \in C$. Show that the degree of λ on B is zero. Conclude that λ is not ample.

Exercise 6.5. Conclude from the previous two results that the intersection of the ample cone with the plane spanned by δ and λ is bounded by the rays λ and $11\lambda - \delta$.

We do not know the ample cone of $\overline{\mathrm{M}}_g$ in general. There is, however, a beautiful conjecture due to Fulton that describes the ample cone of $\overline{\mathrm{M}}_{g,n}$ in general.

Definition 6.6. The *dual graph* of a stable curve C is a decorated graph such that

(1) The vertices are in one-to-one correspondence with the irreducible components of C. Each vertex is marked by a non-negative integer equal to the geometric genus of the corresponding component.
(2) For every node of C there is an edge connecting the corresponding vertices.
(3) For every marked point p_i, there is a half-edge emanating from the vertex corresponding to the component containing p_i.

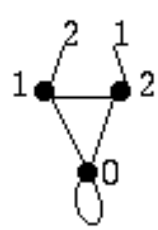

FIGURE 4. The dual graph of a stable curve.

Example 6.7. Figure 4 depicts the dual graph of a 2-pointed, genus 5 stable curve with 3 components C_1, C_2, C_3. C_1 is a nodal rational curve with no marked points. C_2 is a genus one curve containing the marked point p_2 and C_3 is a genus 2 curve containing the marked point p_1. The three irreducible components are pairwise connected by a node.

Exercise 6.8. Show that only finitely many graphs can occur as the dual graphs of n-pointed, genus g stable curves.

Exercise 6.9. List all the dual graphs for the following values of (g, n):

$$(2,0), (3,0), (4,0), (2,1), (2,2), (2,3).$$

There is a stratification of $\overline{\mathrm{M}}_{g,n}$, called the *topological stratification*, where the strata are indexed by the dual graphs and consist of stable curves with that dual graph.

Exercise 6.10. Show that the codimension of a stratum is the number of nodes of a curve contained in the stratum (equivalently, the number of edges in the dual graph).

In particular, the strata consisting of curves with $3g - 4 + n$ nodes form curves in $\overline{\mathrm{M}}_{g,n}$ called *F-curves* (in honor of Faber and Fulton). Of course, every ample divisor has positive degree on each F-curve. Fulton's Conjecture asserts that the converse is also true.

Conjecture 6.11. *(F-conjecture) A divisor D on $\overline{M}_{g,n}$ is ample if and only if $D \cdot C > 0$ for every F-curve on $\overline{M}_{g,n}$.*

A consequence of the conjecture would be that the ample cone of $\overline{\mathrm{M}}_{g,n}$ is a finite, polyhedral cone. There is a nice theorem due to Gibney, Keel and Morrison that asserts that to prove the F-conjecture it suffices to check it for $g = 0$. Unfortunately, even for $g = 0$, the cones involved are combinatorially very complicated.

The locus of *pig tails* is the image of the morphism $\overline{\mathrm{M}}_{0,g+n}/\mathfrak{S}_g \to \overline{\mathrm{M}}_{g,n}$ obtained by attaching the one-pointed, one-nodal genus one curve at the g unordered points.

Theorem 6.12 (Gibney, Keel, Morrison, [GKM])**.** *A divisor D on $\overline{M}_{g.n}$ is NEF if and only if D has non-negative intersection with all the F-curves and the restriction of D to the locus of pig tails is NEF. In particular, the F-conjecture for $g = 0$ implies the F-conjecture for all g.*

We should also remark that the F-conjecture is known for small genera and small numbers of points thanks to the work of Keel, McKernan, Farkas and Gibney (see [KeM], [FaG], [Gi]).

7. The moduli space $\overline{\mathrm{M}}_g$ is of general type when $g \geq 24$

7.1. **The general strategy.** In this section, we would like to sketch the main steps of the proof of the fundamental theorem due to Harris, Mumford and Eisenbud that asserts that $\overline{\mathrm{M}}_g$ is of general type for $g \geq 24$. You can read more about the details in [HMo1] §6.F. The papers [HM], [H] and [EH4] contain the proofs.

Theorem 7.1. *The moduli space of curves $\overline{M}_g$ is of general type if $g \geq 24$.*

The strategy of the proof is to show that the canonical class of the moduli space of curves is big by showing that it is numerically equivalent to the sum of an ample and an effective divisor. We already know that the class of any divisor on the moduli space may be expressed as a linear combination of the classes λ and the boundary divisors δ_i.

We also know that the canonical class of $\overline{\mathrm{M}}_g$ is given by the formula

$$K_{\overline{\mathrm{M}}_g} = 13\lambda - 2\delta - \delta_1.$$

Since $(11 + \epsilon)\lambda - \delta$ is ample, λ is big. Hence, if we could find an effective divisor D with class

$$a\lambda - b_0\delta_{irr} - b_1\delta_1 - \cdots - b_{\lfloor g/2\rfloor}\delta_{\lfloor g/2\rfloor}$$

satisfying the inequalities

$$\frac{a}{b_i} < \frac{13}{2}, \quad \frac{a}{b_1} < \frac{13}{3},$$

we can conclude that the canonical bundle is big. We will have expressed the canonical class as the sum of the classes of a big divisor λ and an effective divisor (a non-negative linear combination of D and the boundary divisors).

In order to encode the inequalities in the previous paragraph, one introduces the notion of the slope. The *slope* of a divisor $[D] = a\lambda - b_0\delta_{irr} - \cdots - b_{\lfloor g/2\rfloor}\delta_{\lfloor g/2\rfloor}$ on $\overline{\mathrm{M}}_g$ is the maximum of a/b_i. The slope of the moduli space $\overline{\mathrm{M}}_g$ is the infimum over all effective divisors D of the slope of D. In order to prove that $\overline{\mathrm{M}}_g$ is of general type, we need to produce divisors of sufficiently small slope.

There are two main difficulties with the approach we have outlined so far. First the construction of effective divisors with small slope is a difficult problem. We will see that the Brill-Noether and Petri divisors will do the job for Theorem 7.1. However, the calculation of these divisor classes requires some work.

The second problem is that even if we show that there are many canonical forms on $\overline{\mathrm{M}}_g$, this does not necessarily prove that the moduli space is of general type. The problem is that $\overline{\mathrm{M}}_g$ is singular. It is possible that canonical forms defined on the smooth locus do not extend to a desingularization. Luckily, this is not the case. All the singularities of $\overline{\mathrm{M}}_g$ are canonical, hence the canonical forms defined on the smooth locus extend to any desingularization. More precisely:

Theorem 7.2. *Let $g \geq 4$. Then for every n, the n-canonical forms defined on the locus of curves without automorphisms extend to n-canonical forms on a desingularization of $\overline{M}_g$.*

A sketch of some ideas. The proof of this theorem would take us too far afield. We will briefly outline some of the main ideas that go into the proof. For a complete argument see [HM].

The moduli space of curves has only finite quotient singularities along the locus of curves with non-trivial automorphisms. Locally analytically, around a point corresponding to a curve C with an automorphism, the moduli space looks like an open set in $\mathbb{C}^{3g-3}$ modulo the action of the automorphism group of C. We may, therefore, analyze the singularities of the moduli space using the Reid-Tai Criterion (see [R] and [T]). Let G be a finite group acting on a finite dimensional vector space V linearly. Let V^0 be the locus where the action is free. The Reid-Tai criterion answers the question of when pluri-canonical forms extend from V^0/G to a desingularization of V/G. For all $g \in G$, let g be conjugate to a matrix $Diag(\zeta^{a_1}, \ldots, \zeta^{a_d})$ where ζ is a primitive m-th root of unity and $0 \leq a_i < m$. If for all $g \in G$ and ζ

$$\sum_{i=1}^{d} \frac{a_i}{m} \geq 1$$

then any pluri-canonical form on V^0/G extends holomorphically to a desingularization of V/G.

In view of the Reid-Tai Criterion, one has to check whether $\sum_{i=1}^{d} \frac{a_i}{m} \geq 1$ holds and, in cases it does not hold, verify by hand that the pluri-canonical sections extend holomorphically to a desingularization. The following theorem characterizes the stable curves that fail to satisfy the Reid-Tai criterion.

Theorem 7.3. *Let C be a stable curve of arithmetic genus $g \geq 4$. Let ϕ be an automorphism of C of order n. Let ζ be a primitive n-th root of unity and suppose that the action of ϕ on $H^0(\Omega_C \otimes \omega_C)$ is given by $Diag(\zeta^{a_1}, \ldots, \zeta^{a_{3g-3}})$. Then one of the following possibilities hold:*

(1) $\sum_{i=1}^{3g-3} \frac{a_i}{m} \geq 1$.

(2) *C is the union of an elliptic or one-nodal rational curve C_1 meeting a curve C_2 of genus $g-1$ at one point. ϕ is the hyperelliptic involution on C_1 and the identity on C_2.*

(3) *C is the union of the elliptic curve C_1 with j invariant 0 meeting a curve C_2 of genus $g-1$ at one point. ϕ is an order 6 automorphism of C_1 and is the identity on C_2.*

(4) *C is the union of the elliptic curve C_1 with j invariant 12^3 meeting a curve C_2 of genus $g-1$ at one point. ϕ is an order 4 automorphism of C_1 and is the identity on C_2.*

The proof of this result rests on a case by case analysis of the possibilities based on a lemma that solves the problem for smooth curves.

Lemma 7.4. *Let C be a smooth curve. Let ϕ be an automorphism of C of order n. Let ζ be a primitive n-th root of unity and suppose that the action of ϕ on $H^0(\Omega_C \otimes \omega_C)$ is given by $Diag(\zeta^{a_1}, \ldots, \zeta^{a_{3g-3}})$. Then one of the following possibilities hold:*

(1) $\sum_{i=1}^{3g-3} \frac{a_i}{m} \geq 1$.
(2) *C is a genus zero or one curve.*
(3) *C is a hyperelliptic curve of genus 2 or 3 and ϕ is the hyperelliptic involution.*
(4) *C is a genus 2 curve which is the double cover of an elliptic curve and ϕ is the involution exchanging the branches.*

The proof of the lemma is based on an analysis of the possibilities using the Riemann-Hurwitz formula.

The final step of the proof is to check by explicit computation that pluri-canonical forms extend to the resolution of the singularities over the loci that do not satisfy the Reid-Tai Criterion.

□

The fact that $\overline{\mathrm{M}}_g$ has canonical singularities allows us to carry out the naive program outlined above. We need effective divisors of small slope. The largest dimensional irreducible component of the locus of curves that admit a degree d map to $\mathbb{P}^r$, where g, r, d satisfy the equality

$$g - (r+1)(g-d+r) = -1,$$

forms a divisor on $\overline{\mathrm{M}}_g$ called the *Brill-Noether divisor*. Its class is calculated in the following theorem.

Theorem 7.5. *If $g+1 = (r+1)(g-d+r)$, then the class of the Brill-Noether divisor on $\overline{M}_g$ is given by*

$$c\left((g+3)\lambda - \frac{g+1}{6}\delta_{irr} - \sum_{i=1}^{\lfloor g/2 \rfloor} i(g-i)\delta_i\right)$$

where c is a positive rational constant.

Unfortunately, this divisor exists only when $g+1$ is composite. When g is composite and $g+1$ is not, every curve admits finitely many degree d maps to $\mathbb{P}^r$, where

$$g - (r+1)(g-d+r) = 0.$$

The number of such maps may be determined by Schubert calculus. We can then try to define a divisor by asking that some of these maps not be distinct. This will essentially be the Petri divisor (we will give a more precise definition below).

Example 7.6. The Petri divisors in $g = 4$ and 6 are fun to describe. Consider a smooth, non-hyperelliptic curve C of genus 4. The canonical model of such a curve is the complete intersection in $\mathbb{P}^3$ of a quadric and a cubic surface. Such a curve lies on a unique quadric surface. If the quadric is a smooth quadric surface, then C possesses two (distinct) g^1_3s. They are given by projection to either of the factors of $\mathbb{P}^1 \times \mathbb{P}^1$. In codimension one, C lies on a quadric cone. Such curves do not admit two distinct g^1_3s. The Petri divisor is simply the closure of the locus of such curves.

Exercise 7.7. Calculate the class of the divisor in $\overline{\mathrm{M}}_4$ given by the closure of the locus of non-hyperelliptic curves whose canonical model lies on a singular quadric surface in $\mathbb{P}^3$.

Example 7.8. Let C be a smooth, non-hyperelliptic curve of genus 6. A general such curve C lies on a Del Pezzo surface of degree 5 and contains 5 distinct g^2_6s corresponding to the ways of blowing down D_5 to $\mathbb{P}^2$ (verify these claims!). If C lies on a Del Pezzo surface with double points, then these g^2_6s are no longer distinct. The Petri divisor is the closure of the locus of such curves.

Exercise 7.9. Calculate the class of this divisor in $\overline{\mathrm{M}}_6$.

In general, the Petri divisor is defined as the closure of the union of codimension one loci in $\overline{\mathrm{M}}_g$ of curves which possess a linear series $V \subset H^0(C, L)$ of degree d and dimension 1 such that the multiplication map

$$V \otimes H^0(C, K \otimes L^{-1}) \to H^0(C, K)$$

is not injective.

Theorem 7.10. *Let $g = 2(d-1)$. Then the class of the Petri divisor is given by*

$$\frac{2(2d-4)!}{d!(d-2)!}\left((6d^2+d-6)\lambda - d(d-1)\delta_{irr} - (2d-3)(3d-2)\delta_1 - \cdots\right),$$

where the coefficients of the remaining boundary divisors are negative and larger in absolute value than that of δ_1 (at least when $d > 4$).

The Brill-Noether and Petri divisors give us the necessary divisors to conclude the proof of Theorem 7.1. When $g \geq 24$ and odd, we can use the Brill-Noether divisor with $r = 1$. The relevant ratio is that of λ and δ_0 and is equal to

$$6 + \frac{12}{g+1}.$$

When $g \geq 24$ this is less than 6.5, hence the canonical class of $\overline{\mathrm{M}}_g$ is big provided $g + 1$ is not prime. The Brill-Noether divisors also take care of the cases $g = 24, 26$. When g is even and greater than or equal to 28, the Petri divisor works to give the conclusion.

We will spend the next section calculating the class of the Brill-Noether divisor. The class of the Petri divisor is harder to compute. You can find the computation in [EH4].

Remark 7.11. Recently, G. Farkas has announced that $\overline{\mathrm{M}}_{22}$ is also of general type. The strategy of his proof is the same. However, he constructs more elaborate effective divisors [Far2], [Far1], [Far3].

This section clarifies the importance of the slope defined in §4. Unfortunately the slope of the moduli space is not known. Initially, Harris and Morrison conjectured that the Brill-Noether divisor should give the divisor with the smallest slope on the moduli space [HMo2]. This conjecture turns out to be false. For example, in $\overline{\mathrm{M}}_{10}$ the closure of the locus of curves whose canonical models occur as hyperplane sections of $K3$-surfaces forms a divisor D_{K3}. The class of this divisor has been calculated by Farkas and Popa [FaP].

$$[D_{K3}] = 7\lambda - \delta_{irr} - \sum_{i=1}^{5} \frac{i(11-i)}{2}\delta_i.$$

In particular, the slope obtained from this divisor is $7 < 7\frac{1}{11}$, which is less than the Brill-Noether slope.

Exercise 7.12. It is somewhat surprising that the locus of curves that are hyperplane sections of $K3$ surfaces forms a divisor in $\overline{\mathrm{M}}_{10}$. Show that the moduli space of $K3$ surfaces with a fixed polarization has dimension 19. It turns out that if a canonical curve C of genus 10 is a hyperplane section of a $K3$-surface, then it is the hyperplane section of a 3-dimensional family of $K3$-surfaces. Using this fact, show that D_{K3} is indeed a divisor.

Exercise 7.13. Show that the slope of $\overline{\mathrm{M}}_2$ is 10. (Hint: Consider a pencil of $(2,3)$-curves on a quadric surface and use the relation between λ, δ_{irr} and δ_1).

Exercise 7.14. Show that the slope of $\overline{\mathrm{M}}_3$ is 9. (Hint: The divisor of hyperelliptic curves has slope 9. On the other hand, considering the curve induced by a general pencil of quartic plane curves, show that the slope cannot be less than 9.)

Exercise 7.15. Show that the slope of $\overline{\mathrm{M}}_4$ is 17/2. (Hint: The divisor of curves whose canonical model lies on a singular quadric surface in $\mathbb{P}^3$ has slope 17/2. On the other hand, by taking a pencil of $(3,3)$ curves on a smooth quadric surface in $\mathbb{P}^3$ show that the slope cannot be less than 17/2.)

Exercise 7.16. Show that the slope of $\overline{\mathrm{M}}_5$ is 8. (Hint: The Brill-Noether divisor has slope 8. Show that no member of the one-parameter family of canonical curves that contain 11 general points and intersect a general line is contained in a cubic scroll. Conclude that the slope cannot be less than 8.)

Exercise 7.17. Show that the slope of $\overline{\mathrm{M}}_6$ is 47/6. (Hint: The Petri divisor has slope 47/6. By taking a pencil of canonical curves in a smooth Del Pezzo surface D_5, show that the slope cannot be less than 47/6.)

Exercise 7.18. Determine the slope of $\overline{\mathrm{M}}_g$ for $g < 12$. (Hint: The slope conjecture of Harris and Morrison is true in this range except for $g = 10$. The divisor class determined by Farkas and Popa gives the largest possible slope in $g = 10$. Use pencils of hyperplane sections on a $K3$ surface to check these statements.)

One way to obtain moving curves in $\overline{\mathrm{M}}_g$ is to consider one-parameter families of canonical curves incident or tangent to general linear spaces. Let

$$C(a_0\Lambda_0, \ldots, a_{g-3}\Lambda_{g-3}, b_1 T\Gamma_1, \ldots, b_{g-2} T\Gamma_{g-2})$$

denote the locus of canonical curves of genus g in $\mathbb{P}^{g-1}$ that are incident to a_i general linear spaces Λ_i of dimension i and are tangent to b_i general linear spaces Γ_i of dimension i such that

$$\sum_{i=0}^{g-2} a_i(g-1-i) + \sum_{i=1}^{g-1} b_i(g-i) = g^2 + 3g - 5.$$

Let the *canonical cone* be the cone in N^1 cut out by the supporting hyperplanes to non-empty curves of this form as a_i and b_i vary over all possible values.

Exercise 7.19. Show that the effective cone of $\overline{\mathrm{M}}_g$ is contained in the canonical cone of $\overline{\mathrm{M}}_g$.

Exercise 7.20. Show that these curves give a uniform way of interpreting some of the calculations described in the exercises above and give a way of generalizing them to higher genera. In particular, show that $C(13\mathbb{P}^0)$ gives the sharp slope bound in $\overline{\mathrm{M}}_3$, $C(9\mathbb{P}^0, 5\mathbb{P}^1)$ gives the sharp slope bound in $\overline{\mathrm{M}}_4$, $C(11\mathbb{P}^0, \mathbb{P}^1)$ gives the sharp slope bound in $\overline{\mathrm{M}}_5$ and $C(15\mathbb{P}^0, \mathbb{P}^1, \mathbb{P}^2)$ gives the sharp slope bound in $\overline{\mathrm{M}}_6$.

Unfortunately, it is hard to compute the canonical cone in general.

Problem 7.21. Determine the canonical cone of $\overline{\mathrm{M}}_g$. In particular, determine the slope bound from the one-parameter families of canonical curves described above.

