§2.7: 1)

$$\begin{cases} Y_1 = \dfrac{X_1}{X_1+X_2} \\ Y_2 = \dfrac{X_1+X_2}{X_1+X_2+X_3} \\ Y_3 = X_1+X_2+X_3 \end{cases} \Rightarrow \begin{cases} x_1 = y_1 y_2 y_3 \\ x_2 = y_2 y_3 - y_1 y_2 y_3 \\ x_3 = y_3 - y_2 y_3 \end{cases} \Rightarrow |J| = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} & \frac{\partial x_1}{\partial y_3} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} & \frac{\partial x_2}{\partial y_3} \\ \frac{\partial x_3}{\partial y_1} & \frac{\partial x_3}{\partial y_2} & \frac{\partial x_3}{\partial y_3} \end{vmatrix} = \begin{vmatrix} y_2y_3 & y_1y_3 & y_1y_2 \\ -y_2y_3 & y_3-y_1y_3 & y_2-y_1y_2 \\ 0 & -y_3 & 1-y_2 \end{vmatrix}$$

$$= -(-y_3)\begin{vmatrix} y_2y_3 & y_1y_2 \\ -y_2y_3 & y_2-y_1y_2 \end{vmatrix} + (1-y_2)\begin{vmatrix} y_2y_3 & y_1y_3 \\ -y_2y_3 & y_3-y_1y_3 \end{vmatrix}$$

$$= y_3\cdot\left[y_2y_3\cdot y_2(1-y_1) + y_1y_2\cdot y_2y_3\right] + (1-y_2)\left[y_2y_3\cdot y_3(1-y_1) + y_1y_3\cdot y_2y_3\right]$$

$$= y_2 y_3^2$$

Also: since $X_1, X_2, X_3 \overset{iid}{\sim} f(x) = e^{-x}$, $0<x<\infty$ $\Rightarrow f(x_1,x_2,x_3) = f(x_1)\cdot f(x_2)\cdot f(x_3) = e^{-x_1-x_2-x_3}$;

- For the distribution of Y_1, Y_2, Y_3:

$f(y_1,y_2,y_3) = |J|\, f(x_1,x_2,x_3) = y_2 y_3^2\cdot e^{-y_3}$; with $0<y_1<1$; $0<y_2<1$; $0<y_3<\infty$;

- For marginal distribution of Y_1, Y_2, Y_3:

$f(y_1) = \int_0^\infty\int_0^1 y_2 y_3^2 e^{-y_3}\, dy_2\, dy_3 = \frac{1}{2}\int_0^\infty y_3^2 e^{-y_3} dy_3 = -\frac{1}{2}\int_0^\infty y_3^2\, de^{-y_3} = -\left[\frac{1}{2} y_3^2 e^{-y_3}\Big|_0^\infty - \int_0^\infty e^{-y_3}\, dy_3\right]$

$= \int_0^\infty e^{-y_3} dy_3 = -e^{-y_3}\Big|_0^\infty = 1$; $0<y_1<1$;

$f(y_2) = \int_0^\infty\int_0^1 y_2 y_3^2 e^{-y_3}\, dy_1\, dy_3 = y_2\int_0^\infty y_3^2 e^{-y_3} dy_3 = 2y_2$; $0<y_2<1$;

$f(y_3) = \int_0^1\int_0^1 y_2 y_3^2 e^{-y_3}\, dy_1 dy_2 = y_3^2 e^{-y_3}\int_0^1 y_2\, dy_2 = \frac{1}{2} y_3^2 e^{-y_3}$; $0<y_3<\infty$;

$\Rightarrow f(y_1,y_2,y_3) = y_2 y_3^2 e^{-y_3} = (1)\cdot(2y_2)\cdot(\frac{1}{2}y_3^2 e^{-y_3}) = f(y_1)\cdot f(y_2)\cdot f(y_3) \Rightarrow$ Y_1, Y_2, Y_3 mutually indpt;

4) $X_1, X_2, X_3 \overset{iid}{\sim} f(x) = e^{-x}$, $x>0$ $\Rightarrow f(x_1,x_2,x_3) = e^{-x_1-x_2-x_3}$; $x_1, x_2, x_3 > 0$;

$$\begin{cases} Y_1 = X_1 \\ Y_2 = X_1+X_2 \\ Y_3 = X_1+X_2+X_3 \end{cases} \Rightarrow \begin{cases} x_1 = y_1 \\ x_2 = y_2 - y_1 \\ x_3 = y_3 - y_2 \end{cases} \Rightarrow |J| = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} & \frac{\partial x_1}{\partial y_3} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} & \frac{\partial x_2}{\partial y_3} \\ \frac{\partial x_3}{\partial y_1} & \frac{\partial x_3}{\partial y_2} & \frac{\partial x_3}{\partial y_3} \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{vmatrix} = 1\cdot\begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix} = 1$$

so the joint distribution of Y_1, Y_2, Y_3 is:

$f(y_1,y_2,y_3) = f(x_1,x_2,x_3)\cdot|J| = e^{-y_3}\cdot 1 = e^{-y_3}$; $y_1, y_2, y_3 > 0$;

§2.8: 2) $X_1, X_2, X_3, X_4 \overset{iid}{\sim} f(x) = 2x$, $0<x<1$.

