1 Exercise 9.9.9 (a) (b) (c) on page 323

Answer: **(a)** The characteristic function of X is

$$\begin{aligned}\phi_X(t) &= E\left(e^{itX}\right) = \int_0^\infty e^{itx} \cdot e^{-x} dx \\ &= \int_0^\infty e^{-(1-it)x} dx \\ &= -\frac{1}{1-it}\, e^{-(1-it)x}\Big|_0^\infty \\ &= \frac{1}{1-it}\end{aligned}$$

because $\lim_{x\to\infty} e^{-(1-it)x} = \lim_{x\to\infty} e^{-x} \cdot e^{itx} = 0$ due to $|e^{itx}| \le 1$.
Let $Y = -X$, then

$$\phi_Y(t) = E\left(e^{itY}\right) = E\left(e^{i(-t)X}\right) = \phi_X(-t) = \frac{1}{1+it}.$$

The answer is "yes". The function $\frac{1}{1+it}$ is a characteristic function.

(b) The characteristic function of X_1 is

$$\begin{aligned}\phi_1(t) &= E\left(e^{itX_1}\right) = e^{it\cdot 1} \cdot P[X_1 = 1] + e^{it\cdot(-1)} \cdot P[X_1 = -1] \\ &= \frac{1}{2}\left(e^{it} + e^{-it}\right) \\ &= \cos t\end{aligned}$$

(c) Yes. Let $X_1, X_2, \ldots, X_{17}$ be iid $\sim X_1$. Then $Y = \sum_{j=1}^{17} X_j$ has characteristic function $(\cos t)^{17}$. □

2 Exercise 9.9.16 on page 325

Proof: **(a)** Since X and Y are iid $N(0,1)$ random variables, then $\frac{X+Y}{\sqrt{2}}$ is also normally distributed with mean 0 and variance $(1+1)/2 = 1$. That is,

$$\frac{X+Y}{\sqrt{2}} \overset{d}{=} X \overset{d}{=} Y.$$

(b) Let $X_1, X_2, \ldots, X_n, \ldots$ be iid $\sim F$. Since $\frac{X+Y}{\sqrt{2}} \overset{d}{=} X \overset{d}{=} Y$, then $\frac{X_1+X_2}{\sqrt{2}} \overset{d}{=} X_1 \overset{d}{=} X$. Similarly,

$$\frac{X_1+X_2+X_3+X_4}{\sqrt{4}} = \frac{\frac{X_1+X_2}{\sqrt{2}} + \frac{X_3+X_4}{\sqrt{2}}}{\sqrt{2}} \overset{d}{=} \frac{X_1+X_2}{\sqrt{2}} \overset{d}{=} X_1 \overset{d}{=} X.$$

Repeat the procedure and get in general, $\forall m \geq 1$,

$$\frac{1}{\sqrt{2^m}} \sum_{j=1}^{2^m} X_j \stackrel{d}{=} X_1 \stackrel{d}{=} X.$$

On the other hand, the classical central limit theorem (Theorem 9.7.1 on page 313) implies that

$$\frac{1}{\sqrt{n}} \sum_{j=1}^{n} X_j \Longrightarrow N(0,1).$$

As a subsequence,

$$\frac{1}{\sqrt{2^m}} \sum_{j=1}^{2^m} X_j \Longrightarrow N(0,1).$$

By Remark 8.1.2 on page 252, weak limits are unique. So both X and Y must have a $N(0,1)$ distribution. □

3 Exercise 9.9.28 on page 328

Proof: **(1)** By Exercise 9.9.9(a), the characteristic function of $F(x) = 1 - e^{-x}, x > 0$ is $1/(1-it)$.

(2) If E_1, E_2 are iid $\sim\ F$, then the characteristic function of $E_1 - E_2$ is

$$\phi_{12}(t) = E\left(e^{it(E_1-E_2)}\right) = E\left(e^{itE_1}\right) \cdot E\left(e^{i(-t)E_2}\right) = \frac{1}{1-it} \cdot \frac{1}{1-i(-t)} = \frac{1}{1+t^2}\ .$$

(3) To show that the characteristic function of the Cauchy density is $\phi(t) = e^{-|t|}$, we first find out the distribution function F_{12} of $E_1 - E_2$. Actually, for any $x \geq 0$,

$$\begin{aligned} P[E_1 - E_2 \leq x] &= \int_0^x \left(\int_0^\infty e^{-x_2} \cdot e^{-x_1} dx_2\right) dx_1 + \int_x^\infty \left(\int_{x_1-x}^\infty e^{-x_2} e^{-x_1} dx_2\right) dx_1 \\ &= \int_0^x e^{-x_1} dx_1 + \int_x^\infty e^{x-2x_1} dx_1 \\ &= 1 - e^x + \frac{1}{2} e^{-x} \\ &= 1 - \frac{1}{2} e^{-x}\ . \end{aligned}$$

For any $x < 0$,

$$P[E_1 - E_2 \leq x] = \int_0^\infty \left(\int_{x_1-x}^\infty e^{-x_2} e^{-x_1} dx_2\right) dx_1 = \int_0^\infty e^{x-2x_1} dx_1 = \frac{1}{2} e^x\ .$$

That is, the distribution function of $E_1 - E_2$

$$F_{12}(x) = \begin{cases} \frac{1}{2} e^x, & \text{if } x < 0 \\ 1 - \frac{1}{2} e^{-x}, & \text{if } x \geq 0 \end{cases}$$

Differentiate it and we get a continuous density function $f_{12}(x) = \frac{1}{2}e^{-|x|}$ of $E_1 - E_2$.

Note that the characteristic function of $E_1 - E_2$ is integrable since $\int_{-\infty}^{\infty} \frac{1}{1+t^2} dt = \pi < \infty$. By Corollary 9.5.1 on page 303, for any $x \in \mathbb{R}$,

$$\frac{1}{2}e^{-|x|} = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-iyx}\frac{1}{1+y^2}dy \ .$$

Then for any $t \in \mathbb{R}$,

$$\begin{aligned} e^{-|t|} &= \int_{\mathbb{R}} e^{-ixt}\frac{1}{\pi(1+x^2)}dx \\ &\overset{y=-x}{=} \int_{\mathbb{R}} e^{ity}\frac{1}{\pi(1+y^2)}dy \\ &= Ee^{itX}, \end{aligned}$$

where X follows Cauchy(0,1). That is, $\phi(t) = e^{-|t|}$ is the characteristic function of Cauchy(0,1).

(4) If X_1, X_2 are iid $\sim$ Cauchy(0,1), then

$$Ee^{it(X_1+X_2)} = \left(Ee^{itX_1}\right)\left(Ee^{itX_2}\right) = e^{-|t|}\cdot e^{-|t|} = e^{-2|t|} = Ee^{it(2X_1)}$$

which indicates $X_1 + X_2 \overset{d}{=} 2X_1$. That is, the convolution of two Cauchy(0,1) densities is a density of the same type. □