Remark 7.22. By the work of Harris-Morrison [HMo2] and Pandharipande [Pa3], we know that the slope of $\overline{\mathrm{M}}_g$ is positive. These papers in fact show that the slope is bounded below by c/g for an appropriate positive constant c. However, we do not know any effective divisor on any $\overline{\mathrm{M}}_g$ whose slope is less than 6. As g tends to infinity, it would be interesting to determine whether the slope tends to zero or remains bounded below by a positive constant. If such a constant exists, it has to be less than or equal to 6 (Show this by calculating the slope of the Brill-Noether divisors!). It is known that the slope of the moduli space A_g of principally polarized abelian varieties of dimension g tends to zero as g tends to infinity. If the slope of $\overline{\mathrm{M}}_g$ is bounded below by a positive constant, then one would obtain equations for the Schottky locus.

The Kodaira dimension of $\overline{\mathrm{M}}_g$ is not known for all g. $\overline{\mathrm{M}}_2$ is rational. Using the explicit description of canonical curves of genus $g \leq 6$, it is easy to see that $\overline{\mathrm{M}}_g$ is unirational when $g \leq 6$. With quite a bit more work, it is possible to see that $\overline{\mathrm{M}}_g$ is unirational for $g \leq 13$ and uniruled for $g \leq 15$.

Exercise 7.23. Using the explicit description of hyperelliptic curves of genus g as double covers of $\mathbb{P}^1$ branched along $2g+2$ points, show that the hyperelliptic locus of genus g curves $\mathcal{H}_g \subset \overline{\mathrm{M}}_g$ is rational. In particular, conclude that $\overline{\mathrm{M}}_2$ is rational.

Exercise 7.24. Show that $\overline{\mathrm{M}}_g$ is unirational for $g = 3, 4, 5, 6$. (In fact, $\overline{\mathrm{M}}_g$ is rational for $g \leq 6$ [KS]).

Exercise 7.25. If you would like a challenge, show that $\overline{\mathrm{M}}_g$ is unirational for $g = 7, 8, 9, 10$. If you are feeling really energetic, show that $\overline{\mathrm{M}}_g$ is unirational for $g \leq 13$ (see [ChR] for $g = 11, 12, 13$.)

Exercise 7.26. Show that the trigonal locus (the locus of curves that admit a degree three map to $\mathbb{P}^1$) is unirational. How about the tetragonal or pentagonal loci?

Problem 7.27. What is the Kodaira dimension of the k-gonal locus in $\overline{\mathrm{M}}_g$? If you fix k and let g tend to infinity, how does the Kodaira dimension of the k-gonal locus vary?

Problem 7.28. Determine the Kodaira dimension of $\overline{\mathrm{M}}_g$ when $15 < g \leq 21$.

It is also known that when $g \geq 2$, $\overline{\mathrm{M}}_{g,n}$ is of general type for n sufficiently large. The best results of this form to date are due to Logan [Lo] and Farkas [Far1]. It would take us too far a field to explore this avenue in detail, but we encourage the reader to consult the literature for more information.

7.2. **The Brill-Noether Theorem.** In this subsection, we will discuss some of the basics of Brill-Noether theory and the theory of limit linear series. Eisenbud and Harris have developed this theory in order to prove theorems like the Brill-Noether Theorem or the Gieseker-Petri Theorem. We will describe their approach to some of these theorems. The best places to start learning about the subject are Chapter 5 of [HMo1] and [ACGH]. Other good references are [GH], [EH1], [EH2], [EH3], [EH4], [K1], [K2].

Brill-Noether theory asks the following fundamental question:

Question 7.29. When can a curve of genus g be represented in $\mathbb{P}^r$ as a non-degenerate curve of degree at most d?

There is an expected answer to this question. We may reformulate the question as follows. When does there exist a degree d line bundle on a curve C of genus g with at least an $(r+1)$-dimensional space of global sections? We can calculate the expected dimension of this locus in $Pic^d(C)$ as follows. Let us twist all the line bundles in $Pic^d(C)$ by $\mathcal{O}_C(np)$ for a sufficiently large n (large enough so that $h^1(C, L \otimes \mathcal{O}_C(np)) = 0$ for all line bundles L of degree d. This, of course, can be achieved by taking $n > 2g-1-d$). Over $Pic^d(C)$ there is a map between the push-forward of the Poincaré bundle and the trivial bundle of rank n given by evaluation at the point p. We are interested in the dimension of the locus where the evaluation map has kernel of dimension at least $r+1$. The expected codimension of the locus is given by $(r+1)(g-d+r)$.

Exercise 7.30. Let $\phi : E^l \to F^m$ be a morphism of vector bundles of rank l and m, respectively, on an n-dimensional variety X. Calculate the expected dimension of the locus on X where the rank of ϕ is less than or equal to k.

Define the *Brill-Noether number* by

$$\rho(g,r,d) = g - (r+1)(g-d+r).$$

By the discussion in the previous paragraph, on a general curve of genus g, we expect there to be a g^r_d if and only if $\rho(g,r,d)$ is non-negative.

Example 7.31. The first instance concerns the existence of non-constant meromorphic functions on Riemann surfaces. In a first course in complex analysis, one learns that every Riemann surface admits a non-constant meromorphic function. One can then ask given a genus g Riemann surface S, what is the smallest degree meromorphic function on S?

(1) If S has genus zero, then there are non-constant meromorphic functions of degree one, namely the Möbius transformations.

(2) If S has genus one or two, then the smallest degree non-constant meromorphic function has degree 2.

(3) If S has genus 3, already the story becomes more complicated. If S is hyperelliptic, then it does admit a meromorphic function of degree 2. However, not all genus 3 curves are hyperelliptic. They do not admit meromorphic functions of degree 2. Non-hyperelliptic curves of genus 3 can be realized as plane quartics in $\mathbb{P}^2$.

Projecting the quartic from a point on the curve gives a meromorphic function of degree 3.

(4) If S is a non-hyperelliptic curve of genus 4, then its canonical image is the complete intersection of a quadric surface and a cubic surface in $\mathbb{P}^3$. By projecting to one of the factors of $\mathbb{P}^1 \times \mathbb{P}^1$ or to the base of the Hirzebruch surface F_2 (in case the quadric is singular), we obtain a map of degree 3 from S to $\mathbb{P}^1$.

(5) If S is a non-hyperelliptic and non-trigonal curve of genus 5, then it is the complete intersection of three quadric hypersurfaces in $\mathbb{P}^4$. Hence, such a curve does not admit a map of degree 3 to $\mathbb{P}^1$ (Exercise: why?). Show, however, that such a curve does admit a map of degree 4 to $\mathbb{P}^1$. (Hint: The intersection of two quadrics is a Del Pezzo surface of degree 4. The map to $\mathbb{P}^2$ blowing down 5 disjoint exceptional curves presents the curve as a five-nodal sextic. Projecting from a node gives the desired map.)

(6) Show that a general curve of genus 6 does not admit a map of degree 2 or 3 to $\mathbb{P}^1$, but does admit a map of degree 4. (Hint: The canonical image of a general curve of genus 6 lies on a degree 5 Del Pezzo surface in $\mathbb{P}^5$.)

(7) In fact, the following proposition, which is the first case of the Brill-Noether Theorem, determines the degree of the smallest degree non-constant meromorphic function on a general Riemann surface of genus g.

Proposition 7.32. *Every Riemann surface of genus g admits a non-constant meromorphic function of degree $\lfloor \frac{g+3}{2} \rfloor$. Moreover, a general Riemann surface of genus g does not admit a non-constant meromorphic function of smaller degree.*

We say that a curve C of genus g has a g^r_d if there exists a line bundle L of degree d on C with $h^0(C, L) \geq r+1$. The Brill-Noether theorem asserts that a general curve has a g^r_d if and only if the Brill-Noether number $\rho(g, r, d)$ is non-negative. In fact, more is true. Let $W(C)^r_d$ be the locus of line bundles in $Pic_d(C)$ that have at least an $(r+1)$-dimensional space of global sections. Then for a general C, the dimension of this locus is given by the Brill-Noether number.

Theorem 7.33 (Brill-Noether, Kempf, Kleiman-Laksov, Griffiths-Harris, Eisenbud-Harris)**.** *Let C be a general curve of genus g. Then the dimension of $W(C)^r_d$ is equal to the Brill-Noether number. In particular, there exists a g^r_d on C if and only if the Brill-Noether number is non-negative. Moreover, in case $\rho(g, r, d) = -1$, the closure of the locus of smooth curves that possess a g^r_d is a divisor in $\overline{M}_g$.*

Remark 7.34. Observe that the Proposition 7.32 is a special case of the Brill-Noether Theorem. If we take $r = 1$, then we see that the Brill-Noether number is non-negative if and only if $d \geq \lfloor \frac{g+3}{2} \rfloor$.

A sketch of the proof. The idea of the proof goes back to Castelnuovo. Let us consider a g-nodal rational curve and try to calculate the dimension of the space of g^r_ds on such a curve. If the dimension is correct, then we have a chance of deducing the theorem for general curves by specializing them to a g-nodal rational curve. A map of degree d to $\mathbb{P}^r$ (where $r < d$) on a g-nodal rational curve amounts to the same thing as the projection of a rational normal curve of degree d from a $\mathbb{P}^{d-r-1}$ that meets g specified secant lines. In other words, we are asking for the dimension of the intersection of g Schubert cycles Σ_r in $\mathbb{G}(d-r-1, d)$. Had these cycles been general, we could conclude that the dimension of the space of g^r_d on a g-nodal rational curve is

$$(d-r)(r+1) - gr.$$

This is precisely the Brill-Noether number.

There are a few problems with the previous idea. First, the Jacobian of a g-nodal curve is not compact, so the limits of g^r_ds on a general curve need not be g^r_ds. The second, more serious problem is that the Schubert cycles Σ_r are not general Schubert cycles, hence, their intersection need not be dimension theoretically transverse. We will completely circumvent the first problem and simultaneously deal with the second problem by specializing to g-cuspidal curves. In other words, we will make the Schubert cycles Σ_r be defined with respect to tangent lines to the rational normal curve. Note that the semi-stable reduction of such a curve is the normalization of the curve with g elliptic tails attached at the points that map to the cusps. In particular, the non-compactness issue disappears.

Theorem 7.35 (Eisenbud-Harris)**.** *Let $p_1, \dots, p_m$ be distinct points on a rational normal curve of degree d in $\mathbb{P}^d$. Let $F_1, \dots, F_m$ be the osculating flags to the rational normal curve defined at these points, respectively. Then Schubert varieties defined with respect to the flags F_i in the Grassmannian, if non-empty, intersect in the expected dimension.*

The proof of this theorem is based on a Plücker formula. Let $V \subset H^0(C, L)$ be a linear series of vector-space dimension $r+1$ on a genus g curve C. Let

$$0 \leq \alpha_0(p) \leq \alpha_1(p) \leq \cdots \leq \alpha_r(p)$$

be the ramification sequence of V at a point p of C. Let $R_i(p)$ be the orders of vanishing of sections in V at p. Recall that the ramification sequence index $\alpha_i(p)$ is defined to be $\alpha_i(p) = R_i(p) - i$. The sum of all the ramification indices over all points of the curve C may be expressed only in terms of the dimension of V, the degree of L and the genus of C as the following proposition indicates.

Proposition 7.36. *Let V be a linear series of degree d and vector-space dimension $r+1$ on a genus g curve. Then the sum of the ramification indices satisfy the following equality*

$$\sum_{j,p} \alpha_j(p) = (r+1)d + \frac{r(r+1)}{2}(2g-2).$$

Proof of Proposition. The Taylor expansions of order r of the sections in V gives a map to the bundle of r-jets of sections of L

$$\alpha : V \otimes \mathcal{O}_C \to P^r(L).$$

Taking the $(r+1)$-st exterior power, we get a map

$$\mathcal{O}_C \to \bigwedge^{r+1} P^r(L).$$

The formula claimed in the proposition arises from calculating the number of zeroes of this map in two different ways. First, using the exact sequence that relates principal parts bundles

$$0 \to L \times K_C^m \to P^m(L) \to P^{m-1}(L) \to 0,$$

we see inductively that

$$\bigwedge^{r+1} P^r(L) \cong L^{r+1} \otimes K_C^{\frac{r(r+1)}{2}}.$$

Therefore, the number of zeros is equal to

$$(r+1)d + \frac{r(r+1)}{2}(2g-2),$$

which is the right hand side of the claimed formula.

On the other hand, we can calculate the number of zeros in local coordinates. At each point $p \in C$, we choose the sections of V that vanish to order $i + \alpha_i(p)$ in terms of a local coordinate t. The order of zeros of the map is the smallest order of vanishing of any linear combination of the $(r+1) \times (r+1)$ minors of the matrix

$$\begin{pmatrix} t^{\alpha_0(p)} & t^{1+\alpha_1(p)} & t^{2+\alpha_2(p)} & \dots \\ \alpha_0(p)t^{\alpha_0(p)-1} & (1+\alpha_1(p))t^{\alpha_1(p)} & \dots & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}.$$

This order is precisely the left hand side of the formula in the proposition. □

In particular, when the genus is equal to zero, we see that the total ramification is equal to $(r+1)(d-r)$. Since the total ramification may not exceed this number, it is now easy to conclude the Eisenbud-Harris Theorem. □

Exercise 7.37. Check that for a map of a rational curve C to have a ramification sequence $\alpha_0, \dots, \alpha_{r+1}$ at p is equivalent to asking the center of the projection to satisfy a Schubert condition (of codimension equal to the sum of the ramification indices) with respect to the osculating flag to C at p. Express the class of the Schubert variety in terms of the ramification sequence.

Another central theorem of curve theory that is amenable to similar (but more difficult) techniques is the Gieseker-Petri Theorem.

Theorem 7.38 (Gieseker-Petri, Eisenbud-Harris, Lazarsfeld)**.** *Let C be a general curve. Let L be any line bundle on C. Then the multiplication map*

$$H^0(C,L) \otimes H^0(C, K \otimes L^{-1}) \to H^0(C,K)$$

is injective.

Suppose that there exists a g^r_d with negative Brill-Noether number. Using Riemann-Roch for curves, we see that

$$h^0(K - g^r_d) = h^0(g^r_d) - d + g - 1 = r + 1 - d + g - 1 = r - d + g.$$

Since the Brill-Noether number is negative, we must have $(r+1)(r-d+g) \geq g+1$. Hence the domain of the map $H^0(C, L) \otimes H^0(C, K \otimes L^{-1})$, where L is the line bundle giving the g^r_d has dimension at least $g + 1$. Consequently, the Petri map cannot be injective. We conclude that for a Gieseker-Petri general curve there does not exist a g^r_d if the Brill-Noether number is negative.

Remark 7.39. In general, the failure of the injectivity cannot be explained by dimension theoretic reasons alone. Consider a non-hyperelliptic genus 4 curve with a canonical form with a single zero (necessarily of multiplicity 6). The Weierstrass sequence for such a point is given as follows:

$$h^0(3p) = 2, \quad h^0(5p) = 3, \quad h^0(6p) = 4.$$

Although the target and the domain vector spaces in $h^0(3p) \otimes h^0(3p) \to h^0(6p)$ have the same dimension, the multiplication map is not an isomorphism since it is not possible to get a section vanishing to order 5 by multiplying sections vanishing to order 3.

Unfortunately, the curves that are easiest to work with are often not general in the sense of Gieseker-Petri. For example, a k-gonal curve, i.e., a curve admitting a non-constant holomorphic map of degree k to $\mathbb{P}^1$ will not satisfy the Gieseker-Petri Theorem if k is small ($k < (g+3)/2$) compared to g.

7.3. **Limit linear series.** In this subsection, we will briefly sketch the theory of limit linear series for curves of compact type developed by Eisenbud and Harris in order to study Brill-Noether theory. Since there are nice expository treatments of this material, our treatment will be brief. One of the main uses of the theory is to describe the closure of the Brill-Noether locus at the boundary of the moduli space. For more details see [HMo1] Chapter 5, [EH1], [EH2], [EH3], [EH4].

Definition 7.40. A curve is of *compact type* if its dual graph is a tree.

Proposition 7.41. *The following conditions on an at-worst-nodal curve C of genus g are equivalent*

(1) *C is of compact type.*
(2) *The sum of the geometric genera of the components of C equals g.*
(3) *The Jacobian of C is compact.*

Proof. If C is of compact type, then its dual graph is a tree. In particular, every irreducible component of C is smooth and any two components meet at most in one point. We can prove the equivalence of 1 and 2 by induction. If the dual graph of C has only one vertex, then the equivalence is obvious. Suppose the result is true for C whose dual graphs have at most k vertices. Take a leaf of the dual graph of C with $k+1$ vertices. If we remove the leaf, the remaining curve is a curve of compact type whose dual graph has k vertices. Hence, the sum of the geometric genera of its components equals its genus. Since the component we removed is attached at one point using the exact sequence

$$0 \to \mathcal{O}_C \to \mathcal{O}_{C_1} \oplus \mathcal{O}_{C_2} \to \mathcal{O}_{C_1 \cap C_2} \to 0$$

we see that

$$h^1(C, \mathcal{O}_C) = h^1(C_1, \mathcal{O}_{C_1}) + h^1(C_2, \mathcal{O}_{C_2}).$$

This completes the proof that 1 implies 2.

To see that 2 implies 1, we observe that by the same exact sequence that the genus of a curves is at least the sum of the genus of its components. If there is a loop, then by the exact sequence the genus of the curve formed by a loop is one more than the sum of its components.

To see the equivalence of these conditions with the condition that the Jacobian is compact, we need to study the group line bundles on a singular curve. Let $\nu : \tilde{C} \to C$ be the normalization of the curve C.

We have an exact sequence

$$0 \to \mathbb{C}^* \to (\mathbb{C}^*)^r \to \Gamma(C) \to \mathrm{Pic}(C) \to \mathrm{Pic}(\tilde{C}) \to 0,$$

where r is the number of irreducible components of C. Consequently, $J(C)$ is compact if and only if the number of points lying over the singular points of the curve is two less than twice the number of irreducible components. But the latter can only happen if and only if the dual graph of the curve is a tree. This proves the equivalence of the conditions.

□

The importance of curves of compact type arises from the fact that one can develop a theory of limits of line bundles on such curves. In fact, one can develop such a theory on tree-like curves. A Deligne-Mumford stable curve is *tree-like* if after normalizing the curve at its non-separating nodes one obtains a curve of compact type. In other words, a tree-like curve differs from curves of compact type in that the irreducible components may have internal nodes.

The main difficulty. Let $\mathcal{X} \to B$ be a one-parameter family of curves such that the total space of the family is smooth, all the fibers but the central fiber are smooth curves and the central fiber is a reducible nodal curve with smooth components. Given a line bundle L on $\mathcal{X} - X_0$, we can always extend it to the total space. Since $\mathcal{X} - X_0$ is smooth, the line bundle L corresponds to a Cartier divisor on $\mathcal{X} - X_0$. We can take the closure of this divisor in $\mathcal{X}$ to obtain a Cartier divisor on $\mathcal{X}$ (Note that here we use the smoothness of the total space). Since Cartier divisors correspond to line bundles, there is a corresponding line bundle $\tilde{L}$ extending L.

Unfortunately, the extension is not unique. This is the main technical difficulty of the subject. Suppose the central fiber $X_0 = Y \cup Z$. If we twist $\tilde{L}$ by $\mathcal{O}_{\mathcal{X}}(mY)$ or $\mathcal{O}_{\mathcal{X}}(mZ)$, we do not change the line bundle L on $\mathcal{X} - X_0$; however, we obtain a different line bundle on the total space.

Definition 7.42 (Limit linear series). Let C be a curve of compact type. A *limit linear series* D of degree d and dimension r on C is a linear series $|V_Y|$ of degree d and dimension r on every irreducible component of C called the *aspect* of D on Y, such that for any two components Y and Z of C meeting at a node p the aspects V_Y and V_Z satisfy

$$a_i(V_Y, p) + a_{r-i}(V_Z, p) \geq d.$$

The limit linear series is *refined* if these inequalities are equalities for every i. The limit linear series is *crude* if one inequality is strict.

Using the Plücker formulae, one may generalize the Brill-Noether theorem to curves of compact type. In fact, one can generalize further to tree-like curves.

