Stat 401 HW#8 Solution §2.3: 4, 7, 8; §2.4: 1, 3, 10;

§2.3: **4)** $p(x_1,x_2)=\begin{cases}\dfrac{x_1+2x_2}{18}, & (x_1,x_2)=(1,1),(1,2),(2,1),(2,2).\\ 0. & \end{cases}$

since $p_{X_1}(x_1)=\sum_{x_2} p(x_1,x_2)=\dfrac{x_1+2(1)}{18}+\dfrac{x_1+2(2)}{18}=\dfrac{2x_1+6}{18}.$

$p(x_1,x_2)$	x_2		
x_1	1	2	$p_{X_1}(x_1)$
1	$\frac{1+2(1)}{18}=\frac{1}{6}$	$\frac{1+2(2)}{18}=\frac{5}{18}$	$\frac{4}{9}$
2	$\frac{2+2(1)}{18}=\frac{2}{9}$	$\frac{2+2(2)}{18}=\frac{1}{3}$	$\frac{5}{9}$
$p_{X_2}(x_2)$	$7/18$	$11/18$	1

so: conditional pmf

$$= P(X_2 \mid X_1=x_1) = \frac{P(X_1=x_1, X_2=x_2)}{P(X_1=x_1)} = \frac{(x_1+2x_2)/18}{(2x_1+6)/18} = \frac{x_1+2x_2}{2x_1+6};$$

- $$E(X_2 \mid X_1=x_1)=\sum_{x_2=1,2} x_2\cdot p(x_2 \mid X_1=x_1) = 1\cdot\frac{\frac{x_1+2(1)}{18}}{\frac{x_1+2(1)}{18}+\frac{x_1+2(2)}{18}} + 2\cdot\frac{\frac{x_1+2(2)}{18}}{\frac{x_1+2(1)}{18}+\frac{x_1+2(2)}{18}} = \frac{3x_1+10}{2x_1+6}$$

- $$E(X_2^2 \mid X_1=x_1)=\sum_{x_2=1,2} x_2^2\cdot p(x_2 \mid X_1=x_1) = 1\cdot\frac{\frac{x_1+2(1)}{18}}{\frac{x_1+2(1)}{18}+\frac{x_1+2(2)}{18}} + 2^2\cdot\frac{\frac{x_1+2(2)}{18}}{\frac{x_1+2(1)}{18}+\frac{x_1+2(2)}{18}} = \frac{5x_1+18}{2x_1+6}$$

$$Var(X_2 \mid X_1=x_1)=E(X_2^2 \mid X_1=x_1)-[E(X_2 \mid X_1=x_1)]^2=\frac{5x_1+18}{2x_1+6}-\left(\frac{3x_1+10}{2x_1+6}\right)^2=\frac{x_1^2+6x_1+8}{(2x_1+6)^2}$$

- $$EX_1=\sum_{x_1=1,2} x_1 p(x_1)=1\cdot\frac{4}{9}+2\cdot\frac{5}{9}=\frac{14}{9};\quad EX_2=\sum_{x_2=1,2} x_2 p(x_2)=1\cdot\frac{7}{18}+2\cdot\frac{11}{18}=\frac{29}{18};$$

$$E(3X_1-2X_2)=3EX_1-2EX_2=3\cdot\frac{14}{9}-2\cdot\frac{29}{18}=\frac{13}{9}$$

7) $p(x_1,x_2)=\dfrac{3x_1+x_2}{24}$; $(x_1,x_2)=(1,1),(1,2),(2,1),(2,2)$; zero elsewhere.

		x_2		
		1	2	$p_{X_1}(x_1)$
x_1	1	$\frac{3(1)+1}{24}=\frac{1}{6}$	$\frac{3(1)+2}{24}=\frac{5}{24}$	$\frac{9}{24}=\frac{3}{8}$
	2	$\frac{3(2)+1}{24}=\frac{7}{24}$	$\frac{3(2)+2}{24}=\frac{8}{24}$	$\frac{15}{24}=\frac{5}{8}$
$p_{X_2}(x_2)$		$\frac{11}{24}$	$\frac{13}{24}$	1

$$E(X_2 \mid x_1=1)=\sum_{x_2=1,2} x_2 p(x_2 \mid x_1=1) = 1\cdot\frac{\frac{1}{6}}{\frac{9}{24}}+2\cdot\frac{\frac{5}{24}}{\frac{9}{24}}=\frac{14}{9};$$

