Stat 401. §5.1: 2, 3, 5; §5.2: 1, 2

§5.1: 2) $Y_n \sim b(n,p)$. $\sigma = \sqrt{np(1-p)}$

(a) To prove $\frac{Y_n}{n} \xrightarrow{P} p$: For all $\varepsilon > 0$: $\mathbb{P}(|\frac{Y_n}{n} - p| \geq \varepsilon) = \mathbb{P}(|Y_n - np| \geq n\varepsilon)$

$$= \mathbb{P}(|Y_n - np| \geq \frac{n\varepsilon}{\sqrt{np(1-p)}} \cdot \sqrt{np(1-p)})$$

$$\overset{\text{Chebyshev's}}{\leq} \frac{np(1-p)}{n^2\varepsilon^2} = \frac{p(1-p)}{n\varepsilon^2} \to 0 \text{ as } n \to \infty$$

(b) To prove $1 - \frac{Y_n}{n} \xrightarrow{P} 1-p$: By theorem 5.1.4;

Let $g(x) = 1-x$. is a continuous function at p.

Since $\frac{Y_n}{n} \xrightarrow{P} p$; we have $g(\frac{Y_n}{n}) \xrightarrow{P} g(p)$. i.e. $1 - \frac{Y_n}{n} \xrightarrow{P} 1-p$;

(c) To prove $\frac{Y_n}{n} \cdot (1 - \frac{Y_n}{n}) \xrightarrow{P} p(1-p)$:

From theorem 5.1.5 & Part (a), (b); we will have $\frac{Y_n}{n}(1-\frac{Y_n}{n}) \xrightarrow{P} p(1-p)$; #

3) $W_n \sim (\mu, \frac{b}{n^p})$, $p > 0$. To show $W_n \xrightarrow{P} \mu$; By chebyshev's inequality $\mathbb{P}(|X-\mu| \geq k\sigma) \leq \frac{1}{k^2}$.

For all $\varepsilon > 0$: $\mathbb{P}(|W_n - \mu| \geq \varepsilon) = \mathbb{P}(|W_n - \mu| \geq \varepsilon\sqrt{\frac{n^p}{b}} \cdot \sqrt{\frac{b}{n^p}}) \leq \frac{1}{(\varepsilon\sqrt{\frac{n^p}{b}})^2} = \frac{b}{n^p\varepsilon^2} \to 0$ as $n \to \infty$ #

5) $X_1, \cdots, X_n \overset{iid}{\sim} f(x) = e^{-(x-\theta)}$, $x > \theta$. $Y_n = \min\{X_1, \cdots, X_n\}$.

First, the cdf of Y_n: $F_X(x) = \mathbb{P}(X \leq x) = \int_\theta^x e^{-(t-\theta)}dt = e^\theta \cdot [-e^{-t}|_\theta^x] = 1 - e^{-(x-\theta)}$, $x > \theta$

$$F_{Y_n}(y) = \mathbb{P}(Y_n \leq y) = 1 - \mathbb{P}(Y_n > y) = 1 - \mathbb{P}(X_1 > y, \cdots, X_n > y)$$

$$\overset{iid}{=} 1 - [\mathbb{P}(X_i > y)]^n$$

$$= 1 - [1 - F_X(y)]^n$$

$$= 1 - [1 - (1 - e^{-(y-\theta)})]^n$$

$$= 1 - e^{-n(y-\theta)} ; \quad y > \theta;$$

Then; to show $Y_n \xrightarrow{P} \theta$, for all $\varepsilon > 0$:

$$\mathbb{P}(|Y_n - \theta| \geq \varepsilon) \overset{Y_n > \theta}{=} \mathbb{P}(Y_n - \theta \geq \varepsilon) = \mathbb{P}(Y_n \geq \theta + \varepsilon)$$

$$= 1 - [1 - e^{-n(\theta + \varepsilon - \theta)}]$$

$$= e^{-n\varepsilon} \to 0 \text{ as } n \to \infty. \quad \#$$

§5.2: 1) $\bar{X}_n = \frac{1}{n}\sum X_i$. $X_i \sim N(\mu, \sigma^2)$.

$$M_{\bar{X}_n}(s) = \mathbb{E}(e^{s\bar{X}_n}) = \mathbb{E}[\exp(\frac{s}{n}\sum_{i=1}^n X_i)]$$

$$= \mathbb{E}[\prod_{i=1}^n \exp(\frac{s}{n}X_i)] = [\mathbb{E}(\exp(\frac{s}{n}X_i))]^n = [\exp(\mu \cdot \frac{s}{n} + \frac{\sigma^2}{2}\frac{s^2}{n^2})]^n$$

$$= \exp(\mu s + \frac{\sigma^2 s^2}{2n}) \xrightarrow{n \to \infty} \exp(\mu s)$$

$$= M_\mu(s)$$

Hence, $\bar{X}_n \xrightarrow{D} \mu$; #

2) $Y_1 = \min\{X_1, \cdots, X_n\}$. $X_i \overset{iid}{\sim} f(x) = e^{-(x-\theta)}$, $\theta < x$. $Z_n = n(Y_1 - \theta)$.

To find the limiting distribution of Z_n,

From §5.1. problem 5), CDF of Y_1 is $F_{Y_1}(y_1) = 1 - e^{-n(y_1-\theta)}$; $y_1 > \theta$.

$\mathbb{P}(Z_n \le z) = \mathbb{P}(n(Y_1-\theta) \le z)$

$= \mathbb{P}(Y_1 \le \theta + \frac{z}{n}) = 1 - e^{-n(\theta + \frac{z}{n} - \theta)} = 1 - e^{-z}$. $z > 0$

So the limiting distribution of Z_n is:

$\lim_{n\to\infty} F_{Z_n}(z) = 1 - e^{-z}$; $z > 0$;

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