Theorem 7.43. *Let C be a tree-like curve. Suppose the following about the irreducible components of Y:*

(1) *If the genus of Y is 1, then Y meets the rest of the curve in one point.*
(2) *If the genus of Y is 2, then Y meets the rest of the curve in one point which is not a Weierstrass point.*
(3) *If the genus of Y is three or more, then Y meets the rest of the curve at general points*

If $p_1, \dots, p_r$ are general points of C or arbitrary smooth points on rational components of C, then for any ramification sequence at the points p_i, the dimension of the special linear series with the given ramification sequences at the points has the expected dimension.

Remark 7.44. For our purposes, the important corollary of the theorem is that if we consider the pull-back of the Brill-Noether divisor to $\overline{\mathrm{M}}_{0,n}$ and $\overline{\mathrm{M}}_{2,1}$ via the map that attaches g fixed elliptic curves at the marked points and the map that attaches a fixed genus $g-2$ curve, respectively, the pull-back to $\overline{\mathrm{M}}_{0,n}$ is zero while the pull-back to $\overline{\mathrm{M}}_{2,1}$ is supported on the Weierstrass divisor.

7.4. **Calculating the classes of the Brill-Noether divisors.** In this subsection, we complete our discussion of the proof of Theorem 7.1 by calculating the class of the Brill-Noether divisors. For the rest of this section, assume that the Brill-Noether divisor has the following expression in terms of the standard generators

$$a\lambda - b_0\delta_{irr} - \sum_{i=1}^{\lfloor g/2 \rfloor} b_i \delta_i.$$

We calculate the class by pulling-back the Brill-Noether divisor to $\overline{\mathrm{M}}_{2,1}$ and $\overline{\mathrm{M}}_{0,g}$. Using the first pull-back, we obtain the relations

$$a = 5b_1 - 2b_2 \quad \text{and} \quad b_{irr} = \frac{b_1}{2} - \frac{b_2}{6}.$$

Using the second pull-back, we obtain for $i > 1$ the relations

$$b_i = \frac{i(g-i)}{g-1} b_1.$$

Solving for all the coefficients in terms of b_1, we obtain the class of the Brill-Noether divisors up to a positive constant. (One can determine the constant, but we do not need this for proving Theorem 7.1.)

Theorem 7.45. *If $g+1=(r+1)(g-d+r)$, then the class of the Brill-Noether divisor on $\overline{\mathrm{M}}_g$ is given by*

$$c\left((g+3)\lambda - \frac{g+1}{6}\delta_{irr} - \sum_{i=1}^{\lfloor g/2\rfloor} i(g-i)\delta_i\right)$$

where c is a positive rational constant.

To conclude the proof, we need to show the claimed relations between the coefficients. First, consider the map

$$at_{g-2} : \overline{\mathrm{M}}_{2,1} \to \overline{\mathrm{M}}_g$$

obtained by attaching a fixed genus $g-2$ curve with a marked point to curves of genus 2 with a marked point along their marked points. The theory of limit linear series shows that the pull-back of the Brill-Noether divisor is a multiple of the divisor W on $\overline{\mathrm{M}}_{2,1}$ obtained by taking the closure of the locus in $\mathrm{M}_{2,1}$ where the marked point is a Weierstrass point. The first set of relations are obtained by comparing the class of W and the pull-backs of the standard generators by at_{g-2}.

Claim 7.46. *The class of the Weierstrass divisor W is given by*

$$W = 3\omega - \lambda - \delta_1,$$

where ω is the class of the relative dualizing sheaf on $M_{2,1}$.

The pull-back of λ by at_{g-2} is λ on $\overline{\mathrm{M}}_{2,1}$. Similarly the pull-backs of δ_{irr} and δ_1 by at_{g-2} are δ_{irr} and δ_1 on $\overline{\mathrm{M}}_{2,1}$, respectively. By adjunction the pull-back of δ_2 is $-\omega$. It follows that by pulling back the Brill-Noether divisor and using the claim we obtain the relation

$$a\lambda - b_{irr}\delta_{irr} - b_1\delta_1 - b_2\omega = c(3\omega - \lambda - \delta_1).$$

We thus see that $b_2 = 3c$. Next we use the relation

$$10\lambda = \delta_{irr} + 2\delta_1$$

to solve for the other coefficients to obtain the first set of relations.

To calculate the class of the Weierstrass divisor W, we note that a Weierstrass point is a ramification point of the canonical linear series. Using this one can exhibit W as the degeneracy locus of a map between vector bundles.

Exercise 7.47. Carry this out and complete the calculation of the class of W. (Hint: If stuck, see page 338-339 in [HMo1]).

Next, consider the map

$$att : \overline{\mathrm{M}}_{0,g} \to \overline{\mathrm{M}}_g$$

obtained by attaching a fixed one pointed elliptic curve to the marked points. To obtain the required relations among the coefficients of the boundary we consider the pull-back of the Brill-Noether divisors by π. Since the Brill-Noether divisor is disjoint from the image of att, the pull-back of the divisor to $\overline{\mathrm{M}}_{0,g}$ is zero.

We thus obtain the following relation among the coefficients:

$$a\ att^*\lambda - b_0\ att^*\delta_0 - \sum_{i=1}^{\lfloor g/2 \rfloor} b_i\ att^*\delta_i = 0.$$

We have to calculate the pull-backs of the standard divisors by att. Clearly, λ and δ_{irr} pull-back to zero. The pull-backs of the divisors δ_i are the classes δ_i^0 on $\overline{\mathrm{M}}_{0,g}$ (where we place a 0 to remind ourselves that these are the divisors on $\overline{\mathrm{M}}_{0,g}$) provided $i > 1$. The image of att is contained in $\Delta_1 \subset \overline{\mathrm{M}}_g$, so the pull-back of δ_1 is the hardest to calculate. To calculate its class, we take a one-parameter family of curves

$$\pi : C \to B$$

in $\overline{\mathrm{M}}_{0,g}$. We may assume that every member of the family has at most two components and that the total space of the family is smooth. Contracting the components with fewer sections (or either of the components when equal numbers of sections pass through both components), we obtain a $\mathbb{P}^1$ bundle with g sections

$$\tilde{\pi} : \tilde{C} \to B.$$

Since the classes of any two sections differ by a multiple of the fiber class, the difference of two section classes has self-intersection zero.

The pull-back of δ_1 by att is the push-forward to the sum of the squares of the sections σ_i in the original family to the base. The sections γ_i in the projective bundle and in the original family are related by

$$\tilde{\pi}_*(\sum \gamma_i^2) = \pi_*(\sum \sigma_i^2) + \sum_{i=2}^{\lfloor g/2 \rfloor} i\ \delta_i^0.$$

Using that

$$\gamma_i^2 + \gamma_j^2 = 2\gamma_i \cdot \gamma_j$$

we obtain the relation

$$\tilde{\pi}_*(\sum \gamma_i^2) = \sum_{i=2}^{\lfloor g/2 \rfloor} \frac{i(i-1)}{g-1} \delta_i^0.$$

Combining these relations, we obtain that

$$att^*\delta_1 = \sum_{i=2}^{\lfloor g/2 \rfloor} -\frac{i(g-i)}{g-1} \delta_i^0.$$

The class of the Brill-Noether divisors (up to a constant multiple) follows from these calculations.

8. The ample cone of the Kontsevich moduli space

In this section, we discuss the ample cone of $\overline{\mathrm{M}}_{0,n}(\mathbb{P}^r, d)$.

Theorem 8.1. *Let r and d be positive integers, n a nonnegative integer such that $n+d \geq 3$. There is an injective linear map,*

$$v : Pic(\overline{M}_{0,n+d}/\mathfrak{S}_d) \otimes \mathbb{Q} \to Pic(\overline{M}_{0,n}(\mathbb{P}^r, d)) \otimes \mathbb{Q}.$$

The NEF cone of $\overline{M}_{0,n}(\mathbb{P}^r, d)$ is the product of the cone generated by

$$\mathcal{H}, \mathcal{T}, \mathcal{L}_1, \ldots, \mathcal{L}_n$$

and the image under v of the NEF cone of $\overline{M}_{0,n+d}/\mathfrak{S}_d$.

Recall that $\mathcal{H}$ is the class of the divisor of maps whose images intersect a fixed codimension two linear space in $\mathbb{P}^r$ (provided $r > 1$ and $d > 0$). The class $\mathcal{L}_i$ is the pullback of $\mathcal{O}_{\mathbb{P}^r}(1)$ by the i-th evaluation morphism. Fixing a hyperplane $\Pi \subset \mathbb{P}^r$, $\mathcal{T}$ is the class of the divisor parametrizing stable maps $(C, p_1, \ldots, p_i, f)$ for which $f^{-1}(\Pi)$ is not d reduced, smooth points of C. Recall that, in terms of Pandharipande's generators, the class of $\mathcal{T}$ equals,

$$\mathcal{T} = \frac{d-1}{d}\mathcal{H} + \sum_{k=0}^{\lfloor d/2 \rfloor} \frac{k(d-k)}{d} \left(\sum_{A,B} \Delta_{(A,k),(B,d-k)} \right).$$

We now describe the map v that occurs in Theorem 8.1.

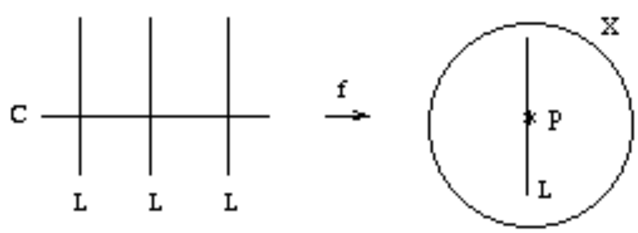

Figure 5. The morphism α.

The morphism $\alpha : \overline{\mathbf{M}}_{0,n+d} \times \mathbb{P}^{r-1} \to \overline{\mathbf{M}}_{0,n}(\mathbb{P}^r, d)$. Fix a point $p \in \mathbb{P}^r$ and a line $L \subset \mathbb{P}^r$ containing p. To every curve C in $\overline{\mathrm{M}}_{0,n+d}$ attach a copy of L at each of the last d marked points and denote the resulting curve by C'. Consider the morphism $f : C' \to \mathbb{P}^r$ that contracts C to p and maps the d rational tails isomorphically to L (see Figure 5). Since the space of lines in $\mathbb{P}^r$ passing through the point p is parameterized by $\mathbb{P}^{r-1}$, there is an induced morphism $\alpha : \overline{\mathrm{M}}_{0,n+d} \times \mathbb{P}^{r-1} \to \overline{\mathrm{M}}_{0,n}(\mathbb{P}^r, d)$.

Since α is invariant for the action of $\mathfrak{S}_d$ permuting the last d marked points, the pullback map determines a homomorphism

$$\alpha^* = (\alpha_1^*, \alpha_2^*) : \mathrm{Pic}(\overline{\mathrm{M}}_{0,n}(\mathbb{P}^r, d)) \to \mathrm{Pic}(\overline{\mathrm{M}}_{0,n+d})^{\mathfrak{S}_d} \times \mathrm{Pic}(\mathbb{P}^{r-1}).$$

We will denote the two projections of α^* by α_1^* and α_2^*.

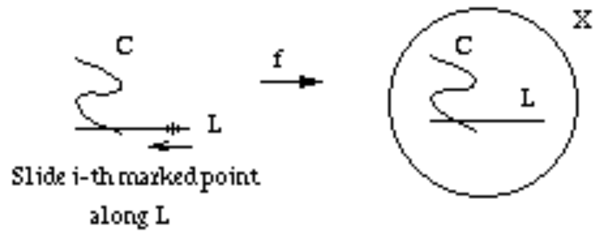

FIGURE 6. The morphism β_i.

The morphisms β_i. For each $1 \leq i \leq n$, there is a morphism $\beta_i : \mathbb{P}^1 \to \overline{\mathrm{M}}_{0,n}(\mathbb{P}^r, d)$ defined as follows. Fix a degree-$(d-1)$, $(n-1)$-pointed curve C containing all except the i-thmarked point. At a general point of C, attach a line L. Attach a line L to C at a general point of C. The resulting degree-d, reducible curve will be the domain of our map. The final, i-th marked point is in L. Varying p_i in L gives a morphism $\beta_i : \mathbb{P}^1 \to \overline{\mathrm{M}}_{0,n}(\mathbb{P}^r, d)$ (see Figure 6). This definition has to be slightly modified in the cases $(n, d) = (1, 1)$ or $(2, 1)$. When $(n, d) = (1, 1)$, we assume that the line L with the varying marked point p_i constitutes the entire stable map. When $(n, d) = (2, 1)$, we assume that the map has L as the only component. One marked point is allowed to vary on L and the remaining marked point is held fixed at a point $p \in L$.

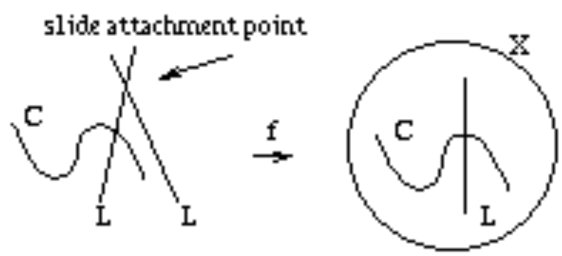

FIGURE 7. The morphism γ.

The morphism γ. If $d \geq 2$, there is a morphism $\gamma : \mathbb{P}^1 \to \overline{\mathrm{M}}_{0,n}(\mathbb{P}^r, d)$ defined as follows. Take two copies of a fixed line L attached to each other at a variable point. Fix a point p in the second copy of L. Let C be a smooth, degree-$(d-2)$, genus 0, $(n+1)$-pointed stable map to $\mathbb{P}^r$ whose $(n+1)$-st point maps to p. Attach this to the second copy of L at p. Altogether, this gives a degree-d, n-pointed, genus 0 stable maps with three irreducible components. The n marked points are the first n marked points of C. The only varying aspect of this family of stable maps is the attachment point of the two copies of L. Varying the attachment point in $L \cong \mathbb{P}^1$ gives a stable maps is parameterized by $\mathbb{P}^1$, hence there is an induced morphism $\gamma : \mathbb{P}^1 \to \overline{\mathrm{M}}_{0,n}(\mathbb{P}^r, d)$ (see Figure 7). When $(n, d) = (1, 2)$, we modify the definition by assuming that the map consists only of the two copies of the line L and the marked point is held fixed at the point p on the second copy of L.

If $d \geq 2$, denote by $P_{r,n,d}$ the Abelian group

$$P_{r,n,d} := \text{Pic}(\overline{\text{M}}_{0,n+d})^{\mathfrak{S}_d} \times \text{Pic}(\mathbb{P}^{r-1}) \times \text{Pic}(\mathbb{P}^1)^n \times \text{Pic}(\mathbb{P}^1).$$

Denote by $u = u_{r,n,d} : \text{Pic}(\overline{\text{M}}_{0,n}(\mathbb{P}^r, d)) \to P_{r,n,d}$ the pull-back map

$$u_{r,n,d} = (\alpha^*, (\beta_1^*, \dots, \beta_n^*), \gamma^*).$$

If $d = 1$, denote by $P_{r,n,1}$ the Abelian group

$$P_{r,n,1} := \text{Pic}(\overline{\text{M}}_{0,n+d})^{\mathfrak{S}_d} \times \text{Pic}(\mathbb{P}^{r-1}) \times \text{Pic}(\mathbb{P}^1)^n$$

and denote by $u = u_{r,n,1} : \text{Pic}(\overline{\text{M}}_{0,n}(\mathbb{P}^r, 1)) \to P_{r,n,1}$ the pull-back map

$$u_{r,n,1} = (\alpha^*, (\beta_1,^*, \dots, \beta_n^*))$$

Theorem 8.1 is equivalent to Theorem 8.2.

Theorem 8.2. *The map $u_{r,n,d} \otimes \mathbb{Q}$: Pic($\overline{M}_{0,n}(\mathbb{P}^r, d)$) $\otimes \mathbb{Q} \to P_{r,n,d} \otimes \mathbb{Q}$ is an isomorphism. The image under $u_{r,n,d} \otimes \mathbb{Q}$ of the ample cone, resp. NEF, eventually free cone of $\overline{M}_{0,n}(\mathbb{P}^r, d)$ equals the product of the ample cones, resp. NEF, eventually free cones of Pic($\overline{M}_{0,n+d}$)$^{\mathfrak{S}_d}$, Pic($\mathbb{P}^{r-1}$), and the factors Pic($\mathbb{P}^1$).*

To apply Theorem 8.2, we need to express the images of the standard generators of $\text{Pic}(\overline{\text{M}}_{0,n}(\mathbb{P}^r, d))$ in terms of the standard generators for $\text{Pic}(\overline{\text{M}}_{0,n+d})^{\mathfrak{S}_d}$, $\text{Pic}(\mathbb{P}^{r-1})$ and $\text{Pic}(\mathbb{P}^1)$ factors. This is summarized in Table 8.

Let $\Pi \subset \mathbb{P}^r$ be a hyperplane not containing the point p used to define the morphisms α and γ. Assume that the degree $d-1$ curve used to define the morphisms β_i is not tangent to Π, and none of the marked points on this curve are contained in Π. Finally, assume that the degree $d-2$ curve used to define the morphism γ is not tangent to Π and none of the marked points are contained in Π.

Denote by $\text{M}_{0,n+d}(\mathbb{P}^r, d)$ the open subset of $\overline{\text{M}}_{0,n+d}(\mathbb{P}^r, d)$ parameterizing stable maps with irreducible domain. Let

$$\text{ev}_{n+1,\dots,n+d} : \text{M}_{0,n+d}(\mathbb{P}^r, d) \to (\mathbb{P}^r)^d$$

be the evaluation morphism associated to the last d marked points. Let $\text{M}_{0,n+d}(\mathbb{P}^r, d)_\Pi$ denote the inverse image of Π^d and by $\overline{\text{M}}_{0,n+d}(\mathbb{P}^r, d)_\Pi$ the closure of $\text{M}_{0,n+d}(\mathbb{P}^r, d)_\Pi$ in $\overline{\text{M}}_{0,n+d}(\mathbb{P}^r, d)$.

$\overline{\text{M}}_{0,n+d}(\mathbb{P}^r, d)_\Pi$ is $\mathfrak{S}_d$-invariant under the action of $\mathfrak{S}_d$ on $\overline{\text{M}}_{0,n+d}(\mathbb{P}^r, d)$ permuting the last d marked points. Denote by

$$\pi : \overline{\text{M}}_{0,n+d}(\mathbb{P}^r, d) \to \overline{\text{M}}_{0,n}(\mathbb{P}^r, d)$$

the forgetful morphism that forgets the last d marked points and stabilizes the resulting family of prestable maps. This is $\mathfrak{S}_d$-invariant. Let

$$\rho : \overline{\text{M}}_{0,n+d}(\mathbb{P}^r, d) \to \overline{\text{M}}_{0,n+d}$$

denote the morphism that stabilizes the universal family of marked prestable curves over $\overline{\text{M}}_{0,n+d}(\mathbb{P}^r, d)$. This is $\mathfrak{S}_d$-equivariant.