$\mathbb{E}X = \int_0^1 x\cdot f(x)\, dx = \int_0^1 x\cdot 2x\, dx = 2\cdot\frac{1}{3}x^3\Big|_0^1 = \frac{2}{3}$; $\mathbb{E}X^2 = \int_0^1 x^2 f(x)\, dx = \int_0^1 2x^3 dx = 2\cdot\frac{1}{4}x^4\Big|_0^1 = \frac{1}{2}$;

$Var(X) = \mathbb{E}X^2 - (\mathbb{E}X)^2 = \frac{1}{2} - \left(\frac{2}{3}\right)^2 = \frac{1}{18}$;

Let $Y = \sum_i X_i = X_1+X_2+X_3+X_4$

so $\mathbb{E}Y = \mathbb{E}X_1 + \mathbb{E}X_2 + \mathbb{E}X_3 + \mathbb{E}X_4 = \frac{2}{3}\cdot 4 = \frac{8}{3}$;

$Var(Y) = \sum_i Var(X_i) = 4\cdot\frac{1}{18} = \frac{2}{9}$;

5) $X_1, \cdots, X_5 \overset{iid}{\sim} f(x) = 6x(1-x);\ 0<x<1$.

$\mathbb{E}X = \int_0^1 x f(x)\,dx = \int_0^1 6x^2 - 6x^3\,dx = (2x^3 - \frac{3}{2}x^4)\Big|_0^1 = \frac{1}{2}$; $\mathbb{E}X^2 = \int_0^1 x^2 f(x)\,dx = \int_0^1 6x^3 - 6x^4\,dx = (\frac{3}{2}x^4 - \frac{6}{5}x^5)\Big|_0^1 = \frac{3}{10}$

$Var(X) = \mathbb{E}X^2 - (\mathbb{E}X)^2 = \frac{3}{10} - (\frac{1}{2})^2 = \frac{1}{20}$;

so $Y = \sum_{i=1}^5 X_i \Rightarrow \mathbb{E}Y = \sum_{i=1}^5 \mathbb{E}(X_i) = \frac{1}{2}\times 5 = \frac{5}{2}$; $Var(Y) = \sum_{i=1}^5 Var(X_i) = \frac{1}{20}\cdot 5 = \frac{1}{4}$;

6) $X_1, \cdots, X_5 \overset{iid}{\sim} f(x) = 4x^3, 0<x<1$.

$\mathbb{E}X = \int_0^1 x\cdot 4x^3\,dx = 4\cdot\frac{1}{5}x^5\Big|_0^1 = \frac{4}{5}$; $\mathbb{E}X^2 = \int_0^1 x^2 4x^3\,dx = 4\cdot\frac{1}{6}x^6\Big|_0^1 = \frac{2}{3}$; $Var(X) = \mathbb{E}X^2 - (\mathbb{E}X)^2 = \frac{2}{3} - (\frac{4}{5})^2 = \frac{2}{75}$

so $\bar{X} = \frac{1}{5}\sum_{i=1}^5 X_i \Rightarrow \mathbb{E}(\bar{X}) = \frac{1}{5}\sum_{i=1}^5 \mathbb{E}(X_i) = \frac{1}{\cancel{5}}\cdot\frac{4}{5}\cdot\cancel{5} = \frac{4}{5}$; $Var(\bar{X}) = \frac{1}{25}\sum_{i=1}^5 Var(X_i) = \frac{1}{25}\cdot\frac{2}{75}\cdot 5 = \frac{2}{375}$.

§3.1: 4) $X_1, X_2, X_3 \overset{iid}{\sim} f(x) = 3x^2, 0<x<1$.

Let p stand for the prob that the variable exceed ½, that is:

$p = P(X_i > \frac{1}{2}) = \int_{\frac{1}{2}}^1 3x^2\,dx = x^3\Big|_{\frac{1}{2}}^1 = \frac{7}{8}$; so we have binomial dist: $X \sim Binom(3, \frac{7}{8})$;

then the prob that two of three variables exceeds ½ is: $P(X=2) = \binom{3}{2}p^2(1-p) = \binom{3}{2}(\frac{7}{8})^2(\frac{1}{8}) = \frac{147}{512}$;

5) Y is the number of successes in n indpt repetitions of a random experiment.

so $Y \sim B(n, p) = B(n, \frac{2}{3})$

① when $n=3$: $Y \sim B(3, \frac{2}{3})$: $P(2 \le Y) = P(Y=2) + P(Y=3) = \binom{3}{2}(\frac{2}{3})^2(\frac{1}{3}) + \binom{3}{3}(\frac{2}{3})^3(\frac{1}{3})^0 = \frac{20}{27}$.

② when $n=5$: $Y \sim B(5, \frac{2}{3})$: $P(3 \le Y) = P(Y=3) + P(Y=4) + P(Y=5) = \binom{5}{3}(\frac{2}{3})^3(\frac{1}{3})^2 + \binom{5}{4}(\frac{2}{3})^4(\frac{1}{3}) + \binom{5}{5}(\frac{2}{3})^5 = \frac{64}{81}$

6) $Y \sim B(n, \frac{1}{4})$.

$P(1 \le Y) = 1 - P(Y=0) = 1 - \binom{n}{0}(\frac{1}{4})^0(\frac{3}{4})^n = 1 - (\frac{3}{4})^n \ge 0.70 \Rightarrow (\frac{3}{4})^n \le 0.30 \Rightarrow n=5$ is the smallest value