Stat 401 HW#6 Solution §2.1: 1, 10, 12, 13, 16.

1) $f(x_1,x_2) = 4x_1x_2$, $0<x_1<1$, $0<x_2<1$.

• $P(0<X_1<\frac{1}{2}, \frac{1}{4}<X_2<1) = \int_0^{\frac{1}{2}}\int_{\frac{1}{4}}^{1} 4x_1x_2\, dx_2\, dx_1 = \int_0^{\frac{1}{2}} 4x_1 \cdot \left(\int_{\frac{1}{4}}^{1} x_2\, dx_2\right) dx_1$

$$= \int_0^{\frac{1}{2}} 4x_1 \cdot \left(\frac{1}{2}x_2^2\Big|_{\frac{1}{4}}^{1}\right) dx_1 = \frac{15}{8}\int_0^{\frac{1}{2}} x_1\, dx_1 = \frac{15}{8}\cdot\frac{1}{2}x_1^2\Big|_0^{\frac{1}{2}} = \frac{15}{8}\times\frac{1}{2}\times\left(\frac{1}{4}\right) = \frac{15}{64}$$

• $P(X_1=X_2) = P(X_1-X_2=0) = 0$

• $P(X_1<X_2) = \int_0^1\left[\int_0^{x_2} 4x_1x_2\, dx_1\right] dx_2 = \int_0^1 4x_2\left(\frac{1}{2}x_1^2\Big|_0^{x_2}\right) dx_2 = 2\int_0^1 x_2^3\, dx_2 = 2\left(\frac{1}{4}x_2^4\Big|_0^1\right) = 2\cdot\frac{1}{4} = \frac{1}{2}$

• $P(X_1\le X_2) = P(X_1=X_2) + P(X_1<X_2) = 0 + \frac{1}{2} = \frac{1}{2}$

10) $f(x_1,x_2) = 15x_1^2x_2$. $0<x_1<x_2<1$.

• Marginal distribution:

- For X_1: $f(x_1) = \int_{x_1}^1 15x_1^2x_2\, dx_2 = 15x_1^2\left(\frac{1}{2}x_2^2\Big|_{x_1}^1\right) = \frac{15}{2}x_1^2(1-x_1^2)$, $0<x_1<1$.
- For X_2: $f(x_2) = \int_0^{x_2} 15x_1^2x_2\, dx_1 = 15x_2\cdot\left(\frac{1}{3}x_1^3\Big|_0^{x_2}\right) = 5x_2^4$; $0<x_2<1$.

• $P(X_1+X_2\le 1) = \int_0^1\left[\int_0^{1-x_2} 15x_1^2x_2\, dx_1\right] dx_2 = \int_0^1 15x_2\cdot\left(\frac{1}{3}x_1^3\Big|_0^{1-x_2}\right) dx_2$

$$= 5\int_0^1 x_2(1-x_2)^3\, dx_2$$

$$= 5\int_0^1 (x_2 - 3x_2^2 + 3x_2^3 - x_2^4)\, dx_2$$

$$= 5\left[\frac{1}{2}x_2^2\Big|_0^1 - \frac{3}{3}x_2^3\Big|_0^1 + \frac{3}{4}x_2^4\Big|_0^1 - \frac{1}{5}x_2^5\Big|_0^1\right] = \frac{1}{4}$$

12) $p(x_1,x_2) = \frac{x_1+x_2}{12}$, $x_1=1,2$; $x_2=1,2$

• $p(x_1) = \sum_{x_2\in D_2} p(x_1, X_2=x_2) = \frac{x_1+1}{12} + \frac{x_1+2}{12} = \frac{2x_1+3}{12}$; $x_1=1,2$;

$EX_1 = \sum_{x_1\in D_1} x_1 p(X_1=x_1) = 1\cdot\frac{2\cdot1+3}{12} + 2\cdot\frac{2\cdot2+3}{12} = 19/12$; $EX_1^2 = \sum_{x_1} x_1^2 p(x_1) = 1^2\cdot\frac{5}{12} + 2^2\cdot\frac{7}{12} = \frac{11}{4}$

• $p(x_2) = \sum_{x_1\in D_1} p(X_1=x_1, x_2) = \frac{1+x_2}{12} + \frac{2+x_2}{12} = \frac{2x_2+3}{12}$; $x_2=1,2$;

$EX_2 = \sum_{x_2\in D_2} x_2 p(X_2=x_2) = 1\cdot\frac{2\cdot1+3}{12} + 2\cdot\frac{2\cdot2+3}{12} = \frac{19}{12}$; $EX_2^2 = \sum_{x_2} p(x_2)\cdot x_2^2 = 1^2\cdot\frac{5}{12} + 2^2\cdot\frac{7}{12} = \frac{11}{4}$

• $EX_1X_2 = \sum_{x_1\in D_1; x_2\in D_2} x_1x_2 p(X_1=x_1, X_2=x_2) = 1\cdot1\cdot\frac{1+1}{12} + 1\cdot2\cdot\frac{1+2}{12} + 2\cdot1\cdot\frac{2+1}{12} + 2\cdot2\cdot\frac{2+2}{12} = \frac{30}{12} = \frac{5}{2}$

