Stat 401 HW #9 Solution §2.5: 1, 2, 3, 8, 9; §2.6: 1(a)(c)(d)(f)(g), 3.

§2.5: 1) $f(x_1,x_2)=\begin{cases}12x_1x_2(1-x_2) & 0<x_1<1,\ 0<x_2<1.\\ 0 & \text{e.w.}\end{cases}$

then: $f(x_1)=\int_0^1 f(x_1,x_2)\,dx_2$ $\qquad f(x_2)=\int_0^1 f(x_1,x_2)\,dx_1$

$=\int_0^1 12x_1x_2(1-x_2)\,dx_2$ $\qquad =\int_0^1 12x_1x_2(1-x_2)\,dx_1$

$=12x_1\int_0^1(x_2-x_2^2)\,dx_2$ $\qquad =12x_2(1-x_2)\int_0^1 x_1\,dx_1$

$=12x_1\left(\frac{1}{2}x_2^2-\frac{1}{3}x_2^3\right)\Big|_0^1$ $\qquad =12x_2(1-x_2)\left(\frac{1}{2}x_1^2\Big|_0^1\right)$

$=2x_1,\ 0<x_1<1.$ $\qquad =6x_2(1-x_2)\quad 0<x_2<1$

$f(x_1,x_2)=12x_1x_2(1-x_2)=2x_1\cdot 6x_2(1-x_2)=f(x_1)\cdot f(x_2).\quad 0<x_1<1,\ 0<x_2<1.$

$\Rightarrow$ x_1 and x_2 are independent.

2) $f(x_1,x_2)=2e^{-x_1-x_2},\quad 0<x_1<x_2,\quad 0<x_2<\infty.$

In this problem, the space $S=\{(x_1,x_2):\ 0<x_1<x_2<\infty\}$ is not a product space. Since the space of positive probability density of X_1 and X_2 is bounded by a curve that is neither a horizontal nor a vertical line.

Therefore, X_1 and X_2 are dependent.

3) $p(x_1,x_2)=\frac{1}{16},\ x_1=1,2,3,4;\ x_2=1,2,3,4;$

$p(x_1)=\sum_{x_2\in D_2}p(x_1,x_2)=p(x_1,1)+\cdots+p(x_1,4)=4\cdot\frac{1}{16}=\frac{1}{4};\ x_1=1,2,3,4;$

$p(x_2)=\sum_{x_1\in D_1}p(x_1,x_2)=p(1,x_2)+\cdots+p(4,x_2)=4\cdot\frac{1}{16}=\frac{1}{4};\ x_2=1,2,3,4;$

$p(x_1)\cdot p(x_2)=\frac{1}{4}\cdot\frac{1}{4}=\frac{1}{16}=p(x_1,x_2);\ x_1=1,2,3,4;\ x_2=1,2,3,4$

$\Rightarrow$ X_1 and X_2 are Independent.

8) $f(x,y)=3x,\ 0<y<x<1;$

Again, space $S=\{(x,y): 0<y<x<1\}$ is not a product space. So X and Y dependent

$f(y)=\int_y^1 f(x,y)\,dx=\int_y^1 3x\,dx=\frac{3}{2}x^2\Big|_y^1=\frac{3}{2}(1-y^2);\quad 0<y<1$

so conditional probability of $Y|X$ is:

$f(x|y)=\frac{f(x,y)}{f(y)}=\frac{3x}{\frac{3}{2}(1-y^2)}=\frac{2x}{1-y^2};\quad 0<y<x<1$

$E(X|y)=\int_{-\infty}^{\infty}x\cdot f(x|y)\,dx=\int_y^1 x\frac{2x}{1-y^2}\,dx=\frac{2}{1-y^2}\cdot\left(\frac{1}{3}x^3\Big|_y^1\right)=\frac{2(1-y^3)}{3(1-y^2)};\quad 0<y<1$

9) Let X be time depature. $0<X<30$: $f(x)=\frac{1}{30}$, $0<x<30$.

Let Y be time travel. $40<Y<50$: $f(y)=\frac{1}{50-40}=\frac{1}{10}$, $40<y<50$.

~~then: $P(X+Y<60)$~~

Since X and Y are independent: $f(x,y)=f(x)\cdot f(y)=\frac{1}{300}$, $0<x<30$, $40<y<50$.

then: $P(X+Y<60)=\int_{40}^{50}\int_0^{60-y}\frac{1}{300}\,dx\,dy=\frac{1}{300}\int_{40}^{50}(60-y)\,dy=\frac{1}{300}(60y-\frac{1}{2}y^2)\Big|_{40}^{50}=\frac{1}{2}$;

§2.6: 1) $f(x,y,z)=2(x+y+z)/3$, $0<x<1$, $0<y<1$, $0<z<1$.

(a) $f(x)=\int_0^1\int_0^1 2(x+y+z)/3\,dy\,dz=\int_0^1\left[\frac{2}{3}(x+z)\cdot y+\frac{1}{3}y^2\right]\Big|_0^1 dz$

$=\int_0^1\left[\frac{2}{3}(x+z)+\frac{1}{3}\right]dz=\left[\frac{1}{3}(2x+1)z+\frac{2}{3}\cdot\frac{1}{2}z^2\right]\Big|_0^1=\frac{2}{3}(x+1),\ 0<x<1$.