8) $f(x,y) = 2\exp\{-(x+y)\}$. $0<x<y<\infty$; zero elsewhere.

$$f_X(x) = \int_x^\infty f(x,y)\,dy = \int_x^\infty 2e^{-x}\cdot e^{-y}\,dy = 2e^{-x}\cdot(-e^{-y})\Big|_x^\infty = 2e^{-2x};\ 0<x<\infty.$$

Conditional pdf of Y on $X=x$ is:

$$f_{Y|X}(y|x) = \frac{f(x,y)}{f(x)} = \frac{\cancel{2}e^{-x-y}}{\cancel{2}e^{-2x}} = e^{x-y};\ 0<x<y<\infty.$$

$$E(Y|X=x) = \int_x^\infty y\cdot e^{x-y}\,dy = e^x\cdot\int_x^\infty ye^{-y}\,dy = -e^x\int_x^\infty y\,de^{-y}$$

$$= -e^{+x}\left(ye^{-y}\Big|_x^\infty - \int_x^\infty e^{-y}\,dy\right) = -e^x\left(-xe^{-x}+0-e^{-x}\right) = x+1.$$

§2.4:

1) (a) $p(x,y)=\frac{1}{3}$; $(x,y)=(0,0),(1,1),(2,2)$; zero elsewhere.

$$EX = \sum_{x=0,1,2} x\,p(x) = 0\cdot\tfrac{1}{3}+1\cdot\tfrac{1}{3}+2\cdot\tfrac{1}{3}=1.$$

$$EY = \sum_{y=0,1,2} y\,p(y) = 0\cdot\tfrac{1}{3}+1\cdot\tfrac{1}{3}+2\cdot\tfrac{1}{3}=1.$$

$$E(XY) = \sum_{x,y} xy\,p(x,y) = 0\cdot0\cdot\tfrac{1}{3}+1\cdot1\cdot\tfrac{1}{3}+2\cdot2\cdot\tfrac{1}{3}=\tfrac{5}{3}$$

$$\sigma_X^2 = EX^2-\mu_X^2 = (0^2+1^2+2^2)\cdot\tfrac{1}{3}-1^2=\tfrac{2}{3};\quad \sigma_Y^2 = EY^2-\mu_Y^2 = (0^2+1^2+2^2)\cdot\tfrac{1}{3}-1^2=\tfrac{2}{3};$$

$$\Rightarrow \rho = \frac{E(XY)-\mu_X\mu_Y}{\sigma_X\sigma_Y} = \frac{\frac{5}{3}-1\cdot1}{\sqrt{\frac{2}{3}}\cdot\sqrt{\frac{2}{3}}} = 1$$

(b) $p(x,y)=\frac{1}{3}$; $(x,y)=(0,2)$; $(1,1)$; $(2,0)$; zero elsewhere.

$$EX = \sum_x x\,p(x) = (0+1+2)\tfrac{1}{3}=1;\quad EY=\sum_y y\,p(y) = (0+1+2)\cdot\tfrac{1}{3}=1;$$

$$EXY = \sum_{x,y} xy\,p(x,y) = 0\cdot2\cdot\tfrac{1}{3}+1\cdot1\cdot\tfrac{1}{3}+2\cdot0\cdot\tfrac{1}{3}=\tfrac{1}{3}$$

$$\sigma_X^2 = EX^2-\mu_X^2 = (0^2+1^2+2^2)\cdot\tfrac{1}{3}-1^2=\tfrac{2}{3};\quad \sigma_Y^2 = EY^2-\mu_Y^2 = (0^2+1^2+2^2)\cdot\tfrac{1}{3}-1^2=\tfrac{2}{3};$$

$$\Rightarrow \rho = \frac{E(XY)-\mu_X\mu_Y}{\sigma_X\cdot\sigma_Y} = \frac{\frac{1}{3}-1\cdot1}{\sqrt{\frac{2}{3}}\cdot\sqrt{\frac{2}{3}}} = -1$$

