Stat 401 HW7 Solution §2.2: 1, 2, 3, 5(a); §2.3: 1, 2.

§2.2: 1) $Y_1 = X_1 - X_2$, $Y_2 = X_1 + X_2$: $(y_1, y_2) = (0,0), (-1,1), (1,1), (0,2)$

$$p(y_1, y_2) = \begin{cases} \left(\frac{2}{3}\right)^{y_2} \cdot \left(\frac{1}{3}\right)^{2-y_2} & ; \ (y_1,y_2) = (0,0), (-1,1), (1,1), (0,2) \\ 0 & ; \ \cancel{\text{zero}} \text{ elsewhere.} \end{cases}$$

2) $p(y_1, y_2) = \frac{y_1}{36}$, $y_2 = 1, 2, 3$; $y_1 = y_2, 2y_2, 3y_2$; zero elsewhere;

So the marginal pmf of Y_1 is: $p_{Y_1}(y_1) = \sum_{y_2} p_{Y_1,Y_2}(y_1, y_2)$, $y_1 = 1, 2, 3, 4, 6, 9$;

y_1	1	2	3	4	6	9
$p(y_1)$	1/36	4/36	6/36	4/36	12/36	9/36

3) $Y_1 = 2X_1$. $Y_2 = X_2 - X_1$. $\Rightarrow \begin{cases} X_1 = \frac{1}{2}Y_1 \\ X_2 = \frac{1}{2}Y_1 + Y_2 \end{cases} \Rightarrow J = \begin{vmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & 1 \end{vmatrix} = \frac{1}{2}$, $|J| = \frac{1}{2}$.

so the joint pdf of Y_1 and Y_2 is:

$$f_{Y_1,Y_2}(y_1, y_2) = \frac{1}{2} \cdot 2e^{-\frac{1}{2}y_1 - (\frac{1}{2}y_1 + y_2)} = \begin{cases} e^{-y_1 - y_2} & ; \ 0 < y_1, y_2 < \infty. \\ 0 & ; \ \text{elsewhere.} \end{cases}$$

5a) $f_{X_1,X_2}(x_1, x_2)$; $-\infty < x_i < \infty$; $i = 1, 2$;

$\begin{cases} Y_1 = X_1 + X_2 \\ Y_2 = X_2 \end{cases} \Rightarrow \begin{cases} X_1 = Y_1 - Y_2 \\ X_2 = Y_2 \end{cases} \Rightarrow J = \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = 1 \Rightarrow |J| = 1$.

$f_{Y_1,Y_2}(y_1, y_2) = f_{X_1,X_2}(y_1 - y_2, y_2) \cdot |J| = f_{X_1,X_2}(y_1 - y_2, y_2)$. $-\infty < y_i < \infty$; $i = 1, 2$.

§2.3: 1) $f(x_1, x_2) = x_1 + x_2$, $0 < x_1 < 1$, $0 < x_2 < 1$.

so $f(x_1) = \int_0^1 (x_1 + x_2)\,dx_2 = \left(x_1 x_2 + \frac{1}{2}x_2^2\right)\Big|_0^1 = x_1 + \frac{1}{2}$, $0 < x_1 < 1$.

$$\Rightarrow f_{X_2|X_1}(x_2|x_1) = \frac{f_{X_1,X_2}(x_1,x_2)}{f_{X_1}(x_1)} = \frac{x_1 + x_2}{x_1 + \frac{1}{2}} = \frac{2x_1 + 2x_2}{2x_1 + 1}, \quad 0 < x_1 < 1, \ 0 < x_2 < 1.$$

then: $E(X_2 | X_1 = x_1) = \int_0^1 x_2 \cdot \frac{2x_1 + 2x_2}{2x_1 + 1}\,dx_2 = \frac{2}{2x_1 + 1} \int_0^1 (x_1 x_2 + x_2^2)\,dx_2$

