Chapter 5: Consistency and Limiting Distributions

A sequence X_n converges to X

($a_n \to a$ convergence of #'s)

(1) $X_n \to X$ as a sequence of #'s.
(not easy to achieve)

(2) $\mathbb{P}\{ |X_n - X| < \varepsilon \} \to 0$ as $n \to \infty$

Convergence in probability.

(3) When it even does not make sense to ask $|X_n - X| < \varepsilon$, we ask the distribution of X_n is getting close to that of X. i.e.

$$F_{X_n}(x) \to F_X(x) \quad \text{as } n \to +\infty$$
$$\text{for all } x \in C(F)$$

<u>Remark</u>: In our discussion, X is mainly a continuous r.v. hence, the above becomes

$$F_{X_n}(x) \to F_X(x) \quad \text{for all } x \in \mathbb{R}$$

Friday, December 02, 2016
9:33 AM

Also, now $F_{X_n}(x) \to F_X(x)$ $\forall x \in \mathbb{R}$ implies

$$\mathbb{P}\{a < X_n \le b\} \to \mathbb{P}\{a < X \le b\}$$

$$F(b) - F(a) \qquad\qquad F(b) - F(a)$$

Eg: $\bar{X}_n \to \mu$ i.p.

Eg: $\sqrt{n}\left(\frac{\bar{X}_n - \mu}{\sigma}\right) \to N(0,1)$ i.d.

Properties of convergence i.p.

(1) $X_n \xrightarrow{P} X$, $Y_n \xrightarrow{P} Y \Rightarrow X_n + Y_n \xrightarrow{P} X+Y$

(2) $X_n \xrightarrow{P} X$, $\Rightarrow aX_n \to aX$

(3) $X_n \xrightarrow{P} X$, $Y_n \xrightarrow{P} Y \Rightarrow X_nY_n \xrightarrow{P} XY$

(4) $X_n \xrightarrow{P} X$, $g \in C(\mathbb{R})$, $g(X_n) \xrightarrow{P} g(X)$

Properties of convergence i.d.

(1) $X_n \xrightarrow{D} X$. $g \in C(\mathbb{R})$, $g(X_n) \xrightarrow{D} g(X)$

(2) $X_n \xrightarrow{D} X$, $Y_n \xrightarrow{D} y \Rightarrow X_n + Y_n \to X + y$

$X_n \cdot Y_n \to Xy$

Friday, December 02, 2016
9:33 AM

But in general the following is not true

$X_n \xrightarrow{D} X$, $Y_n \xrightarrow{D} Y$ then $X_n + Y_n \xrightarrow{D} X+Y$

and $X_n Y_n \xrightarrow{D} XY$

Relation between the two convergence:

$$X_n \xrightarrow{P} X \quad \Rightarrow \quad X_n \xrightarrow{D} X$$

but $X_n \xrightarrow{D} X \quad \not\Rightarrow \quad X_n \xrightarrow{P} X$

$\underline{Eg}$: $\underbrace{\qquad}_{\text{how to use law of large \#'s}}$ $\bar{X}_n \to \mu$ i.p.

Hence $\bar{X}_n(1-\bar{X}_n) \xrightarrow{P} \mu(1-\mu)$

$Y_n \sim b(n, p)$

then $\dfrac{Y_n}{n} \xrightarrow{P} p$ $\qquad \left(\dfrac{Y_n}{n}\right)^2 \xrightarrow{P} p^2$ etc.

$\underline{Eg}$: How to use CLT.

two examples that I showed in class.

Friday, December 02, 2016
9:33 AM

Chapter 3. Some Special Distributions.

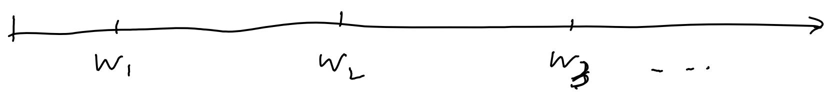
- Bernoulli, Binomial
- Poisson
- Γ
- Normal

Need to know pdf/pmf, mgf, μ, σ^2; all examples in class for computations.

Poisson: $X_1, \dots, X_n$ — Poisson $\mu_1, \dots, \mu_n$ indpt

then $X_1 + \dots + X_n \sim$ Poisson $\mu_1 + \dots + \mu_n$

Relation between Γ and Poisson.



$X_t \sim$ # of events from time 0 to time t.

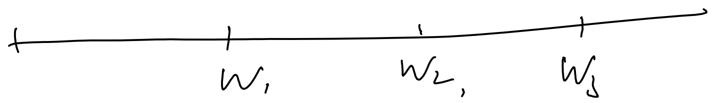
X_t is Poisson with parameter μt

Let W_k be the time that the k-th event happens then

W_k — is gamma with $\alpha = k$ and $\beta = \frac{1}{\mu}$

Friday, December 02, 2016
9:33 AM

In particular



W_1 — exponential distribution.

Indeed, W_1, $W_2 - W_1$, $W_3 - W_2$, ...

are i.i.d exponentials.

This can be confirmed by

<u>Thm</u>: $X_1, X_2, \dots, X_n$ indpt Γ

$\Gamma(\alpha_1, \beta) \quad \Gamma(\alpha_2, \beta) \quad \dots \quad \Gamma(\alpha_n, \beta)$

then $X_1 + X_2 + \dots + X_n \sim \Gamma(\alpha_1 + \dots + \alpha_n, \beta)$

<u>Normal</u>: (Need to know the computations and how to check the table)

<u>Th</u>: $X_1, X_2, \dots, X_n$ indpt.

$X_i \sim N(\mu_i, \sigma_i^2)$

$$a_1X_1 + \dots + a_nX_n \sim N\left(\sum a_i\mu_i, \sum a_i^2\sigma_i^2\right)$$

Friday, December 02, 2016
9:34 AM

Multivariate normal. — Only bivariate normal (X_1, X_2)

Distribution of $\begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$ is determined by

$$\vec{\mu} = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} \quad \text{and} \quad \Sigma \text{ — covariance matrix.}$$

$$\Sigma = \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix}, \quad \sigma_{12} = \rho\sigma_1\sigma_2$$

Two types of questions

1) Given $\vec{\mu}$, Σ, what is the marginal of (X_1, X_2)

$X_1 \sim N(\mu_1, \sigma_1^2)$

$X_2 \sim N(\mu_2, \sigma_2^2)$

2) Given $\vec{\mu}$, Σ, what is the conditional distribution of X_1 given X_2.

Important : For multivariate normal $\begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$ it is true that if
$\mathbb{E} X_1 X_2 = 0$
then X_1, X_2 indpt.