(c) $p(x,y)=\frac{1}{3}$; $(x,y)=(0,0),(1,1),(2,0)$. zero elsewhere.

$$EX=\sum_x x\,p(x) = (0+1+2)\cdot\tfrac{1}{3}=1;\quad EY=\sum_y y\,p(y) = (0+1+0)\cdot\tfrac{1}{3}=\tfrac{1}{3};$$

$$E(XY)=\sum_{x,y} xy\,p(x,y) = 0\cdot0\cdot\tfrac{1}{3}+1\cdot1\cdot\tfrac{1}{3}+2\cdot0\cdot\tfrac{1}{3}=\tfrac{1}{3},$$

$\sigma_X^2 = \mathbb{E}X^2 - \mu_X^2 = (0^2+1^2+2^2)\cdot\frac{1}{3} - 1^2 = \frac{2}{3}$; $\sigma_Y^2 = \mathbb{E}Y^2 - \mu_Y^2 = (0^2+1^2+0^2)\cdot\frac{1}{3} - (\frac{1}{3})^2 = \frac{2}{9}$; 13

$$\Rightarrow \rho = \frac{\mathbb{E}XY - \mathbb{E}X\cdot\mathbb{E}Y}{\sigma_X\sigma_Y} = \frac{\frac{1}{3} - 1\cdot\frac{1}{3}}{\sqrt{\frac{2}{3}}\cdot\sqrt{\frac{2}{9}}} = 0;$$

3) $f(x,y) = 2$; $0<x<y<1$

$f(x) = \int_x^1 f(x,y)\,dy = \int_x^1 2\,dy = 2y\Big|_x^1 = 2(1-x)$; $0<x<1$.

$f(y) = \int_0^y f(x,y)\,dx = \int_0^y 2\,dx = 2x\Big|_0^y = 2y$; $0<y<1$.

$f(x|y) = \frac{f(x,y)}{f(y)} = \frac{2}{2y} = \frac{1}{y}$; $0<x<y$; $0<y<1$;

$f(y|x) = \frac{f(x,y)}{f(x)} = \frac{2}{2(1-x)} = \frac{1}{1-x}$; $0<x<y$; $0<y<1$;

- $\mathbb{E}(Y|X=x) = \int_0^1 y f(y|x)\,dy = \int_x^1 y\cdot\frac{1}{1-x}\,dy = \left[\frac{y^2}{2(1-x)}\right]\Big|_x^1 = \frac{1+x}{2}$; $0<x<1$.

  $\mathbb{E}(X|Y=y) = \int_0^1 x f(x|y)\,dx = \int_0^y x\cdot\frac{1}{y}\,dx = \frac{1}{y}\cdot\frac{1}{2}x^2\Big|_0^y = \frac{y}{2}$; $0<y<1$.

- $\mathbb{E}X = \int_0^1 x f(x)\,dx = \int_0^1 x\cdot 2(1-x)\,dx = (x^2 - \frac{2}{3}x^3)\Big|_0^1 = \frac{1}{3}$

  $\mathbb{E}Y = \int_0^1 y f(y)\,dy = \int_0^1 y\cdot 2y\,dy = \frac{2}{3}y^3\Big|_0^1 = \frac{2}{3}$

  $\mathbb{E}XY = \int_0^1\int_0^y 2xy\,dx\,dy = \int_0^1 2y\cdot(\frac{1}{2}x^2\Big|_0^y)\,dy = \frac{1}{4}y^4\Big|_0^1 = \frac{1}{4}$

  $\mathbb{E}X^2 = \int_0^1 x^2 f(x)\,dx = \int_0^1 x^2\cdot 2(1-x)\,dx = \frac{1}{6}$;