Divisors in $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$	α_1^*	α_2^*	β_i^*	γ^*
$\mathcal{T}$	0	0	0	$\mathcal{O}_{\mathbb{P}^1}(2)$
$\mathcal{H}$	0	$\mathcal{O}_{\mathbb{P}^{r-1}}(d)$	0	0
$\mathcal{L}_i$	0	0	$\mathcal{O}_{\mathbb{P}^1}(1)$	0
$\mathcal{L}_{j\neq i}$	0	0	0	0
$\Delta_{(\emptyset,1),(\underline{n},d-1)}$	c	$\mathcal{O}_{\mathbb{P}^{r-1}}(-d)$	$\mathcal{O}_{\mathbb{P}^1}(-1)$	$\mathcal{O}_{\mathbb{P}^1}(4)$
$\Delta_{(\emptyset,2),(\underline{n},d-2)}$	$\tilde{\Delta}_{(\emptyset,2),(\underline{n},d-2)}$	0	0	$\mathcal{O}_{\mathbb{P}^1}(-1)$
$\Delta_{(\{i\},1),(\{i\}^c,d-1)}$	$\tilde{\Delta}_{(\{i\},1),(\{i\}^c,d-1)}$	0	$\mathcal{O}_{\mathbb{P}^1}(-1)$	0
$\Delta_{(A,d_A),(B,d_B)}$ all others	$\tilde{\Delta}_{(A,d_A),(B,d_B)}$	0	0	0

FIGURE 8. The pull-backs of the standard generators

Denote by $q : \overline{\mathrm{M}}_{0,n+d} \to \overline{\mathrm{M}}_{0,n+d}/\mathfrak{S}_d$ the geometric quotient. The composition

$$q \circ \rho : \overline{\mathrm{M}}_{0,n+d}(\mathbb{P}^r, d)_\Pi \to \overline{\mathrm{M}}_{0,n+d}/\mathfrak{S}_d$$

is $\mathfrak{S}_d$-equivariant. Over the open set O_Π in $\mathrm{M}_{0,n}(\mathbb{P}^r, d)$ of maps f for which $f^{-1}(\Pi)$ is d distinct points of the domain, $\mathrm{M}_{0,n+d}(\mathbb{P}^r, d)_\Pi$ is an $\mathfrak{S}_d$-torsor. Hence there is an induced morphism $\phi'_\Pi : O_\Pi \to \overline{\mathrm{M}}_{0,n+d}/\mathfrak{S}_d$ such that $\phi' \circ \pi = q \circ \rho$.

Definition 8.3. Define U_Π to be the maximal open subset of $\overline{\mathrm{M}}_{0,n}(\mathbb{P}^r, d)$ over which ϕ'_Π extends to a morphism, denoted

$$\phi_\Pi : U_\Pi \to \overline{\mathrm{M}}_{0,n+d}/\mathfrak{S}_d.$$

Define I_Π to be the normalization of the closure in $\overline{\mathrm{M}}_{0,n}(\mathbb{P}^r, d) \times \overline{\mathrm{M}}_{0,n+d}/\mathfrak{S}_d$ of the image of the graph of ϕ'_Π, i.e., I_Π is the normalization of the image of $(\pi, q \circ \rho)$. Define $\widetilde{I}_\Pi$ to be the normalization of the image of (π, ρ) in $\overline{\mathrm{M}}_{0,n}(\mathbb{P}^r, d) \times \overline{\mathrm{M}}_{0,n+d}$. Finally, define $\widetilde{U}_\Pi$ to be the inverse image of U_Π in $\widetilde{I}_\Pi$.

There is a pull-back map of $\mathfrak{S}_d$-invariant invertible sheaves,

$$\rho^* : \mathrm{Pic}(\overline{\mathrm{M}}_{0,n+d})^{\mathfrak{S}_d} \to \mathrm{Pic}(\widetilde{I}_\Pi)^{\mathfrak{S}_d},$$

which further restricts to $\mathrm{Pic}(\widetilde{U}_\Pi)^{\mathfrak{S}_d}$. The pull-back map $\mathrm{Pic}(U_\Pi) \to \mathrm{Pic}(\widetilde{U}_\Pi)^{\mathfrak{S}_d}$ is an isomorphism after tensoring with $\mathbb{Q}$; in fact, both the kernel and cokernel are annihilated by $d!$. Because $\overline{\mathrm{M}}_{0,n+d}/\mathfrak{S}_d$ is a proper scheme and because $\overline{\mathrm{M}}_{0,n}(\mathbb{P}^r, d)$ is separated and normal, by the valuative criterion of properness the complement of U_Π has codimension ≥ 2. Hence, the restriction map $\mathrm{Pic}(\overline{\mathrm{M}}_{0,n}(\mathbb{P}^r,d)) \to \mathrm{Pic}(U_\Pi)$ is an isomorphism.

Definition 8.4. Define $v : \mathrm{Pic}(\overline{\mathrm{M}}_{0,n+d})^{\mathfrak{S}_d} \to \mathrm{Pic}(\overline{\mathrm{M}}_{0,n}(\mathbb{P}^r,d)) \otimes \mathbb{Q}$ to be the unique homomorphism commuting with ρ^* via the isomorphisms above.

The map v is independent of the choice of Π, hence it sends NEF divisors to NEF divisors.

Lemma 8.5. *For every base-point-free invertible sheaf $\mathcal{L}$ in $Pic(\overline{M}_{0,n+d})^{\mathfrak{S}_d}$, $v(\mathcal{L})$ is base-point-free. In particular, for every ample invertible sheaf $\mathcal{L}$, $v(\mathcal{L})$ is NEF. Thus, by Kleiman's criterion, for every NEF invertible sheaf $\mathcal{L}$, $v(\mathcal{L})$ is NEF.*

Proof. For every $[(C,(p_1,\dots,p_n),f)]$ in $\overline{\mathrm{M}}_{0,n}(\mathbb{P}^r,d)$, there exists a hyperplane Π satisfying the conditions above and such that $f^{-1}(\Pi)$ is a reduced Cartier divisor containing none of $p_1,\dots,p_n$. $(C,(p_1,\dots,p_n),f)$ is contained in U_Π. Since $\mathcal{L}$ is base-point-free, there exists a divisor D in the linear system $|\mathcal{L}|$ not containing $\phi_\Pi[(C,(p_1,\dots,p_n),f)]$. By the proof of [Ha, Prop. 6.5(c)], the closure of $\phi_\Pi^{-1}(D)$ is in the linear system $|v(\mathcal{L})|$; and it does not contain $[(C,(p_1,\dots,p_n),f)]$. □

Lemma 8.6. (i) *The images of α, β_i and γ are contained in U_Π.*
(ii) *The morphisms $\phi_\Pi \circ \beta_i$ and $\phi_\Pi \circ \gamma$ are constant morphisms. Therefore $\beta_i^* \circ v$ and $\gamma^* \circ v$ are the zero homomorphism.*
(iii) *The composition of α with ϕ_Π equals $q \circ pr_{\overline{M}_{0,n+d}}$. Therefore*

$$\alpha^* \circ v : Pic(\overline{M}_{0,n+d})^{\mathfrak{S}_d} \to Pic(\overline{M}_{0,n+d})^{\mathfrak{S}_d} \times Pic(\mathbb{P}^{r-1})$$

is the homomorphism whose projection on the first factor is the identity, and whose projection on the second factor is 0.

Proof. **(i):** The image of α is contained in O_Π. Denote by q the intersection point of L and Π.

The image $\beta_i(L-\{q\})$ is contained in O_Π. The stable map $\beta_i(q)$ sends the i-th marked point into Π. Up to labeling the d points of the inverse image of Π, there is only one $(n+d)$-pointed stable map in $\overline{\mathrm{M}}_{0,n+d}(\mathbb{P}^r,d)_\Pi$ that stabilizes to this stable map. It is obtained from $\beta_i(q)$ by removing the i-th marked point from L, attaching a contracted component C' to L at q, containing the i-th marked point and exactly one of the last d marked points, and labeling the $d-1$ points in $C \cap \Pi$ with the remaining $d-1$ marked points.

Similarly, $\gamma(L-\{q\})$ is contained in O_Π. The stable map $\gamma(q)$ has two copies of L attached to each other at q. This appears to be a problem, because the inverse image of $\gamma(q)$ in $\overline{\mathrm{M}}_{0,n+d}(\mathbb{P}^r,d)_\Pi$ is 1-dimensional, isomorphic to $\overline{\mathrm{M}}_{0,4}$. The stable maps have a contracted component C' such that both copies of L are attached to C' and 2 of the d new marked points are attached to C'. The remaining $d-2$ marked points are the points of $C \cap \Pi$. However, the map ρ that stabilizes the resulting prestable $(n+d)$-marked curve is constant on this $\overline{\mathrm{M}}_{0,4}$. Indeed, the first copy of L has no marked points and is attached to C' at one point. So the first step in stabilization will prune L reducing the number of special points on C' from 4 to 3.

(ii): In the family defining β_i, only the i-th marked point on L varies. After adding the d new marked points, L is a 3-pointed prestable curve; marked by the node p, the i-th marked point, and the point q. For every base the only family of genus 0, 3-pointed, stable curves is the constant family. So upon stabilization, this family of genus 0, 3-pointed, stable curves becomes the constant family.

In the family defining γ, only the attachment point of the two copies of L varies. The first copy of L gives a family of 2-pointed, prestable curves; marked by q and the attachment point of the two copies of L. This is unstable. Upon stabilization, the first copy of L is pruned and the marked point q on the first copy is replaced by a marked point on the second copy at the original attachment point. Now the second copy of L gives a family of 3-pointed, prestable curves; marked by the attachment point p of the second and third irreducible components, the attachment point of the first and second components, and q. For the same reason as in the last paragraph, this becomes a constant family.

(iii): Each stable map in $\alpha(\overline{\mathrm{M}}_{0,n+d} \times \mathbb{P}^{r-1})$ is obtained from a genus 0, $(n+d)$-pointed, stable curve $(C_0, (p_1, \ldots, p_n, q_1, \ldots, q_d))$ and a line L in $\mathbb{P}^r$ containing p by attaching for each $1 \leq i \leq n$, a copy C_i of L to C_0 where p in C_i is identified with q_i in C_0. The map to $\mathbb{P}^r$ contracts C_0 to p, and sends each curve C to L via the identity morphism. Denoting by r the intersection point of L and Π, the inverse image of Π consists of the d points $r_1, \ldots, r_d$, where r_i is the copy of r in C_i.

The component C_i is a 2-pointed, prestable curve: marked by the attachment point p of C_i and by r_i. This is unstable. So, upon stabilization, C_i is pruned and the marked point r_i is replaced by a marking on C_0 at the point of attachment of C_0 and C_i, namely q_i. Therefore, up to relabeling of the last d marked points, the result is the genus 0, $(n+d)$-pointed, stable curve we started with, $(C_0, (p_1, \ldots, p_n, q_1, \ldots, q_d))$. □

Lemma 8.7. (i) *The Cartier divisors $\mathcal{T}_\Pi$, $\mathcal{L}_{i,\Pi}$ and $\mathcal{H}_\Lambda$ are NEF.*

(ii) *The pull-backs $\alpha^*(\mathcal{T}_\Pi)$ and $\alpha^*(\mathcal{L}_{i,\Pi})$ are zero. On the other hand, the pull-back $\alpha^*(\mathcal{H}_\Lambda)$ equals $(0, \mathcal{O}_{\mathbb{P}^{r-1}}(d))$ in $Pic(\overline{M}_{0,n+d})^{\mathfrak{S}_d} \times Pic(\mathbb{P}^{r-1})$; if $r = 1$, then $\mathcal{O}_{\mathbb{P}^{r-1}}(1)$ is the trivial invertible sheaf.*

(iii) *Assume $n \geq 1$ so that β_i is defined for $1 \leq i \leq n$. The pull-backs $\beta_i^*(\mathcal{T}_\Pi)$ and $\beta_i^*(\mathcal{H}_\Pi)$ are zero. For $1 \leq j \leq n$ different from i, $\beta_i^*(\mathcal{L}_{j,\Pi})$ is zero. Finally, $\beta_i^*(\mathcal{L}_{i,\Pi})$ is $\mathcal{O}_{\mathbb{P}^1}(1)$.*

(iv) *Assume $d \geq 2$ so that γ is defined. The pull-backs $\gamma^*(\mathcal{H}_\Lambda)$ and $\gamma^*(\mathcal{L}_{i,\Pi})$ are zero, and $\gamma^*(\mathcal{T}_\Pi)$ is $\mathcal{O}_{\mathbb{P}^1}(2)$ in $Pic(\mathbb{P}^1)$.*

Proof. **(i):** By an argument similar to the one in Lemma 8.5, these divisors are base-point-free (whenever they are non-empty). The divisor $\mathcal{H}_\Lambda$ is big if $r \geq 2$, and $\mathcal{T}_\Pi$ is big if $d \geq 2$. The divisors $\mathcal{L}_i$ are not big.

(ii): By the proof of Lemma 8.6, the image of α is in O_Π, which is disjoint from $\mathcal{T}_\Pi$. Also, $\mathrm{ev}_i \circ \alpha$ is the constant morphism with image p, so the inverse image of $\mathcal{L}_i$ is empty. Finally, the pull-back of $\mathcal{H}_\Pi$ equals the pull-back under the diagonal Δ of the Cartier divisor $\sum_{j=1}^{d} \mathrm{pr}_j^{-1}(\Lambda)$ in $(\mathbb{P}^{r-1})$, where Λ is considered as a divisor in $\mathbb{P}^{r-1}$ via projection from p.

(iii): Since the image of β_i is disjoint from $\mathcal{H}_\Pi$, $\mathcal{T}_\Pi$ and $\mathcal{L}_{j,\Pi}$ for $j \neq i$, the corresponding pull-backs are zero. The map $ev_i \circ \beta_i : \mathbb{P}^1 \to \mathbb{P}^r$ embeds $\mathbb{P}^1$ as the line L in $\mathbb{P}^r$, hence $\beta_i^*(\mathcal{L}_{i,\Pi}) = \mathcal{O}_{\mathbb{P}^1}(1)$.

(iv): Since neither the image curve nor the marked points vary under γ, clearly $\gamma^*\mathcal{H}_\Lambda$ and $\gamma^*\mathcal{L}_{i,\Pi}$ are zero. To compute $\gamma^*\mathcal{T}_\Pi$, use [Pa1, Lemma 2.3.1]. □

Proposition 8.8. *The $\mathbb{Q}$-vector space $Pic(\overline{M}_{0,n}(\mathbb{P}^r, d)) \otimes \mathbb{Q}$ is generated by $\mathcal{T}_\Pi$, $\mathcal{H}_\Lambda$, $\mathcal{L}_{i,\Pi}$ for $1 \leq i \leq n$, and the image of v.*

Proof. From Lemmas 8.7 and 8.6 and by pairing with one-parameter families, we see that

$$v(\tilde{\Delta}_{(A,d_A),(B,d_B)}) = \Delta_{(A,d_A),(B,d_B)}$$

unless $(\#A, d_A)$ or $(\#B, d_B)$ equals one of $(0, 2)$ or $(1, 1)$.

$$v(\tilde{\Delta}_{(A,d_A),(B,d_B)}) = \frac{1}{2}\, \mathcal{T} + \Delta_{(A,d_A),(B,d_B)}$$

if $(\#A, d_A)$ or $(\#B, d_B)$ equals $(0, 2)$. Finally,

$$v(\tilde{\Delta}_{(\{i\},1),(\{i\}^c,d-1)}) = \Delta_{(\{i\},1),(\{i\}^c,d-1)} + \mathcal{L}_{i,\Pi}.$$

Consequently, it follows that the classes of the divisors $\mathcal{H}, \mathcal{T}$, $\mathcal{L}_{i,\Pi}$ and the image of v generate the classes of all the boundary divisors in the Kontsevich moduli space. Hence, they generate $\mathrm{Pic}(\overline{\mathrm{M}}_{0,n}(\mathbb{P}^r, d)) \otimes \mathbb{Q}$. □

Proof of Theorem 8.1. Now we can complete the proof of Theorem 8.1. Denote by

$$\widetilde{v} : P_{r,n,d} \otimes \mathbb{Q} \to \mathrm{Pic}(\overline{\mathrm{M}}_{0,n}(\mathbb{P}^r, d)) \otimes \mathbb{Q}$$

the unique homomorphism whose restriction to $\mathrm{Pic}(\overline{\mathrm{M}}_{0,n+d})^{\mathfrak{S}_d}$ is v (see Definition 8.4), whose restriction to $\mathrm{Pic}(\mathbb{P}^{r-1})$ sends $\mathcal{O}_{\mathbb{P}^{r-1}}(1)$ to $[\mathcal{H}_\Lambda]$, whose restriction to the i-th factor of $\mathrm{Pic}(\mathbb{P}^1)^n$ sends $\mathcal{O}_{\mathbb{P}^1}(1)$ to $[\mathcal{L}_i]$ if $n \geq 1$, and whose restriction to the last factor $\mathrm{Pic}(\mathbb{P}^1)$ (assuming $d \geq 2$) sends $\mathcal{O}_{\mathbb{P}^1}(1)$ to $1/2\ [\mathcal{T}_\Pi]$. By Lemma 8.6 (ii), (iii) and by Lemma 8.7, $u \otimes \mathbb{Q} \circ \widetilde{v}$ is the identity map. In particular, $\widetilde{v}$ is injective. By Proposition 8.8, $\widetilde{v}$ is surjective. Thus $\widetilde{v}$ and $u \otimes \mathbb{Q}$ are isomorphisms.

Because α, β_i and γ are morphisms, for every NEF, resp. eventually free, divisor D in $\mathrm{Pic}(\overline{\mathrm{M}}_{0,n}(\mathbb{P}^r, d)) \otimes \mathbb{Q}$, $\alpha^*(D)$, $\beta_i^*(D)$, and $\gamma^*(D)$ are NEF, resp. eventually free. Denote,

$$D_1 = \alpha_1^*(D), \quad a\ [\mathcal{O}_{\mathbb{P}^{r-1}}(1)] = \alpha_2^*(D), \quad b_i\ [\mathcal{O}_{\mathbb{P}^1}(1)] = \beta_i^*(D), \quad c\ [\mathcal{O}_{\mathbb{P}^1}(1)] = \gamma^*(D),$$

where by convention a is defined to be 0 if $r = 1$ and c is defined to be 0 if $d = 1$. If D is NEF, resp. eventually free, D_1 is NEF, resp. eventually free, in $\mathrm{Pic}(\overline{\mathrm{M}}_{0,n+d})^{\mathfrak{S}_d}$, and $a, b_i, c \geq 0$.

Conversely, by Lemma 8.5, for every NEF, respectively, eventually free, divisor D_1 in $\mathrm{Pic}(\overline{\mathrm{M}}_{0,n+d})^{\mathfrak{S}_d}$, $v(D_1)$ is NEF, resp. eventually free. By Lemma 8.7(i), for $a, b_i, c \geq 0$, $a[\mathcal{H}_\Lambda]$, $b_i[\mathcal{L}_{i,\Pi}]$ and $c/2\ [\mathcal{T}_\Pi]$ are NEF and eventually free. Since a sum of NEF, resp. eventually free, divisors is NEF, resp. eventually free, $D = v(D_1) + a\ [\mathcal{H}_\Lambda] + b_i\ [\mathcal{L}_i] + c/2\ [\mathcal{T}_\Pi]$ is NEF, resp. eventually free. Therefore D is NEF if and only if $u \otimes \mathbb{Q}(D)$ is in the product of the NEF cones of the factors. This argument needs to be modified in the obvious way when $(n, d) = (0, 3)$ and $(1, 2)$ to account for the slight variations in the formulae.

Because the interior of a product of cones equals the product of the interiors of the cones, by Kleiman's criterion, D is ample iff $u \otimes \mathbb{Q}(D)$ is contained in the product of the ample cones of the factors. $\square$

Theorem 8.1 has the following important corollary.

Theorem 8.9. *For every integer $r \geq 1$ and $d \geq 2$, there is a contraction,*

$$cont \colon \overline{M}_{0,0}(\mathbb{P}^r, d) \to Y,$$

restricting to an open immersion on the interior $M_{0,0}(\mathbb{P}^r, d)$ and whose restriction to the boundary divisor $\Delta_{k,d-k} \cong M_{0,1}(\mathbb{P}^r, k) \times_{\mathbb{P}^r} M_{0,1}(\mathbb{P}^r, d-k)$ factors through the projection to $\overline{M}_{0,1}(\mathbb{P}^r, d-k)$ for each $1 \leq k \leq \lfloor d/2 \rfloor$. The following divisor is the pullback of an ample divisor on Y,

$$D_{r,d} = \mathcal{T} + \sum_{k=2}^{\lfloor d/2 \rfloor} k(k-1)\Delta_{k,d-k}.$$

In fact, Y can be interpreted as a moduli space of $\lfloor d/2 \rfloor$-pointed stable maps constructed by [MM] (see also [Par]) and it is possible to prove a more precise statement.