• $EX_1\cdot EX_2 = \frac{19}{12}\cdot\frac{19}{12} \neq \frac{5}{2} = EX_1X_2$. No

• $E(2X_1 - 6X_2^2 + 7X_1X_2) = 2EX_1 - 6EX_2^2 + 7EX_1X_2 = 2\cdot\frac{19}{12} - 6\cdot\frac{11}{4} + 7\cdot\frac{5}{2} = \frac{25}{6}$;

13) $f(x_1,x_2)=4x_1x_2,\ 0<x_1<1.\ 0<x_2<1.$

- $f(x_1)=\int_0^1 4x_1x_2\,dx_2 = 4x_1\cdot\frac{1}{2}x_2^2\Big|_0^1 = 2x_1,\ 0<x_1<1$

  $\mathbb{E}X_1=\int_0^1 x_1 f(x_1)\,dx_1=\int_0^1 x_1\cdot 2x_1\,dx_1 = \frac{2}{3}x_1^3\Big|_0^1=\frac{2}{3}$

  $\mathbb{E}X_1^2 = \int_0^1 x_1^2 f(x_1)\,dx_1=\int_0^1 x_1^2\cdot 2x_1\,dx_1=\frac{1}{2}x_1^4\Big|_0^1=\frac{1}{2}$

- x_1 and x_2 are symmetric. so $f(x_2)=2x_2$; $0<x_2<1$

  $\mathbb{E}X_2=\frac{2}{3}$

  $\mathbb{E}X_2^2=\frac{1}{2}$

- $\mathbb{E}X_1X_2=\int_0^1\int_0^1 x_1x_2(4x_1x_2)\,dx_1\,dx_2 = 4\int_0^1 x_2^2\left(\int_0^1 x_1^2\,dx_1\right)dx_2$

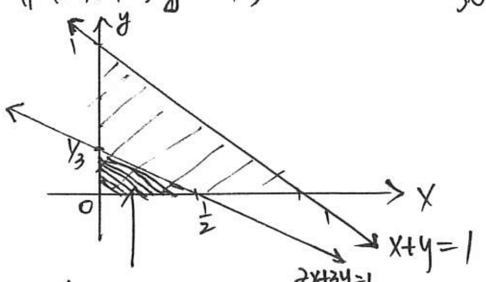
  $= 4\int_0^1\frac{1}{3}x_2^2\,dx_2 = \frac{4}{3}\cdot\frac{1}{3}x_2^3\Big|_0^1 = \frac{4}{9}$

- $\mathbb{E}X_1\cdot\mathbb{E}X_2 = \frac{2}{3}\cdot\frac{2}{3}=\frac{4}{9}=\mathbb{E}X_1X_2$. Yes

- $\mathbb{E}(3X_2-2X_1^2+6X_1X_2)=3\mathbb{E}X_2-2\mathbb{E}X_1^2+6\mathbb{E}X_1X_2 = 3\cdot\frac{2}{3}-2\cdot\frac{1}{2}+6\cdot\frac{4}{9}=\frac{11}{3}$

16) $f(x,y)=6(1-x-y)$; $x+y<1.\ 0<x,\ 0<y.$

- $\mathbb{P}(2x+3y<1)$



$\mathbb{P}(2x+3y<1)$

so $\mathbb{P}(2x+3y<1)=\int_0^{\frac{1}{2}}\int_0^{\frac{1-2x}{3}}6(1-x-y)\,dy\,dx$

$=6\int_0^{\frac{1}{2}}\left(y-xy-\frac{1}{2}y^2\right)\Big|_0^{\frac{1-2x}{3}}dx$

$=6\int_0^{\frac{1}{2}}\left[\frac{1-2x}{3}-\frac{x-2x^2}{3}-\frac{(1-2x)^2}{18}\right]dx$

$=\frac{1}{3}\int_0^{\frac{1}{2}}(5-14x+8x^2)\,dx$

$=\frac{1}{3}\left(5x-7x^2+\frac{8}{3}x^3\right)\Big|_0^{\frac{1}{2}}=\frac{13}{36}$

- $\mathbb{E}(xy+2x^2)=\int_0^1\int_0^{1-x}(xy+2x^2)6(1-x-y)\,dy\,dx$

$=6\int_0^1\int_0^{1-x}(xy-3x^2y-xy^2+2x^2-2x^3)\,dy\,dx$

$=6\int_0^1\left[\frac{1}{2}xy^2-\frac{3}{2}x^2y^2-\frac{1}{3}xy^3+2x^2y-2x^3y\right]\Big|_0^{1-x}dx$

$=6\int_0^1\left(\frac{5}{6}x^4-\frac{3}{2}x^3+\frac{1}{2}x^2+\frac{1}{6}x\right)dx$

$=\int_0^1(5x^4-9x^3+3x^2+x)\,dx$

$=\left(\frac{5}{5}x^5-\frac{9}{4}x^4+\frac{3}{3}x^3+\frac{1}{2}x^2\right)\Big|_0^1$

$=\frac{1}{4}$