§1.9. 1) (a) $p(x) = \frac{3!}{x!(3-x)!}\left(\frac{1}{2}\right)^3$; $x=0,1,2,3$, zero elsewhere.

- $\mathbb{E}(X) = \sum_{x=0}^{3} x\,p(x) = 0\cdot\frac{3!}{0!(3-0)!}\left(\frac{1}{2}\right)^3 + 1\cdot\frac{3!}{1!(3-1)!}\left(\frac{1}{2}\right)^3 + 2\cdot\frac{3!}{2!(3-2)!}\left(\frac{1}{2}\right)^3 + 3\cdot\frac{3!}{3!(3-3)!}\left(\frac{1}{2}\right)^3 = \frac{3}{2}$.
- $\mathbb{E}(X^2) = \sum_{x=0}^{3} x^2 p(x) = 0\cdot\frac{3!}{0!(3-0)!}\left(\frac{1}{2}\right)^3 + 1^2\cdot\frac{3!}{1!(3-1)!}\left(\frac{1}{2}\right)^3 + \frac{3!}{2!(3-2)!}\left(\frac{1}{2}\right)^3\cdot 2^2 + 3^2\cdot\frac{3!}{3!(3-3)!}\left(\frac{1}{2}\right)^3 = 3$.
- $V(X) = \mathbb{E}(X^2) - (\mathbb{E}X)^2 = 3 - \left(\frac{3}{2}\right)^2 = \frac{3}{4}$

(b) $f(x) = 6x(1-x)$; $0<x<1$. zero elsewhere.

- $\mathbb{E}(X) = \int_0^1 x f(x)\,dx = \int_0^1 (6x^2 - 6x^3)\,dx = \left(2x^3 - \frac{3}{2}x^4\right)\Big|_0^1 = \frac{1}{2}$.
- $\mathbb{E}(X^2) = \int_0^1 x^2 f(x)\,dx = \int_0^1 (6x^3 - 6x^4)\,dx = \left(\frac{3}{2}x^4 - \frac{6}{5}x^5\right)\Big|_0^1 = \frac{3}{10}$.
- $V(X) = \mathbb{E}(X^2) - (\mathbb{E}X)^2 = \frac{3}{10} - \left(\frac{1}{2}\right)^2 = \frac{1}{20}$

(c) $f(x) = 2/x^3$, $1<x<\infty$, zero elsewhere.

- $\mathbb{E}(X) = \int_1^\infty x f(x)\,dx = \int_1^\infty x\cdot\frac{2}{x^3}\,dx = \int_1^\infty 2x^{-2}\,dx = -2x^{-1}\Big|_1^\infty = 2$
- $\mathbb{E}(X^2) = \int_1^\infty x^2 f(x)\,dx = \int_1^\infty \frac{2}{x}\,dx = \left(2\ln|x|\right)\Big|_1^\infty = \infty$ D.N.E
- $V(X) = \mathbb{E}(X^2) - (\mathbb{E}X)^2$ D.N.E

2) $p(x) = \left(\frac{1}{2}\right)^x$, $x = 1,2,3,\cdots$

- MGF: $M(t) = \mathbb{E}(e^{tX}) = \sum_{x=1}^\infty e^{tx}\left(\frac{1}{2}\right)^x = \sum_{x=1}^\infty \left(\frac{e^t}{2}\right)^x = \frac{\frac{e^t}{2}}{1-e^t/2} = \frac{e^t}{2-e^t}$, $\frac{e^t}{2}<1$; $\Rightarrow t<\ln 2$;
- $M'(t) = \frac{e^t(2-e^t) - e^t(-e^t)}{(2-e^t)^2} = \frac{2e^t}{(2-e^t)^2}$;

$\mathbb{E}(X) = M'(0) = \frac{2\cdot e^0}{(2-e^0)^2} = 2$;

$M''(t) = \frac{2e^t(2-e^t)^2 - 2e^t\cdot 2(2-e^t)(-e^t)}{(2-e^t)^4} = \frac{2e^t(2+e^t)}{(2-e^t)^3}$;

$\mathbb{E}(X^2) = M''(0) = \frac{2\cdot e^0(2+e^0)}{(2-e^0)^3} = 6$

$V(X) = \mathbb{E}(X^2) - (\mathbb{E}X)^2 = M''(0) - [M'(0)]^2 = 6 - 2^2 = 2$.

18) $F(x) = \begin{cases} 0 & x<0 \\ \frac{x}{8} & 0 \le x < 2 \\ \frac{x^2}{16} & 2 \le x < 4 \\ 1 & 4 \le x \end{cases}$ $\quad f(x) = F'(x) = \begin{cases} 0 & x<0 \\ \frac{1}{8} & 0 \le x < 2 \\ \frac{x}{8}; & 2 \le x < 4 \\ 0 & 4 \le x \end{cases}$

- $\mathbb{E}(X) = \int_0^2 x \cdot \frac{1}{8} dx + \int_2^4 x \cdot \frac{x}{8} dx = \frac{1}{16} x^2 \Big|_0^2 + \frac{1}{24} x^3 \Big|_2^4 = \cancel{\tfrac{12}{3x}}\ \frac{31}{12}$
- $\mathbb{E}(X^2) = \int_0^2 x^2 \cdot \frac{1}{8} dx + \int_2^4 x^2 \cdot \frac{x}{8} dx = \frac{1}{24} x^3 \Big|_0^2 + \frac{1}{32} x^4 \Big|_2^4 = \frac{47}{6}$
- $V(X) = \mathbb{E}(X^2) - \mathbb{E}^2(X) = \frac{47}{6} - \left(\frac{31}{12}\right)^2 = \frac{167}{144}$

§1.10: 2) By Markov Inequality: $P(y \geq c) = \frac{\mathbb{E}(y)}{c}$

so $P(X \geq 2\mu) = \frac{\mathbb{E}(X)}{2\mu} = \frac{\mu}{2\mu} = \frac{1}{2}$;

3) Chebyshev's Inequality: $P(|X-\mu| \geq \varepsilon) \leq \frac{\sigma^2}{\varepsilon^2}$

Here: $P(-2 < X < 8) = P(-2-\mu < X-\mu < 8-\mu)$

$\overset{\mu=3}{=} P(-2-3 < X-\mu < 8-3)$

$= P(-5 < X-\mu < 5)$

$= 1 - P(|X-\mu| \geq 5)$

$\sigma^2 = \mathbb{E}X^2 - (\mathbb{E}X)^2 = 13 - 3^2 = 4$

$\geq 1 - \frac{4}{5^2} = 1 - \frac{4}{25} = \frac{21}{25}$.

i.e. the lower bound is $\frac{21}{25}$;