# Stat 501 Probability Theory I (Final) Fall 2019

Name: ______________________ UIN: ________________

Problem 1. Let $\mathscr{C}$ be a collection of subsets of $\Omega$. Denote by $\sigma(\mathscr{C})$ the $\sigma$-field generated by $\mathscr{C}$ (and by $\mathscr{A}(\mathscr{C})$ the field generated by $\mathscr{C}$, respectively). Prove

$$\sigma(\mathscr{C}) = \sigma(\mathscr{A}(\mathscr{C})).$$

Problem 2. Suppose $X_n$ is a sequence of random variables that is Cauchy in probability. Show that there is a subsequence $X_{n_k}$ of $X_n$ such that $\lim_{n\to\infty} X_{n_k}$ exists almost surely.

Problem 3. Fix any $p \geq 1$. Let $a_1, ..., a_n$ and $b_1, ..., b_n$ be real numbers. Prove that

$$\left( \sum_{i=1}^{n} |a_i + b_i|^p \right)^{1/p} \leq \left( \sum_{i=1}^{n} |a_i|^p \right)^{1/p} + \left( \sum_{i=1}^{n} |b_i|^p \right)^{1/p}.$$

(Hint: Minkowski inequality for a proper probability space.)

Problem 4. Let $(\Omega, \mathcal{B}, \mathbb{P})$ be a probability space, and $X, Y$ be two bounded random variables. This problem is to show that if

$$\int_B X d\mathbb{P} = \int_B Y d\mathbb{P}, \quad \text{for all } B \in \mathcal{B},$$

then $X = Y$ almost surly. The proof is divided into the following steps.

(a) Note that the two events $A_1 = \{X > Y\}$ and $A_2 = \{X < Y\}$ are in $\mathcal{B}$. Show that $\int_{A_1} (X - Y) d\mathbb{P} = 0$ and $\int_{A_2} (Y - X) d\mathbb{P} = 0$.

(b) Use (a) to show that $\mathbb{P}(A_i) = 0, i = 1, 2$.

(c) Use (b) to conclude $X = Y$ almost surely.

Problem 5. Suppose $\mu$ is a probability measure on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$ such that for all intervals of the form $(a, b]$, $-\infty \le a \le b < +\infty$, we have

$$\mu((a,b]) = \int_{(a,b]} f d\lambda. \tag{1}$$

Here, $\lambda$ is the Lebesgue measure on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$, and $f \in L^1(\lambda)$ is a given function .

The aim of this problem is to show, under assumption (1), that

$$\mu(B) = \int_B f d\lambda, \tag{2}$$

for all $B \in \mathscr{B}(\mathbb{R})$. The proof is divided into two steps.

(a) Set

$$\mathscr{C} = \{B \in \mathscr{B}(\mathbb{R}), \quad \text{equation (2) holds for } B\}.$$

Show that $\mathscr{C}$ is a $\lambda$-system.

(b) Use (a) to show that (2) holds for all $B \in \mathscr{B}(\mathbb{R})$.