Definition 8.10. Let k be an integer $0 \leq k \leq d$. Fix a rational number $0 < \epsilon < 1$. The *moduli space of k-stable maps* $\overline{\mathrm{M}}_{0,0}(\mathbb{P}^r, d, k)$ is the coarse moduli scheme associated to the functor $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d, k)$ which associates to a scheme S the set of isomorphism classes of the data

$$(\pi : C \to S, \mu : C \dashrightarrow \mathbb{P}^r, L, e)$$

such that

(1) $\pi : C \to S$ is a flat family of connected, projective, at-worst-nodal curves of arithmetic genus zero.
(2) L is a line bundle on C that has degree d on each fiber of π and $e : \mathcal{O}_C^{r+1} \to L$ is a morphism that determines the rational map $\mu : C \dashrightarrow \mathbb{P}^r$.
(3) $\omega_{C/S}^{d-k+\epsilon} \otimes L$ is relatively ample over S.
(4) Let E be the cokernel of e. Then the restriction of E to each fiber of π is a skyscraper sheaf and $\dim(E_p) \leq d-k$ for every $p \in C_s$. Furthermore, if $0 < \dim(E_p)$, then $p \in C_s$ is a smooth point of C_s.

When $k = d$, $\overline{\mathrm{M}}_{0,0}(\mathbb{P}^r, d, d) = \overline{\mathrm{M}}_{0,0}(\mathbb{P}^r, d)$. Furthermore, there are natural contraction morphisms $\overline{\mathrm{M}}_{0,0}(\mathbb{P}^r, d, k+1) \to \overline{\mathrm{M}}_{0,0}(\mathbb{P}^r, d, k)$. Hence, the contraction in Theorem 8.9 can be factored into a sequence of explicit contractions, where the intermediate spaces have nice modular interpretations.

Exercise 8.11. Show that for $\overline{\mathrm{M}}_{0,0}(\mathbb{P}^2, 2)$ the contraction is the natural projection map to the $\mathbb{P}^5$ of dual conics in $(\mathbb{P}^2)^*$.

Exercise 8.12. Generalize Theorem 8.1 to the case when the target is a Grassmannian. More generally, extend the theorem to the case when the target is a partial flag variety.

Exercise 8.13. Describe the ample cone of $\overline{\mathrm{M}}_{0,0}(\mathbb{P}^r, d)$ for $2 \leq d \leq 5$ by explicitly describing the extremal rays that span the cone.

Exercise 8.14. Describe the ample cone of $\overline{\mathrm{M}}_{0,0}(G(k,n),d)$ for $2 \leq d \leq 5$ by explicitly describing the extremal rays that span the cone.

Exercise 8.15. Formulate an analogue of the F-conjecture for the Kontsevich moduli space $\overline{\mathrm{M}}_{0,n}(\mathbb{P}^r,d)$. Show that the conjecture for the Kontsevich moduli space is equivalent to the conjecture for $\overline{\mathrm{M}}_{0,n+d}/\mathfrak{S}_d$.

9. The stable base locus decomposition of the Kontsevich moduli space and other birational models

The minimal model program for a moduli space $\overline{\mathrm{M}}$ consists of the following steps.

(1) Determine the cones of ample and effective divisors on $\overline{\mathrm{M}}$.
(2) Decompose the effective cone of $\overline{\mathrm{M}}$ according to the stable base loci of the divisors.
(3) For an effective divisor D in each chamber of the stable base locus decomposition, describe
$$\overline{\mathrm{M}}(D) = \mathrm{Proj}\left(\bigoplus_{m\geq 0} H^0(\overline{\mathrm{M}}, mD)\right),$$
assuming that the section ring is finitely generated.
(4) Describe explicitly a sequence of divisorial contractions and flips that transform $\overline{\mathrm{M}}$ to $\overline{\mathrm{M}}(D)$. Use this description, to find a modular interpretation of $\overline{\mathrm{M}}(D)$ (if possible).

We have so far concentrated on the first step of the program. In this section, we will run the entire program on a few concrete examples of Kontsevich moduli spaces. In the next section, we will discuss the program for the moduli space of curves.

The following Lemma is often useful in identifying birational models (see [L1], §2.1.B).

Lemma 9.1. *Let X be a normal, projective variety. Let L be a semi-ample line bundle on X.*

(1) *Then the section ring $\oplus_{m\geq 0} H^0(X, L^{\otimes m})$ of L is finitely generated. Let*
$$M(L) = \mathit{Proj}(\oplus_{m\geq 0} H^0(X, L^{\otimes m}))$$
and let $f : X \to M(L)$. Then $f_*\mathcal{O}_X \cong \mathcal{O}_{M(L)}$.
(2) *If there exists a morphism $g : X \to Y$ such that every curve contracted by f is also contracted by g, then g factors through f. In particular, if L' is an ample line bundle on Y, then g factors through $M(g^*L')$ and $M(g^*L')$ is the normalization of Y.*

9.1. **Degree two maps to projective space.**

As a warm-up example, we describe the Mori theory of $\overline{\mathrm{M}}_{0,0}(\mathbb{P}^2,2)$. The Neron-Severi space of $\overline{\mathrm{M}}_{0,0}(\mathbb{P}^2,2)$ is generated by the divisors $\mathcal{H}$ and $\Delta = \Delta_{1,1}$. We have already seen the following:

- The effective cone is the cone bounded by Δ and $D_{deg} = \frac{1}{4}(3\mathcal{H} - \Delta)$.
- The NEF cone is the closed cone bounded by $\mathcal{H}$ and $\mathcal{T} = \frac{1}{4}(3\mathcal{H} + \Delta)$.

The effective cone decomposes into three chambers.

- If D is in the chamber $[D_{deg}, \mathcal{H})$, then the stable base locus of D is divisorial consisting of D_{deg}.
- If D is in the chamber $[\mathcal{H}, \mathcal{T}]$, then the stable base locus of D is empty.
- Finally, if D is in the chamber $(\mathcal{T}, \Delta]$, then the stable base locus of D is divisorial consisting of the boundary divisor Δ.

Exercise 9.2. Verify the stable base locus decomposition of the effective cone of $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^2, 2)$ by carrying out the following steps.

(1) Let C_1 be the curve induced in $\overline{\mathrm{M}}_{0,0}(\mathbb{P}^2, 2)$ by a general pencil of double covers of a fixed line $L \in \mathbb{P}^2$. Show that $C_1 \cdot \mathcal{H} = 0$ and $C_1 \cdot \Delta > 0$. Conclude that $C_1 \cdot D < 0$ if $D \in [D_{deg}, \mathcal{H})$. Show that curves with class C_1 cover D_{deg}. Conclude that D_{deg} must be in the base locus of any divisor with class $a\mathcal{H} + b\Delta$ with $b < 0$.
(2) Conversely, show that since $D \in [D_{deg}, \mathcal{H})$ is a non-negative linear combination of D_{deg} and $\mathcal{H}$ and $\mathcal{H}$ is base-point-free, the stable base locus of D is contained in D_{deg}.
(3) Let C_2 be the curve in $\overline{\mathrm{M}}_{0,0}(\mathbb{P}^2, 2)$ obtained by attaching a line $M \subset \mathbb{P}^2$ to a pencil of lines in $\mathbb{P}^2$ at their base-point. Show that $C_2 \cdot \mathcal{H} = 1$ and $C_2 \cdot \Delta = -1$. Conclude that $C_2 \cdot D < 0$ for $D \in (\mathcal{T}, \Delta)$. Show that curves with class C_2 cover Δ. Conclude that Δ must be in the stable base locus of D for $D \in (\mathcal{T}, \Delta)$.
(4) Conversely, since $\mathcal{T}$ is base-point-free, show that for $D \in (\mathcal{T}, \Delta)$, the stable base locus must be contained in Δ.
(5) Finally, show that since both $\mathcal{H}$ and $\mathcal{T}$ are base-point-free, the stable base locus of any divisor $D \in [\mathcal{H}, \mathcal{T}]$ is empty.

For an effective divisor D, let $\overline{\mathrm{M}}_{0,0}(r, d, D)$ denote the model

$$\mathrm{Proj}\left(\bigoplus_{m \geq 0} H^0(\overline{\mathrm{M}}_{0,0}(\mathbb{P}^r, d), mD)\right).$$

Proposition 9.3. *The Kontsevich moduli space $\overline{M}_{0,0}(\mathbb{P}^2, 2)$ admits two divisorial contractions.*

(1) *The model $\overline{M}_{0,0}(2, 2, \mathcal{H})$ is the Hilbert scheme of conics in $\mathbb{P}^2$ and is isomorphic to $\mathbb{P}^5$.*
(2) *The model $\overline{M}_{0,0}(2, 2, \mathcal{T})$ is the space of dual conics in $(\mathbb{P}^2)^*$ and is also isomorphic to $\mathbb{P}^5$.*

Exercise 9.4. Interpret $\overline{\mathrm{M}}_{0,0}(\mathbb{P}^2, 2)$ as the space of complete conics in $\mathbb{P}^5 \times (\mathbb{P}^5)^*$. Show that the two models correspond to the projections to the two factors. Using Lemma 9.1, verify the proposition.

Exercise 9.5. Show that the stable base locus decomposition of $\overline{\mathrm{M}}_{0,0}(\mathbb{P}^r, 2)$ for $r > 2$ also has three chambers with the same descriptions. Show that in the chamber $[D_{deg}, \mathcal{H})$, the stable base locus consists of maps that fail to be birational onto their image. Show that when $r > 2$, this locus is not divisorial. If you are up for a challenge, give modular interpretations to the corresponding models.

9.2. **Degree three maps to projective space.** The Mori theory of $\overline{\mathrm{M}}_{0,0}(\mathbb{P}^3, 3)$ was studied in detail in [Ch1]. We begin by describing the stable base locus decomposition of $\overline{\mathrm{M}}_{0,0}(\mathbb{P}^3, 3)$. Since $\overline{\mathrm{M}}_{0,0}(\mathbb{P}^3, 3)$ has only one boundary divisor, the Neron-Severi space has dimension two. For simplicity, we set $\Delta = \Delta_{1,2}$.

We need to introduce a new effective divisor. This divisor is easiest to introduce in $\overline{\mathrm{M}}_{0,0}(\mathbb{P}^2, 3)$. By Proposition 3.3, it will follow that there is an effective divisor with the same class in $\overline{\mathrm{M}}_{0,0}(\mathbb{P}^3, 3)$. Fix a line $L \subset \mathbb{P}^2$. Let NL denote the locus of maps f that fail to be an isomorphism onto their image over a point of L. The image of a general map in $\overline{\mathrm{M}}_{0,0}(\mathbb{P}^2, 3)$ is a nodal cubic. One can think of NL as the closure of the locus of maps where the node of the cubic lies on the fixed line L.

Exercise 9.6. Show that NL is a divisor with class

$$[NL] = \frac{5}{3}(\mathcal{H} - \frac{1}{5}\Delta).$$

(Hint: Take a general pencil of cubic curves that have a node at a fixed point. Show that the induced curve C_1 in $\overline{\mathrm{M}}_{0,0}(\mathbb{P}^3, 3)$ is disjoint from NL. Show that $C_1 \cdot \mathcal{H} = 1$ and $C_1 \cdot \Delta = 5$. This determines the class up to a constant. Determine the constant by using the curve C_2 in $\overline{\mathrm{M}}_{0,0}(\mathbb{P}^2, 3)$ consisting of a fixed conic attached to a pencil of lines at the base point.)

Exercise 9.7. Using Proposition 3.3, describe the divisor NL directly on $\overline{\mathrm{M}}_{0,0}(\mathbb{P}^3, 3)$. (Hint: Fix a plane $\Lambda \subset \mathbb{P}^3$ and a point $p \in \Lambda$. Show that NL is the class of the divisor of maps such that $f(C) \cap \Lambda$ has two points collinear with p.)

We are now ready to describe the stable base locus decomposition of the effective cone. Figure 9 depicts this decomposition.

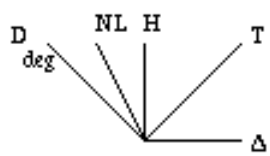

FIGURE 9. The stable base locus decomposition of $\overline{\mathrm{M}}_{0,0}(\mathbb{P}^3, 3)$.

Theorem 9.8. *The stable base locus decomposition of the effective cone of $\overline{M}_{0,0}(\mathbb{P}^3, 3)$ has four chambers.*

1. *The NEF cone is equal to the base-point-free cone and is the chamber bounded by $\mathcal{H}$ and $\mathcal{T}$.*
2. *For a divisor D in the chamber $(\mathcal{T}, \Delta]$, the stable base locus is the boundary divisor Δ.*
3. *For a divisor D in the chamber $[D_{deg}, NL)$, the stable base locus is the divisor D_{deg}.*
4. *For a divisor D in the chamber $[NL, \mathcal{H})$, the stable base locus consists of maps that are not birational onto their image.*

Exercise 9.9. Show that the locus of maps that are not birational onto their image has two components M_3 and $M_{2,1}$. The general point of M_3 consists of a degree three map onto a line. Show that M_3 has dimension 8. A general point of $M_{2,1}$ is a map from a reducible curve onto a pair of intersecting lines where the map has degree two on one component and is an isomorphism on the other component. Show that $M_{2,1}$ has dimension 9.

Exercise 9.10. Verify Theorem 9.8 by carrying out the following steps.

(1) Note that the curve C_1 introduced in Exercise 9.6 is a moving curve class on the divisor D_{deg}. Using the fact that $C_1 \cdot D < 0$ for $D \in [D_{deg}, NL)$, conclude that D_{deg} must be in the stable base locus of the divisors in this chamber. Conversely, expressing any divisor in this chamber as a non-negative linear combination of D_{deg} and $\mathcal{H}$ and using the fact that $\mathcal{H}$ is base-point-free, conclude that the stable base locus of $D \in [D_{deg}, NL)$ is equal to D_{deg}.

(2) Let C_3 be the curve obtained by attaching a general conic to the base-point of a pencil of lines (note that C_3 has the same class as the curve C_2 introduced in Exercise 9.6). Show that $C_3 \cdot D < 0$ for $D \in (\mathcal{T}, \Delta]$. Since C_3 is a moving curve in Δ, conclude that Δ is in the stable base locus of D in this chamber. Conversely, expressing D as a non-negative linear combination of Δ and $\mathcal{T}$ and using the fact that $\mathcal{T}$ is base-point-free, conclude that the stable base locus of D is equal to the boundary divisor Δ.

(3) Show that a map which is birational onto its image is not in the base locus of NL. Conclude that for a divisor $D \in [NL, \mathcal{H})$, the stable base locus must be contained in the union $M_3 \cup M_{2,1}$. Conversely, by fixing the image of the map and varying the multiple cover, obtain moving curves in M_3 and $M_{2,1}$ to show that their union must be contained in the stable base locus of $D \in [NL, \mathcal{H})$.

Before we state the next theorem, we need to introduce some notation. Let $H_{3,0}$ denote the component of the Hilbert scheme parameterizing twisted cubic curves in $\mathbb{P}^3$. It is well-known that $H_{3,0}$ is a smooth, projective, twelve dimensional variety [PS]. The space of twisted cubic curves has another natural compactification in $G(3,10)$. The homogeneous ideal of a twisted cubic is generated by three quadratic equations. Let $H(2) \in G(3,10)$ be the closure of the nets of quadrics that are the equations of a smooth twisted cubic curve. It is also known that $H(2)$ is smooth. Since $\dim(H^0(I(2)) = 3$ for every scheme in $H_{3,0}$, there is a well-defined morphism $h : H_{3,0} \to H(2)$. The morphism h realizes $H_{3,0}$ as the blow-up of $H(2)$ along the flag variety $F(1,3;4) \subset H(2)$ [EPS].

The following theorem summarizes the birational models that correspond to the chambers in the stable base locus decomposition.

Theorem 9.11. *Corresponding to the stable base locus decomposition in Theorem 9.8, there are five birational models of* $\overline{M}_{0,0}(\mathbb{P}^3, 3)$.

(1) $\overline{M}_{0,0}(3,3,D)$ *is isomorphic to the Kontsevich moduli space for* $D \in (\mathcal{H}, \mathcal{T})$.

(2) $\overline{M}_{0,0}(3,3,\mathcal{H})$ *is the normalization* $C_{3,0}$ *of the Chow variety of degree-three genus-zero curves in* $\mathbb{P}^3$. *The morphism* $\phi : \overline{M}_{0,0}(\mathbb{P}^3, 3) \to \overline{M}_{0,0}(3,3,\mathcal{H})$ *is a small contraction with exceptional locus* $M_3 \cup M_{2,1}$.

(3) *For* $D \in (NL, \mathcal{H})$, $\overline{M}_{0,0}(3,3,D)$ *is the flip of the morphism* ϕ *and is isomorphic to the component of the Hilbert scheme* $H_{3,0}$ *parameterizing twisted cubics in* $\mathbb{P}^3$.

(4) $\overline{M}_{0,0}(3,3,NL)$ *is a divisorial contraction of* $H_{3,0}$ *contracting the locus of degenerate curves. The model* $\overline{M}_{0,0}(3,3,NL)$ *is isomorphic to* $H(2)$*, the compactification of twisted cubics by nets of quadrics.*

(5) *Finally, there is a divisorial contraction* $\psi : \overline{M}_{0,0}(\mathbb{P}^3, 3) \to \overline{M}_{0,0}(3,3,\mathcal{T})$. *This coincides with the contraction described in the pervious section.* $\overline{M}_{0,0}(3,3,\mathcal{T})$ *is isomorphic to the moduli space of* 2*-stable maps* $\overline{M}_{0,0}(\mathbb{P}^3, 3, 2)$ *constructed by* [MM].

Exercise 9.12. Show that the Chow variety parameterizing rational curves of degree $d \geq 3$ in $\mathbb{P}^r$ ($r \geq 2$) is not normal. (Hint: The divisor class $\mathcal{H}$ gives a map from $\overline{\mathrm{M}}_{0,0}(\mathbb{P}^r, d)$ to the Chow variety of rational curves. Show that this map is birational, but not all the fibers are connected.)

Exercise 9.13. Let $ch : H_{3,0} \to C_{3,0}$ be the Hilbert-Chow morphism from the component of the Hilbert scheme parameterizing twisted cubics to the normalization of the Chow variety. Check that for $D \in (NL, \mathcal{H})$, $-D$ is ϕ-ample and D is ch-ample. Conclude that $H_{3,0}$ is the flip of the Kontsevich moduli space over the normalization of the Chow variety.

Exercise 9.14. The previous exercise is the most interesting part of the proof of Theorem 9.11. Part (1) of the theorem is clear since the divisors in this chamber are ample. Using Theorem 9.8, verify parts (2), (4) and (5).

Exercise 9.15. Find the stable base locus decomposition of $\overline{\mathrm{M}}_{0,0}(\mathbb{P}^r, 3)$ for $r > 3$ (Hint: The chambers do not change.) If you are looking for a challenge, describe the birational models that correspond to each of the chambers.

Problem 9.16. Calculate the stable base locus decomposition of $\overline{\mathrm{M}}_{0,0}(\mathbb{P}^r, d)$ in general. Note that this is a very hard problem. For instance, it does include the problem of finding the ample cone of $\overline{\mathrm{M}}_{0,n}/\mathfrak{S}_n$.

