Math 310, Fall 2015
Instructor: Chris Skalit
Exam 1

Write your ***FULL NAME*** *and UIN in all of your answer books. Show* ***ALL*** *work.*

1. (20 points) Find all solutions to the following system of linear equations:

$$\begin{array}{rcrcrcr} 2x_1 & + & x_2 & & & = & 0 \\ 2x_1 & + & 2x_2 & & & = & 2 \\ 3x_1 & + & 2x_2 & + & x_3 & = & 2 \\ 4x_1 & & & + & 3x_3 & = & -1 \end{array}$$

Solution: The augmented matrix for this system is $B = \left[\begin{array}{ccc|c} 2 & 1 & 0 & 0 \\ 2 & 2 & 0 & 2 \\ 3 & 2 & 1 & 2 \\ 4 & 0 & 3 & -1 \end{array}\right]$. Since rref $B = \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$, we have $x_1 = -1$, $x_2 = 2$, and $x_3 = 1$ as our unique solution.

2. Let $A = \begin{bmatrix} 2 & 0 & 4 & 6 \\ 1 & 1 & -1 & -1 \\ 4 & 2 & 2 & 4 \end{bmatrix}$.

   (a) (10 points) Find all $\mathbf{x} \in \mathbb{R}^4$ such that $A\mathbf{x} = \mathbf{0}$. Write your solution in vector parametric form.

   Solution: Recall that if $B = \text{rref } A$, we have $B\mathbf{x} = \mathbf{0}$ if and only if $A\mathbf{x} = \mathbf{0}$. Since $B = \text{rref } A = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & -3 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, we see that in the homogeneous system

$$\begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & -3 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

   $x_1 = -2x_3 - 3x_4$ and $x_2 = 3x_3 + 4x_4$ with x_3 and x_4 free. We can therefore write our solutions as

$$\mathcal{S} = \left\{ \begin{bmatrix} -2x_3 - 3x_4 \\ 3x_3 + 4x_4 \\ x_3 \\ x_4 \end{bmatrix} : x_3, x_4 \in \mathbb{R} \right\} = \left\{ x_3 \begin{bmatrix} -2 \\ 3 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -3 \\ 4 \\ 0 \\ 1 \end{bmatrix} : x_3, x_4 \in \mathbb{R} \right\}$$

(b) (5 points) If $\mathbf{b} = \begin{bmatrix} 2 \\ 2 \\ 6 \end{bmatrix}$, use your solution to part (a) to find all solutions to $A\mathbf{x} = \mathbf{b}$.

Hint: Note that $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ is a solution.

Solution: Recall that all solutions $\mathbf{w}$ to $A\mathbf{x} = \mathbf{b}$ are of the form $\mathbf{w} = \mathbf{x}_0 + \mathbf{v}$ where $\mathbf{v}$ is a solution to the homogeneous system $A\mathbf{x} = \mathbf{0}$. Thus our solution set is

$$\mathcal{S} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 3 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -3 \\ 4 \\ 0 \\ 1 \end{bmatrix} : x_3, x_4 \in \mathbb{R} \right\}$$

(c) (5 points) Does there exist a $\mathbf{v} \in \mathbb{R}^3$ such that the linear system $A\mathbf{x} = \mathbf{v}$ is **inconsistent**? Explain why or why not.

Solution: We know that $A\mathbf{x} = \mathbf{v}$ is consistent for all $\mathbf{v} \in \mathbb{R}^3$ if and only if the columns of A span $\mathbb{R}^3$. This, however, is equivalent to saying that every ROW of rref A contains a pivot. Since this is not the case, we know that we can find a $\mathbf{v}$ for which $A\mathbf{x} = \mathbf{v}$ is inconsistent.

3. (a) (5 points) Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear map defined via $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x + 2y \\ y - x \\ 3x \end{bmatrix}$.

Write down the matrix H such that $T(\mathbf{x}) = H\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^2$.

Solution: The columns of H are just $T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$ and $T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$. Thus, $H = \begin{bmatrix} 1 & 2 \\ -1 & 1 \\ 3 & 0 \end{bmatrix}$.

(b) (10 points) Let $S : \mathbb{R}^3 \to \mathbb{R}^4$ be defined by $S(\mathbf{x}) = B\mathbf{x}$ where $B = \begin{bmatrix} 3 & 3 & -3 \\ -2 & 0 & 1 \\ 0 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix}$.