$= \frac{2}{2x_1 + 1} \left[\frac{1}{2}x_1 x_2^2 + \frac{1}{3}x_2^3\right]\Big|_0^1 = \frac{3x_1 + 2}{6x_1 + 3}$;

$E(X_2^2 | X_1 = x_1) = \int_0^1 x_2^2 \cdot \frac{2x_1 + 2x_2}{2x_1 + 1}\,dx_2 = \frac{2}{2x_1 + 1} \int_0^1 (x_1 x_2^2 + x_2^3)\,dx_2 = \frac{2}{2x_1 + 1} \left(\frac{1}{3}x_1 x_2^3 + \frac{1}{4}x_2^4\right)\Big|_0^1$

$= \frac{4x_1 + 3}{12x_1 + 6}$;

$V(X_2 | X_1 = x_1) = E(X_2^2 | X_1 = x_1) - [E(X_2 | X_1 = x_1)]^2 = \frac{4x_1 + 3}{12x_1 + 6} - \left(\frac{3x_1 + 2}{6x_1 + 3}\right)^2 = \frac{6x_1^2 + 6x_1 + 1}{2(6x_1 + 3)^2}$;

2) $f_{X_1|X_2}(x_1|x_2) = c_1 x_1/x_2^2$, $0<x_1<x_2$, $0<x_2<1$

$f_2(x_2) = c_2 x_2^4$, $0<x_2<1$.

(a) Since $\int_{-\infty}^{\infty} f(x)\,dx = 1$. so we have:

$$\int_0^1\int_0^{x_2} \frac{c_1 x_1}{x_2^2}\,dx_1\,dx_2 = 1 \Rightarrow \int_0^1 \frac{c_1}{x_2^2}\left(\tfrac12 x_1^2\right)\Big|_0^{x_2} dx_2 = 1 \Rightarrow \int_0^1 \tfrac12 c_1\,dx_2 = 1 \Rightarrow \frac{c_1}{2} = 1 \Rightarrow c_1 = 2$$

$$\int_0^1 c_2 x_2^4\,dx_2 = 1 \Rightarrow \cancel{\int_0^1} \cdot c_2 \cdot \tfrac15 x_2^5\Big|_0^1 = \tfrac15 c_2 = 1 \Rightarrow c_2 = 5$$

(b) Joint distribution:

$$f_{X_1,X_2}(x_1,x_2) = f_{X_1|X_2}(x_1|x_2)\cdot f_{X_2}(x_2) = \frac{2x_1}{x_2^2}\cdot 5x_2^4 = 10x_1x_2^2\,;\ 0<x_1<x_2<1\,;$$

(c) $$P\left(\tfrac14 < X_1 < \tfrac12 \,\middle|\, X_2 = \tfrac58\right) = \int_{\frac14}^{\frac12} f_{X_1|X_2}\left(x_1 \middle| x_2 = \tfrac58\right) dx_1$$

$$= \int_{\frac14}^{\frac12} 2x_1/\left(\tfrac58\right)^2 dx_1 = \frac{64}{25}\cdot x_1^2\Big|_{\frac14}^{\frac12} = \frac{64}{25}\left(\tfrac14 - \tfrac1{16}\right) = \frac{12}{25}$$

(d) $P\left(\tfrac14 < X_1 < \tfrac12\right)$

$$f_{X_1}(x_1) = \int_{x_1}^1 10x_1x_2^2\,dx_2 = 10x_1\cdot\tfrac13 x_2^3\Big|_{x_1}^1 = \tfrac{10}{3}x_1(1-x_1^3),\quad 0<x_1<1$$

So $$P\left(\tfrac14 < X_1 < \tfrac12\right) = \int_{\frac14}^{\frac12} \tfrac{10}{3}x_1(1-x_1^3)\,dx_1$$

$$= \tfrac{10}{3}\left(\tfrac12 x_1^2 - \tfrac15 x_1^5\right)\Big|_{\frac14}^{\frac12}$$

$$= \frac{449}{1536}$$