Note that a corollary of our discussion is that the Kontsevich moduli spaces $\overline{\mathrm{M}}_{0,0}(\mathbb{P}^r, 2)$ and $\overline{\mathrm{M}}_{0,0}(\mathbb{P}^r, 3)$ are Mori dream spaces. Since the two examples we discussed in detail are Fano, this follows by general principles (by [BCHM] a log-Fano variety is a Mori dream space). However, for large r, these Kontsevich moduli spaces are not Fano.

Problem 9.17. For which r, d is $\overline{\mathrm{M}}_{0,0}(\mathbb{P}^r, d)$ log Fano?

Exercise 9.18. Show that the canonical class of $\overline{\mathrm{M}}_{0,0}(\mathbb{P}^r, d)$ is given by

$$-\frac{(r+1)(d+1)}{2d}\mathcal{H} + \sum_{k=1}^{\lfloor d/2 \rfloor} \left(\frac{(r+1)i(d-i)}{2d} - 2 \right) \Delta_{k,d-k}.$$

Determine for which r, d the Kontsevich moduli space is Fano.

Problem 9.19. Show (or disprove) that the Kontsevich moduli spaces $\overline{\mathrm{M}}_{0,0}(\mathbb{P}^r, d)$ are Mori dream spaces.

9.3. **Degree two maps to Grassmannians.** In our examples so far the dimension of the Neron-Severi space has been two. We give one final example where the dimension of the Neron-Severi space is three.

Let $G(k,n)$ denote the Grassmannian of k-dimensional subspaces of an n-dimensional vector space V. Let λ denote a partition with k parts satisfying $n-k \geq \lambda_1 \geq \cdots \geq \lambda_k \geq 0$. Let λ^* denote the partition dual to λ with parts $\lambda_i^* = n-k-\lambda_{k-i+1}$. Let $F_\bullet : F_1 \subset \cdots \subset F_n$ denote a flag in V. The Schubert cycle σ_λ is the Poincaré dual of the class of the Schubert variety Σ_λ defined by

$$\Sigma_\lambda(F_\bullet) = \{[W] \in G(k,n) \mid \dim(W \cap F_{n-k+i-\lambda_i}) \geq i\}.$$

Schubert cycles form a $\mathbb{Z}$-basis for the cohomology of $G(k,n)$.

Let $\overline{\mathrm{M}}_{0,0}(G(k,n),d)$ denote the Kontsevich moduli space of stable maps to $G(k,n)$ of Plücker degree d. As usual, let

$$\pi : \overline{\mathrm{M}}_{0,1}(G(k,n),d) \to \overline{\mathrm{M}}_{0,0}(G(k,n),d)$$

be the forgetful morphism and let

$$e : \overline{\mathrm{M}}_{0,1}(G(k,n),d) \to G(k,n)$$

be the evaluation morphism. Let us summarize the results scattered in several exercises above.

- Let $\mathcal{H}_{\sigma_{1,1}} = \pi_* e^*(\sigma_{1,1})$ and $\mathcal{H}_{\sigma_2} = \pi_* e^*(\sigma_2)$. Geometrically, $\mathcal{H}_{\sigma_{1,1}}$ (resp., $\mathcal{H}_{\sigma_2}$) is the class of the divisor of maps f in $\overline{\mathrm{M}}_{0,0}(G(k,n),d)$ whose image intersects a fixed Schubert cycle $\Sigma_{1,1}$ (resp., Σ_2).
- Let $\mathcal{T}_{\sigma_1}$ denote the class of the divisor of maps that are tangent to a fixed hyperplane section of $G(k,n)$.
- Let D_{deg} denote the class of the divisor $\mathcal{D}_{deg}$ of maps in $\overline{\mathrm{M}}_{0,0}(G(k,k+d),d)$ whose image is contained in a sub-Grassmannian $G(k,k+d-1)$ embedded in $G(k,k+d)$ by an inclusion of the ambient vector spaces. More generally, for $n \geq k+d$, let D_{deg} denote the class of the divisor of maps f in $\overline{\mathrm{M}}_{0,0}(G(k,n),d)$ such that the projection of the span of the linear spaces parameterized by the image of f from a fixed linear space of dimension $n-k-d$ has dimension less than $k+d$.
- If k divides d, then let D_{unb} be the closure $\mathcal{D}_{unb}$ of the locus of maps f with irreducible domains for which the pull-back of the tautological bundle $f^*(S)$ has unbalanced splitting (i.e., $f^*(S) \neq \oplus_{i=1}^k \mathcal{O}_{\mathbb{P}^1}(-d/k)$).
- If k does not divide d, let $d = kq + r$, where r is the smallest non-negative integer that satisfies the equality. The subbundle of the pull-back of the tautological bundle of rank $k-r$ and degree $-q(k-r)$ induces a rational map

$$\phi : \overline{\mathrm{M}}_{0,0}(G(k,k+d),d) \dashrightarrow \overline{\mathrm{M}}_{0,0}(F(k-r,k;k+d),q(k-r),d).$$

The natural projection $\pi_{k-r} : F(k-r,k;k+d) \to G(k-r,k+d)$ from the two-step flag variety to the Grassmannian induces a morphism

$$\psi : \overline{\mathrm{M}}_{0,0}(F(k-r,k,k+d),q(k-r),d) \to \overline{\mathrm{M}}_{0,0}(G(k-r,k+d),q(k-r)).$$

The maps whose linear spans intersect a linear space of codimension $(q+1)(k-r)$ is a divisor D in $\overline{\mathrm{M}}_{0,0}(G(k-r,k+d),q(k-r))$. Let $D_{unb} = \phi^*\psi^*([D])$.

Then in Exercises 2.35, 3.16 and 8.12, we have seen the following:

Theorem 9.20. *Let $\overline{M}_{0,0}(G(k,n),d)$ denote the Kontsevich moduli space of stable maps to $G(k,n)$ of Plücker degree d. Then:*

(1) [Theorem 1, [Opr]] *The Picard group $Pic(\overline{M}_{0,0}(G(k,n),d)) \otimes \mathbb{Q}$ is generated by the divisor classes $\mathcal{H}_{\sigma_{1,1}}$, $\mathcal{H}_{\sigma_2}$, and the classes of the boundary divisors $\Delta_{k,d-k}$, $1 \leq k \leq \lfloor d/2 \rfloor$.*

(2) [Theorem 1.1, [CS]] *There is an explicit, injective linear map*

$$v : Pic(\overline{M}_{0,d}/\mathfrak{S}_d) \otimes \mathbb{Q} \to Pic(\overline{M}_{0,0}(G(k,n),d)) \otimes \mathbb{Q}$$

that maps base-point-free divisors and NEF divisors to base-point-free divisors and NEF divisors, respectively. A divisor class D in $\overline{M}_{0,0}(G(k,n),d)$ is NEF if and only if D can be expressed as a non-negative linear combination of $\mathcal{H}_{\sigma_{1,1}}, \mathcal{H}_{\sigma_2}, T_{\sigma_1}$ and $v(D')$, where D' is a NEF divisor in $\overline{M}_{0,d}/\mathfrak{S}_d$.

(3) [Theorem 1.2, [CS]] *A divisor class D in $\overline{M}_{0,0}(G(k,k+d),d)$ is effective if and only if it can be expressed as a non-negative linear combination of D_{deg}, D_{unb} and the boundary divisors $\Delta_{k,d-k}$, $1 \leq k \leq \lfloor d/2 \rfloor$.*

Remark 9.21. If we identify the Neron-Severi space of $\overline{\mathrm{M}}_{0,0}(G(k,n),d)$ with the vector space spanned by the divisor classes $\mathcal{H}_{\sigma_{1,1}}$, $\mathcal{H}_{\sigma_2}$, and the classes of the boundary divisors $\Delta_{k,d-k}$, $1 \leq k \leq \lfloor d/2 \rfloor$, then the effective cone of $\overline{\mathrm{M}}_{0,0}(G(k,n),d)$ is contained in the effective cone of $\overline{\mathrm{M}}_{0,0}(G(k,n+1),d)$, with equality if $n \geq k+d$. Hence, Part (3) of Theorem 9.20 determines the effective cone of $\overline{\mathrm{M}}_{0,0}(G(k,n),d)$ for every $n \geq k+d$.

Exercise 9.22. Show that the canonical class of $\overline{\mathrm{M}}_{0,0}(G(k,n),d)$

$$K = \left(\frac{n}{2} - k - 1 - \frac{n}{2d}\right)\mathcal{H}_{\sigma_{1,1}} + \left(k - \frac{n}{2} - 1 - \frac{n}{2d}\right)\mathcal{H}_{\sigma_2} + \sum_{i=1}^{\lfloor d/2 \rfloor}\left(\frac{ni(d-i)}{2d} - 2\right)\Delta_{i,d-i}.$$

(Hint: If stuck, see Theorem 1.1 of [dJS2].) Determine for which k, n and d, $\overline{\mathrm{M}}_{0,0}(G(k,n),d)$ is Fano.

Exercise 9.23. Let $2 \leq k \leq n-2$. Let $\Delta = \Delta_{1,1}$. Show that the divisors $\mathcal{T}, D_{deg}$ and D_{unb} on $\overline{\mathrm{M}}_{0,0}(G(k,n),2)$ have the following expressions.

$$\begin{aligned} \mathcal{T} &= \frac{1}{2}\left(\mathcal{H}_{\sigma_{1,1}} + \mathcal{H}_{\sigma_2} + \Delta\right) \\ D_{deg} &= \frac{1}{4}(-\mathcal{H}_{\sigma_{1,1}} + 3\mathcal{H}_{\sigma_2} - \Delta) \\ D_{unb} &= \frac{1}{4}(3\mathcal{H}_{\sigma_{1,1}} - \mathcal{H}_{\sigma_2} - \Delta) \end{aligned}$$

Most questions about the divisor theory of $\overline{\mathrm{M}}_{0,0}(G(k,n),2)$ can be reduced to studying the divisor theory of $\overline{\mathrm{M}}_{0,0}(G(2,4),2)$. Let W be a four-dimensional subspace of V. Let U be a $(k-2)$-dimensional subspace of V such that $U \cap W = 0$. Given a two-dimensional subspace Λ of W, the span of Λ and U is a k-dimensional subspace of V. Hence, there is an inclusion $i : G(2,4) \to G(k,n)$, which induces a morphism

$$\phi : \overline{\mathrm{M}}_{0,0}(G(2,4),2) \to \overline{\mathrm{M}}_{0,0}(G(k,n),2).$$

It is easy to see that

$$\phi^*(\mathcal{H}_{\sigma_{1,1}}) = \mathcal{H}_{\sigma_{1,1}},\ \phi^*(\mathcal{H}_{\sigma_2}) = \mathcal{H}_{\sigma_2},\ \phi^*(\Delta) = \Delta.$$

Under this correspondence, the stable base-locus-decomposition of $\overline{\mathrm{M}}_{0,0}(G(k,n),2)$ and $\overline{\mathrm{M}}_{0,0}(G(2,4),2)$ coincide, as will become clear below. Many of our constructions will be extended from $G(2,4)$ to $G(k,n)$ via the morphism ϕ. The reader who wishes to specialize $G(k,n)$ to $G(2,4)$ in this section will not lose much generality.

Exercise 9.24. The geometry of $\overline{\mathrm{M}}_{0,0}(G(2,4),2)$ is closely related to the geometry of quadric surfaces in $\mathbb{P}^3$. Show that the lines parameterized by a point in $\overline{\mathrm{M}}_{0,0}(G(2,4),2)$ sweep out a degree two surface in $\mathbb{P}^3$. The maps parameterized by a point in $\mathcal{D}_{deg}$ sweep out a plane two-to-one. The maps parameterized by a general point in $\mathcal{D}_{unb}$ sweep out a quadric cone.

Notation 9.25. Let $\mathcal{Q}[\lambda]$ denote the closure of the locus of maps f in $\overline{\mathcal{M}}_{0,0}(G(k,n),2)$ with irreducible domain such that the map f factors through the inclusion of some Schubert variety Σ_λ in $G(k,n)$.

Exercise 9.26. Show that $\mathcal{Q}[(1)^*]$ denotes the locus of maps two-to-one onto a line in the Plücker embedding of $G(k,n)$. Show that the union of $\mathcal{Q}[(1,1)^*]$ and $\mathcal{Q}[(2)^*]$ in $\overline{\mathrm{M}}_{0,0}(G(k,n),2)$ is the locus of maps f such that the span of f is contained in $G(k,n)$. In particular, the linear spaces parameterized by a general map in $\mathcal{Q}[(1,1)^*]$ sweep out a $\mathbb{P}^k$ two-to-one. The linear spaces parameterized by a general map in $\mathcal{Q}[(2)^*]$ sweep out a k-dimensional cone over a conic curve.

For our calculations of the stable base locus, we will introduce many curve classes and compute their intersections with divisor classes. For the convenience of the reader, we summarize this information in the following table. The first column contains the curve classes in the order that they will be introduced below. The next three columns contain the intersection numbers of these curve classes with the divisors $\mathcal{H}_{\sigma_{1,1}}, \mathcal{H}_{\sigma_2}$ and Δ, respectively. Finally, the last column describes the subvariety of $\overline{\mathrm{M}}_{0,0}(G(k,n),2)$ covered by effective curves in that class. The reader may wish to verify Theorem 9.30 for themselves using this table.

Curve class C	$C\cdot\mathcal{H}_{\sigma_{1,1}}$	$C\cdot\mathcal{H}_{\sigma_2}$	$C\cdot\Delta$	Deformations cover
C_1	1	0	3	$\mathcal{Q}[(1,1)^*]$
C_2	0	1	3	$\mathcal{Q}[(2)^*]$
C_3	1	1	2	$\overline{\mathcal{M}}_{0,0}(G(k,n),2)$
C_4	2	0	0	$\mathcal{Q}[(1,1)^*]$
C_5	0	2	0	$\mathcal{Q}[(2)^*]$
C_6	1	0	-1	Δ
C_7	0	1	-1	Δ
C_8	0	0	>0	$\mathcal{Q}[(1)^*]$

In order to understand the stable base locus decomposition of $\overline{\mathrm{M}}_{0,0}(G(k,n),2)$, we need to introduce one more divisor class. Set $N=\binom{n}{k}$. Let $p:\overline{\mathrm{M}}_{0,0}(G(k,n),2)\dashrightarrow G(3,N)$ denote the rational map, defined away from the locus of double covers of a line in $G(k,n)$, sending a stable map to the $\mathbb{P}^2$ spanned by its image in the Plücker embedding of $G(k,n)$. This map gives rise to a well-defined map p^* on Picard groups. Let $P=p^*(\mathcal{O}_{G(3,N)}(1))$. Geometrically, P is the class of the closure of the locus of maps f such that the linear span of the image of f (viewed in the Plücker embedding of $G(k,n)$) intersects a fixed codimension three linear space in $\mathbb{P}^{N-1}$.

Lemma 9.27. *The divisor class P is equal to*

$$P=\frac{1}{4}(3\mathcal{H}_{\sigma_{1,1}}+3\mathcal{H}_{\sigma_2}-\Delta).$$

Proof. The formula for the class P follows from Lemma 3.4. However, since we will later need the curve classes introduced here, we recall the proof. The divisor class P can be computed by intersecting with test families. Let $\lambda=(1,1)^*$ and $\mu=(2)^*$ be the partitions dual to $(1,1)$ and (2), respectively. In the Plücker embedding of $G(k,n)$, both Σ_λ and Σ_μ are linear spaces of dimension two. Let C_1 and C_2, respectively, be the curves in $\overline{\mathcal{M}}_{0,0}(G(k,n),2)$ induced by a general pencil of conics in a fixed Σ_λ, respectively, Σ_μ. Let $\tilde{C}_3$ be the curve in $\overline{\mathcal{M}}_{0,0}(G(2,4),2)$ induced by a general pencil of conics in a general codimension two linear section of $G(2,4)$ in its Plücker embedding. Let $C_3=\phi(\tilde{C}_3)$. The following intersection numbers are easy to compute.

$$C_1\cdot\mathcal{H}_{\sigma_{1,1}}=1,\quad C_1\cdot\mathcal{H}_{\sigma_2}=0,\quad C_1\cdot\Delta=3,\quad C_1\cdot P=0$$
$$C_2\cdot\mathcal{H}_{\sigma_{1,1}}=0,\quad C_2\cdot\mathcal{H}_{\sigma_2}=1,\quad C_2\cdot\Delta=3,\quad C_2\cdot P=0$$
$$C_3\cdot\mathcal{H}_{\sigma_{1,1}}=1,\quad C_3\cdot\mathcal{H}_{\sigma_2}=1,\quad C_3\cdot\Delta=2,\quad C_3\cdot P=1$$

The class P is determined by these intersection numbers. □

Exercise 9.28. Verify the intersection numbers in the previous lemma.

Notation 9.29. Given two divisor classes D_1,D_2, let $c(D_1D_2)$ (respectively, $c(\overline{D}_1\overline{D}_2)$) denote the open (resp., closed) cone in the Neron-Severi space spanned by positive (resp., non-negative) linear combinations of D_1 and D_2. Let $c(D_1\overline{D}_2)$ denote the cone spanned by linear combinations

$$c(D_1\overline{D}_2)=\{aD_1+bD_2|a\geq 0,b>0\}.$$

The domain in $\mathbb{R}^3$ bounded by the divisor classes $D_1, D_2, \ldots, D_r$ is the open domain bounded by $c(\overline{D}_1\overline{D}_2), c(\overline{D}_2\overline{D}_3), \ldots, c(\overline{D}_r\overline{D}_1)$.

Theorem 9.30 and Figure 10 describe the eight chambers in the stable base locus decomposition of $\overline{\mathrm{M}}_{0,0}(G(k,n),2)$. In the figure, we draw a cross-section of the three-dimensional cone and mark each chamber with the corresponding number that describes the chamber in the theorem.

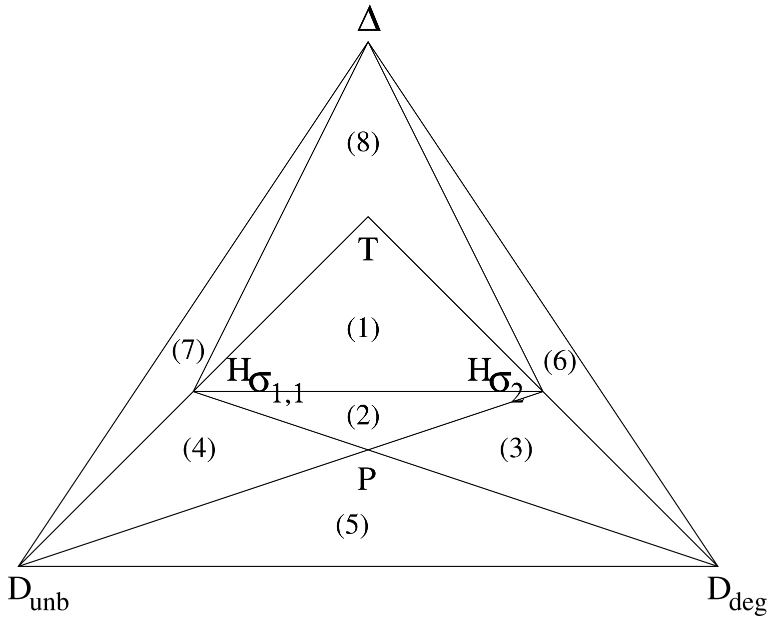

FIGURE 10. The stable base locus decomposition of $\overline{\mathrm{M}}_{0,0}(G(k,n),2)$.