X, Y, Z are symmetric.

So $f(y)=\frac{2}{3}(y+1)$, $0<y<1$; $f(z)=\frac{2}{3}(z+1)$, $0<z<1$.

(b) $f(x)\cdot f(y)\cdot f(z)=\frac{8}{27}(x+1)(y+1)(z+1)\neq\frac{2}{3}(x+y+z)=f(x,y,z)$. So X, Y, Z dependent.

(d) $E(X^2YZ+3XY^4Z^2)=\int_0^1\int_0^1\int_0^1(x^2yz+3xy^4z^2)\cdot\frac{2}{3}(x+y+z)\,dx\,dy\,dz$

$=\frac{2}{3}\int_0^1\int_0^1(\frac{1}{4}x^4yz+x^3y^4z^2+\frac{1}{3}x^3y^2z+\frac{3}{2}x^2y^5z^2+\frac{1}{3}x^3yz^2+\frac{3}{2}x^2y^4z^3)\Big|_0^1 dy\,dz$

$=\frac{2}{3}\int_0^1\int_0^1(\frac{1}{4}yz+y^4z^2+\frac{1}{3}y^2z+\frac{3}{2}y^5z^2+\frac{1}{3}yz^2+\frac{3}{2}y^4z^3)\,dy\,dz$

$=\frac{2}{3}\int_0^1\left[(\frac{1}{8}y^2z+\frac{1}{5}y^5z^2+\frac{1}{9}y^3z+\frac{1}{4}y^6z^2+\frac{1}{6}y^2z^2+\frac{3}{10}y^5z^3)\Big|_0^1\right]dz$

$=\frac{2}{3}\int_0^1(\frac{3}{10}z^3+\frac{37}{60}z^2+\frac{17}{72}z)\,dz=\frac{2}{3}(\frac{3}{40}z^4+\frac{37}{180}z^3+\frac{17}{144}z^2)\Big|_0^1=\frac{287}{1080}$

(f) Conditional prob of X, Y | Z is:

$f(x,y|z)=\frac{f(x,y,z)}{f(z)}=\frac{\cancel{\frac{2}{3}}(x+y+z)}{\cancel{\frac{2}{3}}(z+1)}=\frac{x+y+z}{z+1}$; $0<x<1$, $0<y<1$, $0<z<1$

$E(X+Y|z)=\int_0^1\int_0^1(x+y)\cdot\frac{x+y+z}{z+1}\,dx\,dy=\frac{1}{z+1}\int_0^1\int_0^1(x^2+2xy+xz+y^2+yz)\,dx\,dy$

$=\frac{1}{z+1}\int_0^1\left[\frac{1}{3}x^3+x^2y+\frac{1}{2}x^2z+y^2x+yzx\right]\Big|_0^1 dy$

$=\frac{1}{z+1}\int_0^1(y^2+yz+y+\frac{1}{2}z+\frac{1}{3})\,dy$

$=\frac{1}{z+1}\cdot(\frac{1}{3}y^3+\frac{1}{2}y^2z+\frac{1}{2}y^2+\frac{1}{2}zy+\frac{1}{3}y)\Big|_0^1$

$=\frac{6z+7}{6(z+1)}$, $0<z<1$

(g) $f(y,z)=\int_0^1\frac{2}{3}(x+y+z)\,dx=\left[\frac{1}{3}x^2+\frac{2}{3}(y+z)x\right]\Big|_0^1=\frac{2}{3}(y+z+\frac{1}{2})$; $0<y<1$, $0<z<1$

$f_{X|Y,Z}(x|y,z)=\frac{f(x,y,z)}{f(y,z)}=\frac{\cancel{\frac{2}{3}}(x+y+z)}{\cancel{\frac{2}{3}}(y+z+\frac{1}{2})}=\frac{2(x+y+z)}{2y+2z+1}$; $0<x<1$, $0<y<1$, $0<z<1$.

$E(X|y,z)=\int_0^1 x\cdot\frac{2(x+y+z)}{2y+2z+1}\,dx=\frac{2}{2y+2z+1}\cdot\left[\frac{1}{3}x^3+(y+z)\cdot\frac{1}{2}x^2\right]\Big|_0^1=\frac{3y+3z+6}{6y+6z+3}$; $0<y<1$, $0<z<1$

3) X_1, X_2, X_3, X_4 indpt. $f(x) = 3(1-x)^2,\ 0<x<1.$ $Y = \min(X_1, X_2, X_3, X_4).$

$P(X_i \le y) = \int_0^y 3(1-x)^2\,dx \overset{z=1-x}{=\!=} -\int_1^{1-y} 3z^2\,dz = -z^3\Big|_1^{1-y} = 1-(1-y)^3;\quad 0<y<1.$

then:

CDF: $P(Y \le y) = 1 - P(Y>y) = 1 - P(X_i > y,\ i=1,2,3,4)$

$= 1 - [P(X_i > y)]^4$

$= 1 - [1 - P(X_i \le y)]^4$

$= 1 - [\cancel{1} - (\cancel{1} - (1-y)^3)]^4$

$= 1 - [(1-y)^3]^4$

$= 1-(1-y)^{12};\quad 0<y<1$

PDF: $f(y) = 0 - 12(1-y)^{11}\cdot(-1) = 12(1-y)^{11};\ 0<y<1;$ #