  $\mathbb{E}Y^2 = \int_0^1 y^2 f(y)\,dy = \int_0^1 y^2\cdot 2y\,dy = \frac{1}{2}y^4\Big|_0^1 = \frac{1}{2}$;

  $\sigma_X^2 = \mathbb{E}X^2 - (\mathbb{E}X)^2 = \frac{1}{18}$; $\sigma_Y^2 = \mathbb{E}Y^2 - (\mathbb{E}Y)^2 = \frac{1}{18}$

  $$\rho = \frac{\mathbb{E}XY - \mathbb{E}X\cdot\mathbb{E}Y}{\sigma_X\sigma_Y} = \frac{\frac{1}{4} - \frac{1}{3}\cdot\frac{2}{3}}{\frac{1}{18}} = \frac{1}{2};$$

10)

(x_1,x_2)	$(0,0)$	$(0,1)$	$(0,2)$	$(1,1)$	$(1,2)$	$(2,2)$
$p(x_1,x_2)$	$\frac{1}{12}$	$\frac{2}{12}$	$\frac{1}{12}$	$\frac{3}{12}$	$\frac{4}{12}$	$\frac{1}{12}$

- $p_1(x_1=0) = \frac{1}{12} + \frac{2}{12} + \frac{1}{12} = \frac{1}{3}$

  $p_1(x_1=1) = \frac{3}{12} + \frac{4}{12} = \frac{7}{12}$

  $p_1(x_1=2) = \frac{1}{12}$

$$\Rightarrow p_1(x_1) = \begin{cases} \frac{1}{3}, & x_1 = 0. \\ \frac{7}{12}, & x_1 = 1. \\ \frac{1}{12}, & x_1 = 2. \end{cases}$$

$p_2(X_2=0)=\frac{1}{12}$;

$p_2(X_2=1)=\frac{2}{12}+\frac{3}{12}=\frac{5}{12}$;

$p_2(X_2=2)=\frac{1}{12}+\frac{4}{12}+\frac{1}{12}=\frac{6}{12}=\frac{1}{2}$;

$$p_2(X_2=x_2)=\begin{cases}\frac{1}{12}, & x_2=0.\\ \frac{5}{12}, & x_2=1.\\ \frac{1}{2}, & x_2=2.\end{cases}$$

$\mu_1=\mathbb{E}X_1=0\cdot\frac{1}{3}+1\cdot\frac{7}{12}+2\cdot\frac{1}{12}=\frac{9}{12}=\frac{3}{4}$; $\mu_2=\mathbb{E}X_2=0\cdot\frac{1}{12}+1\cdot\frac{5}{12}+2\cdot\frac{1}{2}=\frac{17}{12}$

$\mathbb{E}X_1^2=0^2\cdot\frac{1}{3}+1^2\cdot\frac{7}{12}+2^2\cdot\frac{1}{12}=\frac{11}{12}$; $\mathbb{E}X_2^2=0^2\cdot\frac{1}{12}+1^2\cdot\frac{5}{12}+2^2\cdot\frac{1}{2}=\frac{29}{12}$

$\sigma_X^2=\mathbb{E}X_1^2-(\mathbb{E}X_1)^2=\frac{17}{48}$; $\sigma_Y^2=\mathbb{E}X_2^2-(\mathbb{E}X_2)^2=\frac{59}{144}$; $\mathbb{E}XY=0\cdot\frac{1}{12}+0\cdot\frac{2}{12}+0\cdot\frac{1}{12}+1\cdot\frac{3}{12}+2\cdot\frac{4}{12}+4\cdot\frac{1}{12}=\frac{5}{4}$

$$\Rightarrow \rho=\frac{\mathbb{E}XY-\mathbb{E}X\cdot\mathbb{E}Y}{\sigma_X\cdot\sigma_Y}=\frac{\frac{5}{4}-\frac{3}{4}\cdot\frac{17}{12}}{\sqrt{\frac{17}{48}}\cdot\sqrt{\frac{59}{144}}}$$

$$=0.4922$$