Theorem 9.30. *The stable base locus decomposition of the effective cone of $\overline{M}_{0,0}(G(k,n),2)$ is described as follows:*

(1) *In the closed cone spanned by non-negative linear combinations of $\mathcal{H}_{\sigma_{1,1}}, \mathcal{H}_{\sigma_2}$ and $\mathcal{T}$, the stable base locus is empty.*
(2) *In the domain bounded by $\mathcal{H}_{\sigma_{1,1}}, \mathcal{H}_{\sigma_2}$ and P union $c(\mathcal{H}_{\sigma_{1,1}}\overline{P}) \cup c(\mathcal{H}_{\sigma_2}\overline{P})$, the stable base locus consists of the locus $\mathcal{Q}[(1)^*]$ of maps two-to-one onto a line in $G(k,n)$.*
(3) *In the domain bounded by $\mathcal{H}_{\sigma_2}, D_{deg}$ and P union $c(\mathcal{H}_{\sigma_2}\overline{D}_{deg}) \cup c(P\overline{D}_{deg})$, the stable base locus consists of the locus $\mathcal{Q}[(1,1)^*]$.*
(4) *In the domain bounded by $\mathcal{H}_{\sigma_{1,1}}, D_{unb}$ and P union $c(\mathcal{H}_{\sigma_{1,1}}\overline{D}_{unb}) \cup c(P\overline{D}_{unb})$, the stable base locus consists of the locus $\mathcal{Q}[(2)^*]$.*
(5) *In the domain bounded by P, D_{deg} and D_{unb} union $c(D_{deg}D_{unb})$, the stable base locus consists of the union $\mathcal{Q}[(1,1)^*] \cup \mathcal{Q}[(2)^*]$.*
(6) *In the domain bounded by $\mathcal{H}_{\sigma_2}, D_{deg}$ and Δ union $c(D_{deg}\Delta)$, the stable base locus consists of the union of the boundary divisor and $\mathcal{Q}[(1,1)^*]$.*
(7) *In the domain bounded by $\mathcal{H}_{\sigma_{1,1}}, D_{unb}$ and Δ union $c(D_{unb}\Delta)$, the stable base locus consists of the union of the boundary divisor and $\mathcal{Q}[(2)^*]$.*

(8) *Finally, in the domain bounded by* $\mathcal{H}_{\sigma_{1,1}}, \mathcal{T}, \mathcal{H}_{\sigma_2}$ *and* Δ *union* $c(\mathcal{H}_{\sigma_2}\overline{\Delta}) \cup c(\mathcal{H}_{\sigma_{1,1}}\overline{\Delta})$ *the stable base locus consists of the boundary divisor.*

Proof. The reader should notice the symmetry across the vertical axis in Figure 10. The Grassmannians $G(k,n)$ and $G(n-k,n)$ are isomorphic. This isomorphism induces an isomorphism

$$\psi : \overline{\mathrm{M}}_{0,0}(G(k,n),2) \to \overline{\mathrm{M}}_{0,0}(G(n-k,n),2)$$

which interchanges $\mathcal{H}_{\sigma_{1,1}}$, D_{unb} with $\mathcal{H}_{\sigma_2}$ and D_{deg}, respectively, and gives rise to the symmetry in the figure. The stable base locus of a divisor $\psi^*(D)$ is equal to the inverse image under ψ of the stable base locus of D. We will often group the divisors that are symmetric under ψ and use the symmetry to simplify our calculations.

Since the effective cone of $\overline{\mathrm{M}}_{0,0}(G(k,n),2)$ is generated by non-negative linear combinations of D_{deg}, D_{unb} and Δ, the stable base locus of any divisor has to be contained in the union of the stable base loci of D_{deg}, D_{unb} and the boundary divisor. We first check that the loci described in the theorem are in the stable base locus of the claimed divisors. To show that a variety X is in the base locus of a linear system $|D|$, it suffices to cover X by curves C that have negative intersection with D.

Express a general divisor $D = a\mathcal{H}_{\sigma_{1,1}} + b\mathcal{H}_{\sigma_2} + c\Delta$. Recall from the proof of Lemma 9.27 that C_1 and C_2 are the curves induced by pencils of conics in Σ_λ and Σ_μ, respectively, where $\lambda = (1,1)^*$ and $\mu = (2)^*$. The intersection numbers of C_1 and C_2 with D are

$$C_1 \cdot D = a + 3c, \quad C_2 \cdot D = b + 3c.$$

Since curves in the class C_1 (resp., C_2) cover $\mathcal{Q}[(1,1)^*]$ (resp., $\mathcal{Q}[(2)^*]$), we conclude that $\mathcal{Q}[(1,1)^*]$ (resp., $\mathcal{Q}[(2)^*]$) is in the base locus of the linear system $|D|$ if $a + 3c < 0$ (resp., $b + 3c < 0$). In other words, $\mathcal{Q}[(1,1)^*]$ is in the restricted base locus of the divisors contained in the interior of the cone generated by D_{deg}, D_{unb} and $D_{deg} + \Delta/3$ and in $c(D_{unb}\overline{D}_{deg})$. Similarly, $\mathcal{Q}[(2)^*]$ is in the restricted base locus of a divisor contained in the interior of the cone generated by D_{deg}, D_{unb} and $D_{unb} + \Delta/3$ and in $c(D_{deg}\overline{D}_{unb})$.

Let C_4 and C_5 be the curves induced in $\overline{\mathcal{M}}_{0,0}(G(k,n),2)$ by the one parameter family of conics tangent to four general lines in a fixed Σ_λ and Σ_μ, respectively. It is straightforward to see that

$$C_4 \cdot D = 2a, \quad C_5 \cdot D = 2b.$$

Curves of type C_4 and C_5 cover $\mathcal{Q}[(1,1)^*]$ and $\mathcal{Q}[(2)^*]$, respectively. Consequently, if $a < 0$ (resp., $b < 0$) $\mathcal{Q}[(1,1)^*]$ (resp., $\mathcal{Q}[(2)^*]$) is in the restricted base locus of $|D|$. We conclude that $\mathcal{Q}[(1,1)^*]$ is in the restricted base locus of any divisor contained in the region bounded by $D_{deg}, \Delta, \mathcal{H}_{\sigma_2}$ and D_{unb} and in $c(\Delta\overline{D}_{deg}) \cup c(D_{unb}\overline{D}_{deg})$. Similarly, $\mathcal{Q}[(2)^*]$ is in the restricted base locus of any divisor contained in the region bounded by $D_{unb}, \Delta, \mathcal{H}_{\sigma_{1,1}}$ and D_{deg} and in $c(\Delta\overline{D}_{unb}) \cup c(D_{deg}\overline{D}_{unb})$.

Next let C_6 and C_7 be the curves induced by attaching a line at the base point of a pencil of lines in Σ_λ and Σ_μ, respectively. These curves have the following intersection numbers with D:

$$C_6 \cdot D = a - c, \quad C_7 \cdot D = b - c.$$

Since deformations of the curves in the same class as C_6 and C_7 cover the boundary divisor, we conclude that the boundary divisor is in the base locus of $|D|$ if $a - c < 0$ or if $b - c < 0$. Hence, the boundary divisor is in the base locus of the divisors contained in the region bounded by $D_{unb}, \mathcal{T}, D_{deg}$ and Δ and in $c(D_{unb}\overline{\Delta}) \cup c(D_{deg}\overline{\Delta})$.

Finally, consider the one-parameter family C_8 of two-to-one covers of a line l in $G(k, n)$ branched along a fixed point $p \in l$ and a varying point $q \in l$. Then

$$C_8 \cdot D = c.$$

Curves in the class C_8 cover the locus of double covers of a line. Hence, if $c < 0$, then the locus of double covers of a line have to be contained in the restricted base locus. Note that since the locus of double covers of a line is contained in both $\mathcal{Q}[(1,1)^*]$ and $\mathcal{Q}[(2)^*]$, any divisor containing the latter in the base locus also contains the locus of double covers. Hence, the locus of double covers is contained in the base locus of every effective divisor contained in the complement of the closed cone generated by $\mathcal{H}_{\sigma_{1,1}}, \mathcal{H}_{\sigma_2}$ and Δ. In particular, this locus is contained in the base locus of divisors contained in the region bounded by $\mathcal{H}_{\sigma_{1,1}}, \mathcal{H}_{\sigma_2}$ and P and in $c(\mathcal{H}_{\sigma_{1,1}}\overline{P}) \cup c(\mathcal{H}_{\sigma_2}\overline{P})$.

We have verified that the loci described in the theorem are in the base locus of the corresponding divisors. We will next show that the divisors listed in the theorem contain only the listed loci in their stable base locus. The divisors $\mathcal{H}_{\sigma_{1,1}}, \mathcal{H}_{\sigma_2}$ and $\mathcal{T}$ are base-point-free ([CS] §5). Hence, for divisors contained in the closed cone generated by $\mathcal{H}_{\sigma_{1,1}}, \mathcal{H}_{\sigma_2}$ and $\mathcal{T}$ the base locus is empty.

Next, note that the base locus of the linear system $|P|$ is exactly the locus of double covers of a line. The rational map p in the definition of P is a morphism in the complement of the locus of double covers of a line. If the image of a map f is a degree two curve in $G(k, n)$, then in the Plücker embedding of $G(k, n)$ the image spans a unique plane. In $\mathbb{P}^{N-1}$, we can always find a codimension three linear space Γ not intersecting Λ. Hence, f is not in the indeterminacy locus of the map to $G(3, N)$ and there is a section of $\mathcal{O}_{G(3,N)}(1)$ not containing the image of f. It follows that f is not in the base locus of $|P|$. By the argument two paragraphs above, the locus of degree two maps onto a line is in the base locus of P. We conclude that in the region bounded by $P, \mathcal{H}_{\sigma_{1,1}}$ and $\mathcal{H}_{\sigma_2}$ and in $c(\mathcal{H}_{\sigma_2}\overline{P}) \cup c(\mathcal{H}_{\sigma_{1,1}}\overline{P})$ the stable base locus consists of the locus of double covers of a line.

For a divisor contained in the region bounded by D_{unb}, P and $\mathcal{H}_{\sigma_{1,1}}$ and in $c(P\overline{D}_{unb}) \cup c(\mathcal{H}_{\sigma_{1,1}}\overline{D}_{unb})$, the stable base locus must be contained in the stable base locus of D_{unb} since every divisor in this region is a non-negative linear combination of D_{unb} and base-point-free divisors. Similarly, for a divisor contained in the region bounded by D_{deg}, P and $\mathcal{H}_{\sigma_2}$ and in $c(P\overline{D}_{deg}) \cup c(\mathcal{H}_{\sigma_2}\overline{D}_{deg})$, the base locus must be contained in the stable base locus D_{deg}. In the region bounded by D_{deg}, D_{unb} and P and in $c(D_{unb}D_{deg})$, the base locus must be contained in the union of the stable base loci of D_{deg} and D_{unb}. The (stable) base locus of D_{deg} is $\mathcal{Q}[(1,1)^*]$ and the (stable) base locus of D_{unb} is $\mathcal{Q}[(2)^*]$. The linear spaces parameterized by a degree two map to $G(k, n)$ span a linear space of dimension at most $k + 2$. As long as they span a linear space of dimension $k + 2$, then the projection from a general linear space of codimension $k + 2$ still spans a linear space of dimension $k+2$, hence the corresponding map is not in the base locus of D_{deg}. By symmetry, as long as the intersection of all the linear spaces parameterized by the degree two map does not contain a $k-1$ dimensional linear space, then the map is not contained in the base locus of D_{unb}. Hence, the claims in parts (3), (4) and (5) of the theorem follow. Similarly, in the region bounded by D_{unb}, Δ and $\mathcal{H}_{\sigma_{1,1}}$ and in $c(D_{unb}\Delta)$, the base locus must be contained in the union of $\mathcal{Q}[(2)^*]$ and the boundary divisor. In the region bounded by D_{deg}, Δ and $\mathcal{H}_{\sigma_2}$ and in $c(D_{deg}\Delta)$, the base locus must be contained in the union of $\mathcal{Q}[(1,1)^*]$ and the boundary divisor. We conclude the equality in these two cases as well. Finally, in the region bounded by $\Delta, \mathcal{H}_{\sigma_{1,1}}$ and $\mathcal{H}_{\sigma_2}$ the base locus has to be contained in the boundary divisor. Hence in the complement of the closed cone spanned by $\mathcal{H}_{\sigma_{1,1}}, \mathcal{T}$ and $\mathcal{H}_{\sigma_2}$ the base locus must equal the boundary divisor by the calculations above. This completes the proof of the theorem. □

Next, we describe the birational models of $\overline{\mathcal{M}}_{0,0}(G(k,n),2)$ that correspond to the chambers in the decomposition. For a big rational divisor class D, let ϕ_D denote the birational map

$$\phi_D : \overline{\mathcal{M}}_{0,0}(G(k,n),2) \dashrightarrow \mathrm{Proj}(\oplus_{m\geq 0}(H^0(\mathcal{O}(\lfloor mD \rfloor)))).$$

Proposition 9.31. *The Kontsevich moduli space $\overline{\mathcal{M}}_{0,0}(G(k,n),2)$ admits the following morphisms:*

(1) $\phi_{t\mathcal{H}_{\sigma_{1,1}}+(1-t)\mathcal{H}_{\sigma_2}}$*, for $0<t<1$, is a morphism from $\overline{\mathcal{M}}_{0,0}(G(k,n),2)$ to the normalization of the Chow variety, which is an isomorphism in the complement of $\mathcal{Q}[(1)^*]$, the locus of double covers of a line in $G(k,n)$, and contracts $\mathcal{Q}[(1)^*]$ so that the locus of double covers with the same image line maps to a point.*

(2) $\phi_{\mathcal{H}_{\sigma_{1,1}}}$ *and* $\phi_{\mathcal{H}_{\sigma_2}}$ *give two morphisms from $\overline{\mathcal{M}}_{0,0}(G(k,n),2)$ to two contractions of the normalization of the Chow variety, where $\phi_{\mathcal{H}_{\sigma_{1,1}}}$ (resp., $\phi_{\mathcal{H}_{\sigma_2}}$), in addition to the double covers of a line, also contract the boundary divisor and $\mathcal{Q}[(2)^*]$ (resp., $\mathcal{Q}[(1,1)^*]$). Any two maps f, f' in the boundary for which the image is contained in the union of the same Schubert varieties $\Sigma_{(2)^*}$ (resp., $\Sigma_{(1,1)^*}$) map to the same point under $\phi_{\mathcal{H}_{\sigma_{1,1}}}$ (resp., $\phi_{\mathcal{H}_{\sigma_2}}$). Similarly, the stable maps in $\mathcal{Q}[(2)^*]$ (resp., $\mathcal{Q}[(1,1)^*]$) with image contained in a fixed Schubert variety $\Sigma_{(2)^*}$ (resp., $\Sigma_{(1,1)^*}$) map to the same point under $\phi_{\mathcal{H}_{\sigma_{1,1}}}$ (resp., $\phi_{\mathcal{H}_{\sigma_2}}$).*

(3) *If D is in the domain bounded by $\mathcal{H}_{\sigma_{1,1}}, \mathcal{H}_{\sigma_2}$ and $\mathcal{T}$, then D is ample and gives rise to an embedding of $\overline{\mathcal{M}}_{0,0}(G(k,n),2)$.*

Proof. The NEF cone of $\overline{\mathcal{M}}_{0,0}(G(k,n),2)$, which coincides with the base-point-free cone, is the closed cone spanned by $\mathcal{H}_{\sigma_{1,1}}, \mathcal{H}_{\sigma_2}$ and $\mathcal{T}$. We, therefore, obtain morphisms for sufficiently high and divisible multiples of each of the rational divisors in this cone. The last part of the proposition follows by Kleiman's Theorem which asserts that the interior of the NEF cone is the ample cone.

The curves in the class C_8 have intersection number zero with any divisor of the form $t\mathcal{H}_{\sigma_{1,1}} + (1-t)\mathcal{H}_{\sigma_2}$. Since these curves cover the locus of double covers of a fixed line, we conclude that the maps obtained from these divisors contract the locus of double covers of a fixed line to a point. The class H of the divisor of maps whose image intersects a codimension two linear space in projective space gives rise to the Hilbert-Chow morphism on $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^{N-1}, 2)$. This morphism has image the normalization of the Chow variety and is an isomorphism away from the locus of maps two-to-one onto their image. The Plücker embedding of $G(k,n)$ induces an inclusion of $\overline{\mathcal{M}}_{0,0}(G(k,n),2)$ in $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^{N-1}, 2)$. The pull-back of H under this inclusion is $\mathcal{H}_{\sigma_{1,1}} + \mathcal{H}_{\sigma_2}$. By symmetry, there is no loss of generality in assuming that $0 < t \leq 1/2$. We can write

$$t\mathcal{H}_{\sigma_{1,1}} + (1-t)\mathcal{H}_{\sigma_2} = t(\mathcal{H}_{\sigma_{1,1}} + \mathcal{H}_{\sigma_2}) + (1-2t)\mathcal{H}_{\sigma_2}.$$

Since $\mathcal{H}_{\sigma_2}$ is base-point-free, the first part of the proposition follows.

The cases of $\phi_{\mathcal{H}_{\sigma_{1,1}}}$ and $\phi_{\mathcal{H}_{\sigma_2}}$ are almost identical, so we concentrate on $\phi_{\mathcal{H}_{\sigma_{1,1}}}$. $\mathcal{H}_{\sigma_{1,1}}$ has intersection number zero with the curve classes C_5, C_7 and C_8. Curves in the class C_5 cover the locus $\mathcal{Q}[(2)^*]$. Curves in the class C_7 cover the boundary divisor and curves in the class C_8 cover $\mathcal{Q}[(1)^*]$. We conclude that these loci are contracted by $\phi_{\mathcal{H}_{\sigma_{1,1}}}$. Part (2) of the proposition follows from these considerations. We observe that the locus of degree two curves whose span does not lie in $G(k,n)$ admit three distinct Chow compactifications depending on whether one uses the codimension two class $\sigma_{1,1}$, σ_2 or $a\sigma_{1,1} + b\sigma_2$ with $a, b > 0$. The three models are the normalization of these Chow compactifications. □

Theorem 9.32. (1) *The birational model corresponding to the divisor $\mathcal{T}$ is the space of weighted stable maps $\overline{\mathcal{M}}_{0,0}(G(k,n),1,1)$. $\phi_{\mathcal{T}}$ is an isomorphism away from the boundary divisor and contracts the locus of maps with reducible domain $f : C_1 \cup C_2 \to G(k,n)$ that have $f(C_1 \cap C_2) = p$ for some fixed $p \in G(k,n)$ to a point.*

(2) *For $D \in c(\mathcal{H}_{\sigma_{1,1}}\mathcal{T})$ or $D \in c(\mathcal{H}_{\sigma_2}\mathcal{T})$ the morphism ϕ_D is an isomorphism away from the boundary divisor. On the boundary divisor, for $D \in c(\mathcal{H}_{\sigma_{1,1}}\mathcal{T})$ (resp., in $c(\mathcal{H}_{\sigma_2}\mathcal{T})$) the morphism contracts the locus of line pairs that are contained in the same pair of intersecting linear spaces with class $\Sigma_{n-k-1,\dots,n-k-1}$ (resp., $\Sigma_{n-k,\dots,n-k}$) to a point. These morphisms are flops of each other over $\phi_{\mathcal{T}}$.*

Exercise 9.33. Show that curves in the class C_6 have intersection number zero with a divisor class D in $c(\overline{\mathcal{H}_{\sigma_{1,1}}\mathcal{T}})$. Similarly, show that curves in the class C_7 have intersection number zero with D in $c(\overline{\mathcal{H}_{\sigma_2}\mathcal{T}})$. Conclude that ϕ_D contracts the loci claimed in the theorem. curves. Check that both of these admit further small contractions to the image of $\phi_{\mathcal{T}}$.

For the next lemma and theorem, we assume that the target is $G(2,4)$. Recall that the Plücker map embeds $G(2,4)$ as a smooth quadric hypersurface in $\mathbb{P}^5$. The orthogonal Grassmannian $OG(3,6)$ parameterizes planes contained in a smooth quadric hypersurface in $\mathbb{P}^5$, hence can be interpreted as parameterizing planes contained in $G(2,4)$. $OG(3,6)$ has two isomorphic connected components (distinguished depending on whether the plane has cohomology class $\sigma_{1,1}$ or σ_2).

