Stat 401 HW3 Solution

§1.5 : 2) (a) $p(x) = c\left(\frac{2}{3}\right)^x$, $x = 1, 2, 3, \cdots$, zero elsewhere.

To be a pmf, one should have $\sum_{x=1}^{\infty} p(x) = 1$.

$$\sum_{x=1}^{\infty} c\left(\frac{2}{3}\right)^x = c\left[\sum_{x=1}^{\infty}\left(\frac{2}{3}\right)^x\right] = c\left[\left(\frac{2}{3}\right) + \left(\frac{2}{3}\right)^2 + \cdots\right] = c \cdot \frac{\frac{2}{3}}{1-\frac{2}{3}} = 2c = 1 \Rightarrow \boxed{c = \frac{1}{2}}$$

(b) $p(x) = cx$, $x = 1, 2, 3, 4, 5, 6$, zero elsewhere.

$$\sum_{x=1}^{6} p(x) = \sum_{x=1}^{6} (cx) = c(1+2+3+4+5+6) = 21c = 1 \Rightarrow \boxed{c = \frac{1}{21}}$$

5) Select 5 cards at random without replacement:

(a) $p(x) = \dfrac{\binom{13}{x}\binom{39}{5-x}}{\binom{52}{5}}$; $x = 0, 1, 2, 3, 4, 5$; zero elsewhere.

(b) $$P(X \le 1) = P(X=0) + P(X=1) = \frac{\binom{13}{0}\binom{39}{5}}{\binom{52}{5}} + \frac{\binom{13}{1}\binom{39}{4}}{\binom{52}{5}} = \frac{\binom{39}{5} + \binom{13}{1}\binom{39}{4}}{\binom{52}{5}}$$

6) $f(x) = \frac{2}{9}x$, $x \in D = \{x : 0 < x < 3\}$

$P_X(D) = \int_D f(x)\,dx$

$D_1 = \{x : 0 < x < 1\}$: $P_X(D_1) = \int_0^1 \frac{2}{9}x\,dx = \frac{2}{9} \cdot \frac{1}{2}x^2\Big|_0^1 = \boxed{\frac{1}{9}}$

$D_2 = \{x : 2 < x < 3\}$: $P_X(D_2) = \int_2^3 \frac{2}{9}x\,dx = \frac{2}{9} \cdot \frac{1}{2}x^2\Big|_2^3 = \boxed{\frac{5}{9}}$.

$P_X(D_1 \cup D_2) = \{x : P(D_1) + P(D_2) - P(D_1 \cap D_2) = \frac{1}{9} + \frac{5}{9} - 0 = \frac{6}{9} = \boxed{\frac{2}{3}}$

7) $D = \{x : 0 < x < 1\}$; $D_1 = \{x : 0 < x < \frac{1}{2}\}$; $D_2 = \{x : \frac{1}{2} \le x < 1\}$

$D = D_1 \cup D_2$ and $D_1 \cap D_2 = 0$.

$P_X(D) = P_X(D_1 \cup D_2) = P_X(D_1) + P_X(D_2) = 1 \Rightarrow P_X(D_2) = 1 - P_X(D_1) = 1 - \frac{1}{4} = \boxed{\frac{3}{4}}$.

§1.6 : 2) (a) If $X = 1$, $P(X=1) = \frac{1}{10}$

If $X = 2$, $P(X=2) = \frac{9}{10} \times \frac{1}{9} = \frac{1}{10}$

If $X = 3$, $P(X=3) = \frac{9}{10} \times \frac{8}{9} \times \frac{1}{8} = \frac{1}{10}$

$\vdots$

$$P(X) = \frac{\binom{9}{x-1} \cdot 1}{\binom{10}{x-1}\,11-x} = \frac{1}{10}, \quad x = 1, 2, 3, \cdots, 10;\ \text{zero elsewhere.}$$

(b) $P(X \le 4) = P(X=1) + P(X=2) + P(X=3) + P(X=4) = \frac{1}{10} \times 4 = \frac{2}{5}$.

8) $D_y = \{1^3, 2^3, 3^3, \dots\}$

$P(Y=y) = P(X^3 = y) = P(X = y^{\frac{1}{3}}) = \left(\frac{1}{2}\right)^{y^{\frac{1}{3}}}$; $y \in D_y$; zero elsewhere.

9) $D_y = \{0, 1\}$

$P(Y=0) = P(X^2=0) = P(X=0) = \frac{1}{3}$.

$P(Y=1) = P(X^2=1) = P(X=\pm 1) = P(X=1) + P(X=-1) = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$

So $P(Y=y) = \begin{cases} \frac{1}{3}, & y=0. \\ \frac{2}{3}, & y=1. \\ 0, & \text{elsewhere}. \end{cases}$

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