Stat 401 Solution for HW1

§1.2: 1) (a) $C_1=\{0,1,2\}$. $C_2=\{2,3,4\}$

$C_1\cup C_2=\{0,1,2,3,4\}$; $C_1\cap C_2=\{2\}$.

(b) $C_1=\{x: 0<x<2\}$. $C_2=\{x: 1\le x<3\}$.

$C_1\cup C_2=\{x: 0<x<3\}$; $C_1\cap C_2=\{x: 1\le x<2\}$

(c) $C_1=\{(x,y): 0<x<2,\ 1<y<2\}$, $C_2=\{(x,y): 1<x<3,\ 1<y<3\}$

$C_1\cup C_2=\{(x,y): 0<x<3,\ 1<y<3\}$

$C_1\cap C_2=\{(x,y): 1<x<2,\ 1<y<2\}$.

11). • $Q(C_1)=\int_{\frac14}^{\frac34} 6x(1-x)\,dx=\left[6\cdot\frac12x^2-6\cdot\frac13x^3\right]\Big|_{\frac14}^{\frac34}=(3x^2-2x^3)\Big|_{\frac14}^{\frac34}=\frac{11}{16}$

• $Q(C_2)=\int_{\frac12}^{\frac12} 6x(1-x)\,dx=0$

• $Q(C_3)=\int_0^{10} 6x(1-x)\,dx=\int_0^1 6x(1-x)\,dx=1$ because $f(x)=0$ when $1\le x<10$.

12). $Q(C)=\iint_C (x^2+y^2)\,dx\,dy$

• $C_1=\{(x,y): -1\le x\le 1,\ -1\le y\le 1\}$

$Q(C_1)=\int_{-1}^1\int_{-1}^1 (x^2+y^2)\,dx\,dy=\int_{-1}^1\left(\left[x^2y+\frac13y^3\right]\Big|_{y=-1}^{y=1}\right)dx=\int_{-1}^1\left(2x^2+\frac23\right)dx=\left(\frac{2x^3}{3}+\frac23x\right)\Big|_{-1}^{1}=\frac83$

• $C_2=\{(x,y): -1\le x=y\le 1\}$

$Q(C_2)=\int_{-1}^1\Big[\underbrace{\int_x^x (x^2+y^2)\,dy}_{="0"}\Big]dx=0$

(It's a line so 2nd integral of line is 0)

• $C_3=\{(x,y): x^2+y^2\le 1\}$

$$Q(C_3)=\iint_{C_3}(x^2+y^2)\,dx\,dy\ \overset{\text{Let } x=r\cos\theta,\ y=r\sin\theta}{\underset{dx\,dy=r\,dr\,d\theta}{=\!=\!=}}\ \int_0^{2\pi}\int_0^1 r^2\cdot r\,dr\,d\theta$$

$$=\int_0^{2\pi}\left[\frac14r^4\Big|_0^1\right]d\theta$$

$$=\int_0^{2\pi}\frac14\,d\theta$$

$$=\frac14\theta\Big|_0^{2\pi}=\frac{\pi}{2}$$

15) $C = \{(x, y, z): \cancel{0<} x^2+y^2+z^2 \le 1\}$.

let $x = r\sin\theta\cos\phi$
$y = r\sin\theta\sin\phi$
$z = r\cos\theta$

so: $x^2+y^2+z^2 = (r\sin\theta\cos\phi)^2 + (r\sin\theta\sin\phi)^2 + (r\cos\theta)^2$
$= r^2(\underbrace{\sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi} + \cos^2\theta)$
$= r^2(\sin^2\theta + \cos^2\theta)$
$= r^2$

$dx\,dy\,dz = r^2\sin\theta\,dr\,d\theta\,d\phi$;

$\Rightarrow Q(C) = \iiint_C r \cdot r^2 \sin\theta\,dr\,d\theta\,d\phi$

$= \int_0^{2\pi}\int_0^{\pi}\int_0^1 r^3\sin\theta\,dr\,d\theta\,d\phi$

$= \int_0^{2\pi}\int_0^{\pi} \cdot \sin\theta \cdot (\frac{1}{4}r^4\Big|_0^1)\,d\theta\,d\phi$

$= \frac{1}{4}\int_0^{2\pi}\int_0^{\pi} \sin\theta\,d\theta\,d\phi$

$= \frac{1}{4}\int_0^{2\pi} \cdot (-\cos\theta)\Big|_0^{\pi}\,d\phi$

$= \frac{1}{4} \cdot \int_0^{2\pi} \cdot 2\,d\phi = \frac{1}{4} \cdot 2 \cdot (2\pi - 0) = \pi$

§1.3: 1) $C_1 = \{1, 2, 3, 4\}$. $C_2 = \{3, 4, 5, 6\} \Rightarrow C_1 \cup C_2 = \{1, 2, 3, 4, 5, 6\}$. $C_1 \cap C_2 = \{3, 4\}$

so $P(C_1) = \frac{1}{6}+\frac{1}{6}+\frac{1}{6}+\frac{1}{6} = \frac{2}{3}$ $\quad P(C_1 \cap C_2) = \frac{1}{6}+\frac{1}{6} = \frac{1}{3}$

$P(C_2) = \frac{1}{6}+\frac{1}{6}+\frac{1}{6}+\frac{1}{6} = \frac{2}{3}$

$P(C_1 \cup C_2) = P(C_1) + P(C_2) - P(C_1 \cap C_2) = \frac{2}{3}+\frac{2}{3}-\frac{1}{3} = 1$

2). $C_1 = \{13 \text{ hearts}\}$. $C_2 = \{4 \text{ kings}\}$

so $C_1 \cap C_2 = \{\text{the king in heart}\}$

$P(C_1) = 13 \times \frac{1}{52} = \frac{1}{4}$

$P(C_2) = 4 \times \frac{1}{52} = \frac{1}{13}$

$P(C_1 \cap C_2) = \frac{1}{52}$

$P(C_1 \cup C_2) = P(C_1) + P(C_2) - P(C_1 \cap C_2) = \frac{1}{4} + \frac{1}{13} - \frac{1}{52} = \frac{16}{52} = \frac{4}{13}$

4) $C = C_1 \cup C_2 \quad P(C) = P(C_1 \cup C_2) = 1$

$P(C_1) = 0.8$; $P(C_2) = 0.5$;

$$P(C_1 \cup C_2) = P(C_1) + P(C_2) - P(C_1 \cap C_2)$$
$$1 = 0.8 + 0.5 - P(C_1 \cap C_2)$$
$$P(C_1 \cap C_2) = \boxed{0.3}$$

5) $C_1 = \{c: 0 < c < \infty\}$ so $P(C_1) = \int_{C_1} e^{-x}\,dx = \int_0^\infty e^{-x}\,dx = -e^{-x}\Big|_0^\infty = 0 - (-1) = 1$

$C = \{c: 4 < c < \infty\}$: $P(C) = \int_C e^{-x}\,dx = \int_4^\infty e^{-x}\,dx = -e^{-x}\Big|_4^\infty = 0 - (-e^{-4}) = e^{-4}$

$C^c = \{c: 0 < c \le 4\}$: $P(C^c) = \int_{C^c} e^{-x}\,dx = \int_0^4 e^{-x}\,dx = -e^{-x}\Big|_0^4 = -e^{-4} - (-1) = 1 - e^{-4}$

$$P(C \cup C^c) = P(C) + P(C^c) - P(C \cap C^c)$$
$$= P(C) + P(C^c) - P(\phi)$$
$$= e^{-4} + (1 - e^{-4}) - 0$$
$$= 1$$

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