Lemma 9.34. *Let $OG_{\sigma_{1,1}}(G(2,4))$ and $OG_{\sigma_2}(G(2,4))$ denote the two connected components of the orthogonal Grassmannian $OG(3,6)$ parametrizing projective planes contained in the Plücker embedding of $G(2,4)$. Then the Hilbert scheme $Hilb_{2x-1}(G(2,4))$ corresponding to the Hilbert polynomial $2x-1$ is isomorphic to the blow-up of $G(3,6)$ along*

$OG(3,6)$. The blow-down morphism

$$\pi : Hilb_{2x-1}(G(2,4)) \to G(3,6)$$

factors through

$$\pi_{1,1} : Hilb_{2x-1}(G(2,4)) \to Bl_{OG_{\sigma_{1,1}}} G(3,6)$$

and

$$\pi_2 : Hilb_{2x-1}(G(2,4)) \to Bl_{OG_{\sigma_2}} G(3,6).$$

Proof. Consider the universal family $I \subset G(3,6) \times \mathbb{P}^5$ over the Grassmannian admitting two natural projections ϕ_1 and ϕ_2 to $G(3,6)$ and $\mathbb{P}^5$, respectively. The bundle $\phi_{1*}\phi_2^*\mathcal{O}_{\mathbb{P}^5}(2)$ is naturally identified with $Sym^2 S^*$. Since $OG(3,6)$ is defined by the vanishing of a general section of $\phi_{1*}\phi_2^*\mathcal{O}_{\mathbb{P}^5}(2)$, we can identify the normal bundle of $OG(3,6)$ at a point Λ of $OG(3,6)$ with $Sym^2 S^*|_\Lambda$. $Hilb_{2x-1}(\mathbb{P}^5)$ is naturally identified with $\mathbb{P}(Sym^2(S^*)) \to G(3,6)$. Then $Hilb_{2x-1}(G(2,4))$ is given by

$$\{([C],[\Lambda]) \mid [\Lambda] \in G(3,6), C \subset \Lambda \cap G(2,4), [C] \in Hilb_{2x-1}(\Lambda)\}.$$

The projection to $G(3,6)$ is clearly an isomorphism away from $OG(3,6)$. Over $OG(3,6)$ the fiber of the Hilbert scheme is identified with the projectivization of $Sym^2 S^*$. It follows that $Hilb_{2x-1}(G(2,4))$ is isomorphic to the blow-up of $G(3,6)$ along $OG(3,6)$. Since $OG(3,6)$ has two connected components, this leads to two exceptional divisors that can be blown-down independently. The Lemma follows from these considerations. □

Theorem 9.35. *The rational maps corresponding to the divisors D in the cone generated by $\mathcal{H}_{\sigma_{1,1}}, \mathcal{H}_{\sigma_2}$ and P are as follows.*

(1) *Let $0 < t < 1$. The Hilbert scheme $Hilb_{2x-1}(G(2,4))$ is the flip of $\overline{\mathcal{M}}_{0,0}(G(2,4),2)$ over the Chow variety $Chow_{t\mathcal{H}_{\sigma_{1,1}}+(1-t)\mathcal{H}_{\sigma_2}}$. For D in the domain bounded by $\mathcal{H}_{\sigma_{1,1}}, \mathcal{H}_{\sigma_2}$ and P, the rational transformation ϕ_D equals*

$$\overline{\mathcal{M}}_{0,0}(G(2,4),2) \dashrightarrow Hilb_{2x-1}(G(2,4))$$

(2) *For $D \in c(\mathcal{H}_{\sigma_{1,1}}P)$, the rational transformation ϕ_D equals*

$$\overline{\mathcal{M}}_{0,0}(G(2,4),2) \dashrightarrow Bl_{OG_{\sigma_{1,1}}} G(3,6)$$

(3) *For $D \in c(\mathcal{H}_{\sigma_2}P)$, the rational transformation ϕ_D equals*

$$\overline{\mathcal{M}}_{0,0}(G(2,4),2) \dashrightarrow Bl_{OG_{\sigma_2}} G(3,6).$$

(4) *The rational transformation ϕ_P equals*

$$\overline{\mathcal{M}}_{0,0}(G(2,4),2) \dashrightarrow G(3,6).$$

Proof. Consider the incidence correspondence consisting of triples (C, C^*, Λ), where Λ is a plane in $\mathbb{P}^5$, C is a connected, arithmetic genus zero, degree two curve in $G(2,4) \cap \Lambda$ and C^* is a dual conic of C in Λ. This incidence correspondence admits a map both to $\overline{\mathcal{M}}_{0,0}(G(2,4),2)$ and to $Hilb_{2x-1}(G(2,4))$ by projection to the first two and by projection to the first and third factors, respectively. The projection to the first factor gives a morphism to the Chow variety. Note that this projection is an isomorphism away from the locus where C is supported on a line. The morphism to the Chow variety is a small contraction in the case of both the Hilbert scheme and the Kontsevich moduli space. The fiber over a point corresponding to a double line in the morphism from the Hilbert scheme to the Chow variety is isomorphic to $\mathbb{P}^1$ corresponding to the choice of plane Λ everywhere tangent to the Plücker embedding of $G(2,4)$ in $\mathbb{P}^5$. The fiber over a point corresponding to a double line in the morphism from $\overline{\mathcal{M}}_{0,0}(G(2,4),2)$ to the Chow variety is isomorphic to $\mathbb{P}^2 = Sym^2(\mathbb{P}^1)$ corresponding to double covers of $\mathbb{P}^1$. Note in both the Hilbert scheme and the Kontsevich moduli space the morphisms to the Chow variety are small contractions. The locus of double lines in the Hilbert scheme (respectively, in $\overline{\mathcal{M}}_{0,0}(G(2,4),2)$) has codimension 3 (respectively, 2). Finally, note that for D in the domain bounded by $\mathcal{H}_{\sigma_{1,1}}, \mathcal{H}_{\sigma_2}$ and P, $-D$ is ample on the fibers of the projection of $\overline{\mathcal{M}}_{0,0}(G(2,4),2)$ to the Chow variety and D is ample on the fibers of the projection of the Hilbert scheme to the Chow variety. We conclude that $Hilb_{2x-1}(G(2,4))$ is the flip of $\overline{\mathcal{M}}_{0,0}(G(2,4),2)$ over the Chow variety. The rest of the Theorem follows from the previous lemma and the definition of P. □

Remark 9.36. For $G(k,n)$ with $(k,n) \neq (2,4)$, the flip of $\overline{\mathcal{M}}_{0,0}(G(k,n),2)$ corresponding to a divisor D in the domain bounded by $F, \mathcal{H}_{\sigma_{1,1}}$ and $\mathcal{H}_{\sigma_2}$ is no longer the Hilbert scheme, but a divisorial contraction of the Hilbert scheme.

Exercise 9.37. Determine the stable base locus decomposition of $\overline{\mathrm{M}}_{0,0}(G(2,5),3)$. If you want a challenge, determine the stable base locus decomposition of $\overline{\mathrm{M}}_{0,0}(G(k,n),3)$ in general. Interpret the birational models that correspond to the chambers in the decomposition.

As d increases, the dimension of the Neron-Severi space of $\overline{\mathrm{M}}_{0,0}(\mathbb{P}^r,d)$ or $\overline{\mathrm{M}}_{0,0}(G(k,n),d)$ increases linearly. The chamber decompositions of the cones become complicated very quickly. A more approachable problem is to describe the decomposition in various distinguished planes or other small dimensional linear spaces of the Neron-Severi space. For example, for $\overline{\mathrm{M}}_{0,0}(\mathbb{P}^r,d)$, one may consider the plane spanned by $\mathcal{H}, \mathcal{T}$ or the three-dimensional subspace spanned by $K_{\overline{\mathrm{M}}_{0,0}(\mathbb{P}^r,d)}, \mathcal{H}, \mathcal{T}$. A large number of geometrically significant divisors lie in these subspaces. The stable base locus decomposition of the plane in the Neron-Severi space spanned by $\mathcal{H}$ and $\mathcal{T}$ has partially been discussed in [CC2].

10. The first divisorial contraction and the first flip for $\overline{\mathrm{M}}_g$

In this section, we briefly sketch recent progress in the Minimal Model Program for $\overline{\mathrm{M}}_g$ due to Hassett and Hyeon. In [Has], Brendan Hassett initiated a program, first conceived in an extended e-mail correspondence between Hassett and Keel, to understand the canonical model

$$\mathrm{Proj}(\bigoplus_{m\geq 0} H^0(\overline{\mathrm{M}}_g, mK_{\overline{\mathrm{M}}_g}))$$

of the moduli space of curves. By the celebrated result of Birkar, Cascini, Hacon and McKernan [BCHM], the canonical ring is finitely generated. When $g \geq 22$, $\overline{\mathrm{M}}_g$ has pluri-canonical sections, so the canonical model of $\overline{\mathrm{M}}_g$ is a projective variety. (Of course, when $\overline{\mathrm{M}}_g$ is uniruled, the canonical ring is not very interesting. In this section, whenever we refer to the canonical model of $\overline{\mathrm{M}}_g$, we will exclude the cases when $\overline{\mathrm{M}}_g$ is uniruled.) Unfortunately, we do not know the canonical model of $\overline{\mathrm{M}}_g$ for any g. Given the importance of $\overline{\mathrm{M}}_g$ and its centrality in mathematics, it is surprising that we do not even have any guesses as to what the canonical model of $\overline{\mathrm{M}}_g$ may be. Hassett had the idea of starting with $\overline{\mathrm{M}}_g$ and arriving at the canonical model by a sequence of divisorial contractions and flips. He considers the log canonical models

$$\overline{\mathrm{M}}_g(a) = \mathrm{Proj}(\bigoplus_{m\geq 0} H^0(\overline{\mathrm{M}}_g, m(K_{\overline{\mathrm{M}}_g} + a\delta))).$$

When $a = 1$, the corresponding divisor on $\overline{\mathrm{M}}_g$ is ample, hence $\overline{\mathrm{M}}_g(1)$ is the usual Deligne-Mumford moduli space of stable curves. One then decreases the coefficient a of the boundary divisor δ. When $a = 0$, of course, one obtains the canonical model of $\overline{\mathrm{M}}_g$. Hassett's program is to describe the birational changes that occur as a decreases from one to zero.

Surprisingly, in the few steps of the program that have been carried out, the log canonical models $\overline{\mathrm{M}}_g(a)$ themselves have modular interpretations. There are three natural ways of constructing different birational models of a moduli space. First, we can vary the definition of the functor. For example, instead of parameterizing isomorphism classes of stable curves, we might parameterize isomorphism classes of a slightly different class of curves. The moduli space of pseudo-stable curves, which we introduce next, will provide a typical example. Second, if the moduli space is constructed as a G.I.T. quotient, we can vary the linearization and consider the corresponding quotients. As we vary the linearization, we get different birational models (see Theorem 10.12 for some examples). Finally, we can run the minimal model program. It is a beautiful feature of the theory that in the examples worked out so far these three points of view coincide to produce some very rich geometry.

Definition 10.1. A complete curve is *pseudo-stable* if

(1) it is connected, reduced and has only nodes and cusps as singularities;
(2) every subcurve of arithmetic genus one meets the rest of the curve in at least two points;
(3) the dualizing sheaf of the curve is ample.

Exercise 10.2. Show that the stability condition on the dualizing sheaf is equivalent to asking that every subcurve of arithmetic genus zero meets the rest of the curve in three points.

Let $\overline{\mathcal{M}}_g^{ps}$ be the functor that associates to each scheme S of finite type over k, the isomorphism classes of families $f : C \to S$, where f is proper and flat and the geometric fibers of f are pseudo-stable curves of genus g. Let $\overline{\mathrm{M}}_g^{ps}$ denote the coarse moduli scheme. Let $Chow_n$ denote the locally closed subset of the Chow variety of curves of degree $2n(g-1)$ in $\mathbb{P}^{(2n-1)(g-1)-1}$ parameterizing n-canonically embedded curves of genus g (for $n > 1$). D. Schubert [S] proved that a Chow point in $Chow_3$ is G.I.T. stable if and only if the corresponding cycle is a pseudo-stable curve of genus g. Furthermore, he showed that there are no strictly semi-stable points. In fact, he proves that the coarse moduli scheme $Chow_3//SL(5g-5)$ is isomorphic to the coarse moduli scheme $\overline{\mathrm{M}}_g^{ps}$ ([S], Theorem 5.4).

Hassett and Hyeon have identified $\overline{\mathrm{M}}_g^{ps} \cong \overline{\mathrm{M}}_g(9/11)$ as the first divisorial contraction that occurs in Hassett's program sketched above ([HH2]).

Theorem 10.3 ([HH2], Theorem 1.1)**.** *There is a morphism of stacks $\phi : \overline{\mathcal{M}}_g \to \overline{\mathcal{M}}_g^{ps}$ which is an isomorphism in the complement of the boundary divisor δ_1. $\overline{M}_g(9/11)$ is isomorphic to the coarse moduli scheme $\overline{M}_g^{ps}$ and is the divisorial contraction of $\overline{M}_g$ contracting Δ_1.*

Exercise 10.4. Using the Theorem 6.2 of Cornalba and Harris that $K_{\overline{\mathrm{M}}_g} + a\delta$ is ample for $1 \geq a > 9/11$. Take a pencil of cubic curves in $\mathbb{P}^2$. Attach a fixed pointed stable curve of genus $g-1$ at a base point of the pencil of cubic curves. Show that the induced curve C in $\overline{\mathcal{M}}_g$ has $C \cdot 11\lambda - \delta = 0$. Show since curves in the class cover Δ_1, Δ_1 is in the stable base locus of the divisor $(11-\epsilon)\lambda - \delta$ for $1 >> \epsilon > 0$. Conversely, show that Δ_1 is the only stable base locus. Theorem 1.1 of [HH2] gives a complete description of the corresponding divisorial contraction.

Furthermore, Hassett and Hyeon show that the model remains the same until $a = 7/10$.

Theorem 10.5 ([HH2], Theorem 1.2)**.** *For $7/10 < \alpha \leq 9/11$, $\overline{M}_g(\alpha)$ exists as a projective variety and is isomorphic to $\overline{M}_g^{ps}$.*

in [HH1], Hassett and Hyeon analyze the model $\overline{\mathrm{M}}_g(7/10)$. Even a summary of their results is beyond the scope of these notes. However, we state their main theorem.

Definition 10.6. An *elliptic bridge* is a connected subcurve of arithmetic genus one meeting the rest of the curve in two nodes. An *open elliptic chain of length r* is a two-pointed projective curve (C, p, q) such that C consists of a chain $E_1 \cup \cdots \cup E_r$ of r genus one curves with nodes, cusps and tacnodes as singularities. E_i and E_{i+1} intersect in a single tacnode for $r-1 \geq i \geq 1$ and non-consecutive curves do not intersect. p is a smooth point of E_1 and q is a smooth point of E_r and $\omega_C(p+q)$ is ample.

Definition 10.7. Let C be a projective, connected curve of arithmetic genus $g \geq 3$ with nodes, cusps and tacnodes as singularities. *C admits an open elliptic chain* if there exists a subcurve C' of C joined to the rest of the curve only at two nodes p and q such that (C', p, q) is an open elliptic chain. If one of the points, say p, is a tacnode instead of a node, we say *C admits a weak elliptic chain.* If $p = q$, the chain is called a *closed elliptic chain.* If $p = q$ is a tacnode of C, then C is called a *closed weak elliptic chain.*

Definition 10.8. A complete curve C is *c-semi-stable* if

1. C has nodes, cusps and tacnodes as singularities;
2. ω_C is ample;
3. a connected genus one subcurve meets the rest of the curve in at least two distinct points.

C is *c-stable* if it is c-semi-stable and has no tacnodes or elliptic bridges.

Definition 10.9. A complete curve of genus g is *h-semi-stable* if it is c-semi-stable and admits no elliptic chains. It is *h-stable* if it is h-semi-stable and admits no weak elliptic chains.

Let $Chow_{g,2}$ and $Hilb_{g,2}$ denote the closure of the loci of bi-canonically embedded curves of genus g in the Chow variety and the Hilbert scheme, respectively. Let

$$\overline{\mathrm{M}}_g^{cs} = Chow_{g,2}//SL(3g-3)$$

and let

$$\overline{\mathrm{M}}_g^{hs} = Hilb_{g,2}//SL(3g-3).$$

Hassett and Hyeon explicitly analyze the semi-stable locus of the $SL(3g-3)$ action for both quotients. They show that the (semi-)stable locus in the Chow quotient correspond to the bi-canonically embedded c-(semi-)stable curves. The (semi)-stable locus in the Hilbert quotient correspond to bi-canonically embedded h-(semi-)stable curves. They obtain the following characterization of the log canonical models of $\overline{\mathrm{M}}_g$.

Theorem 10.10 ([HH1], Theorem 2.12)**.** *For rational numbers* $7/10 < a \leq 9/11$, *there exists a small contraction*

$$\Psi : \overline{M}_g(a) \to \overline{M}_g(7/10).$$

Then

$$\overline{M}_g(7/10) \cong \overline{M}_g^{cs}$$

and

$$\overline{M}_g(7/10-\epsilon) \cong \overline{M}_g^{hs}$$

such that the induced morphism

$$\Psi^+ : \overline{M}_g(7/10-\epsilon) \to \overline{M}_g(7/10)$$

is the flip of Ψ *for small rational* $\epsilon > 0$.

Remark 10.11. In some small genera, Hassett's program has been completely carried out. For example, in genus three Hyeon and Lee [HL] prove the following beautiful theorem.

Theorem 10.12 (Hyeon-Lee, [HL]).

(1) *There is a small contraction*

$$\Psi : \overline{M}_3^{ps} \to \overline{M}_3(7/10)$$

that contracts the locus of elliptic bridges. $\overline{M}_3(7/10)$ *is isomorphic to the GIT quotient* $Chow_{3,2}//SL(6)$ *of the Chow variety of bicanonical curves.*

(2) *There exists a flip* $\Psi^+ : (\overline{M}_3^{ps})^+ \to \overline{M}_3(7/10)$ *and* $(\overline{M}_3^{ps})^+$ *is isomorphic to* $\overline{M}_3(\alpha)$ *for* $17/28 < \alpha < 7/10$. *Moreover, for* α *in this range,* $\overline{M}_3(\alpha)$ *is isomorphic to the GIT quotient* $Hilb_{3,2}//SL(6)$ *of the Hilbert scheme of bicanonical curves.*

(3) *Finally, there exists a divisorial contraction that contracts the divisor of hyperelliptic curves* D_{hyp}. *The log canonical model is isomorphic to* $\mathbb{P}(\Gamma(\mathcal{O}_{\mathbb{P}^2}(4)))//SL(3)$.

Unfortunately, the GIT analysis required to prove these theorems are beyond the scope of these notes. We conclude by mentioning some open problems.

Problem 10.13. Determine the stable base locus of the canonical linear system on $\overline{\mathrm{M}}_g$.

The two steps of Hassett's program sketched above might give the mistaken impression that these birational models involve only changing the boundary of the moduli space. In fact, there are many loci in the locus of smooth curves that are contained in the stable base locus of the canonical linear series. Hence, the canonical model is not a compactification of M_g.

Exercise 10.14. Let $g \geq 3$, show that the locus of hyperelliptic curves is in the base locus of $\overline{\mathrm{M}}_g$.

Problem 10.15. For a fixed g, determine which k-gonal loci are in the stable base locus of the canonical linear system.

Problem 10.16. Determine the stable base locus decomposition of the effective cone of $\overline{\mathrm{M}}_g$.

Problem 10.17. Determine the canonical model of $\overline{\mathrm{M}}_g$.

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