Determine whether S is one-to-one and/or onto. Justify your answer.

Solution: rref $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. Since there is a pivot in every column, S is one-to-one. Since there isn't a pivot in every row, S is NOT onto.

(c) (10 points) Let $Q : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation such that

$$Q\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad Q\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Compute $Q\left(\begin{bmatrix}5\\1\end{bmatrix}\right)$. **Hint:** It will be helpful (and good for partial credit) to first find constants $a, b \in \mathbb{R}$ such that $\begin{bmatrix}5\\1\end{bmatrix} = a\begin{bmatrix}1\\2\end{bmatrix} + b\begin{bmatrix}1\\-1\end{bmatrix}$.

Solution: Finding a and b amounts to solving the system

$$\begin{array}{rcrcl} a & + & b & = & 5 \\ 2a & - & b & = & 1 \end{array}$$

and we see that $a = 2$ and $b = 3$. We now compute

$$Q\left(\begin{bmatrix}5\\1\end{bmatrix}\right) = Q\left(2\begin{bmatrix}1\\2\end{bmatrix} + 3\begin{bmatrix}1\\-1\end{bmatrix}\right) = 2Q\left(\begin{bmatrix}1\\2\end{bmatrix}\right) + 3Q\left(\begin{bmatrix}1\\-1\end{bmatrix}\right) = \begin{bmatrix}8\\3\end{bmatrix}$$

4. (a) (7 points) Compute the matrix product or explain why the product is undefined.

   (i) $\begin{bmatrix}1 & 2\\-1 & 0\\1 & 1\end{bmatrix}\begin{bmatrix}1 & 2\\3 & 0\end{bmatrix}$

   Solution:

$$\begin{bmatrix}1 & 2\\-1 & 0\\1 & 1\end{bmatrix}\begin{bmatrix}1 & 2\\3 & 0\end{bmatrix} = \begin{bmatrix}7 & 2\\-1 & -2\\4 & 2\end{bmatrix}$$

   (ii) $\begin{bmatrix}1 & 2\\3 & 0\end{bmatrix}\begin{bmatrix}1 & 2\\-1 & 0\\1 & 1\end{bmatrix}$

   Solution: Undefined. The number of columns of the matrix on the left is not equal to the number of rows of the matrix on the right.

   (b) (13 points) Let $D = \begin{bmatrix}-1 & -1 & 0\\2 & 1 & 0\\5 & 1 & 1\end{bmatrix}$. Find D^{-1} (if it exists).

   Solution: We set $E = \left[\begin{array}{ccc|ccc}-1 & -1 & 0 & 1 & 0 & 0\\2 & 1 & 0 & 0 & 1 & 0\\5 & 1 & 1 & 0 & 0 & 1\end{array}\right]$ and compute

   $\text{rref}\, E = \left[\begin{array}{ccc|ccc}1 & 0 & 0 & 1 & 1 & 0\\0 & 1 & 0 & -2 & -1 & 0\\0 & 0 & 1 & -3 & -4 & 1\end{array}\right]$. Thus, $D^{-1} = \begin{bmatrix}1 & 1 & 0\\-2 & -1 & 0\\-3 & -4 & 1\end{bmatrix}$.

   (c) (5 points) Are the columns of D linearly independent? Why or why not?

   Solution: Since D is invertible, its columns are linearly independent.

5. (10 points) Let $C = \begin{bmatrix}1 & 1 & 0\\1 & 2 & 0\\0 & 3 & 1\end{bmatrix}$. Compute the LU decomposition of C. That is, find

matrices of the form $L = \begin{bmatrix} 1 & 0 & 0 \\ * & 1 & 0 \\ * & * & 1 \end{bmatrix}$ and $U = \begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{bmatrix}$ such that $C = LU$.

Solution: We begin by reducing C to an upper-triangular matrix U:

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 3 & 1 \end{bmatrix} \xrightarrow{R2 \mapsto R2 - R1} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix} \xrightarrow{R3 \mapsto R3 - 3R2} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = U$$

We now take the 3×3 identity matrix and apply the inverse of these operations in the reverse order:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R3 \mapsto R3 + 3R2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix} \xrightarrow{R2 \mapsto R2 + R1} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix} = L$$