Math 310, Fall 2015
Instructor: Chris Skalit
Exam II

*Write your **FULL NAME** and UIN in all of your answer books. Show **ALL** work.*

1. Let $A = \begin{bmatrix} 3 & 6 & 3 & 6 \\ 2 & 4 & 4 & 4 \\ -1 & 1 & -1 & -2 \\ 3 & 6 & 3 & 8 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$.

   (a) (10 points) Compute $\det A$. **Hint:** Row reduce before attempting a cofactor expansion.

   Solution: Multiplying the first row by 1/3 gives $\begin{bmatrix} 1 & 2 & 1 & 2 \\ 2 & 4 & 4 & 4 \\ -1 & 1 & -1 & -2 \\ 3 & 6 & 3 & 8 \end{bmatrix}$. Adding suitable multiples of the first row of this matrix to those below it yields $A' = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 0 & 2 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$. We can compute the determinant of A' via cofactor expansion down the first row:

   $$\det(A') = 1 \det\left(\begin{bmatrix} 0 & 2 & 0 \\ 3 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}\right) = (-3)(2)(2) = -12$$

   Finally, by recalling the row operations that transformed A to A', we have $\det(A') = (1/3)\det A$, so $\det A = -36$.

   (b) (1 point) What is $\det B$?

   Solution: Since the columns of B are clearly not linearly independent, B is not invertible and hence $\det B = 0$.

   (c) (4 points) What is $\det(AB)$? Is the matrix product AB invertible? Why or why not?

   Solution: $\det(AB) = (\det A)(\det B) = 0$ by part (b). Hence, AB is not invertible.

2. Let $C = \begin{bmatrix} -1 & -2 & 1 & 4 \\ 1 & 2 & 0 & -1 \\ 2 & 4 & 1 & 1 \end{bmatrix}$. Note that $\text{rref}(C) = \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

   (a) (3 points) Find a basis for $\text{Col}(C)$. What is its dimension?

Solution: Since rref C has pivots in the first and third columns, we know that those columns of C form a basis for the column space, meaning that $\operatorname{Col} C$ has basis $\left\{ \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$ and $\dim(\operatorname{Col} C) = 2$.

(b) (2 points) Find a basis for $\operatorname{Row}(C)$. What is its dimension?

Solution: Since $\operatorname{Row}(A) = \operatorname{Row}(\operatorname{rref} A)$, it's clear that our basis is $\left\{ \begin{bmatrix} 1 & 2 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 3 \end{bmatrix} \right\}$, and $\dim(\operatorname{Row} C) = 2$.

(c) (5 points) Find a basis for $\operatorname{Nul}(C)$. What is its dimension?

Solution: For $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$, the nullspace is the set of solutions to $C\mathbf{x} = \mathbf{0}$, or equivalently, the solution set to $(\operatorname{rref} C)\mathbf{x} = \mathbf{0}$. We obtain the relations

$$\begin{array}{rcl} x_1 + 2x_2 - x_4 & = & 0 \\ x_3 + 3x_4 & = & 0 \end{array},$$

so $$\operatorname{Nul}(C) = \left\{ \begin{bmatrix} -2x_2 + x_4 \\ x_2 \\ -3x_4 \\ x_4 \end{bmatrix} : x_2, x_4 \in \mathbb{R} \right\} = \left\{ x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ -3 \\ 1 \end{bmatrix} : x_2, x_4 \in \mathbb{R} \right\}.$$

Our basis therefore consists of $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -3 \\ 1 \end{bmatrix} \right\}$, and $\dim(\operatorname{Nul} C) = 2$.

(d) (5 points) Suppose that M is a 7×8 matrix whose column space has dimension 5. What are $\dim(\operatorname{Nul}(M))$ and $\dim(\operatorname{Row}(M))$?

Solution: M defines a linear transformation $\mathbb{R}^8 \to \mathbb{R}^7$. The rank nullity theorem says that $\dim(\operatorname{Nul} M) + \dim(\operatorname{Col} M) = 8$, so $\dim(\operatorname{Nul} M) = 3$. The column space and row space of a matrix have the same dimension, so $\dim(\operatorname{Row} M) = 5$.

3. Let V be a real, 3-dimensional vector space with bases $\mathcal{A} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ and $\mathcal{B} = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$. Suppose that the following relations are known:

$$\begin{array}{rcl} \mathbf{v}_1 & = & \mathbf{w}_3 \\ \mathbf{v}_2 & = & \mathbf{w}_1 + \mathbf{w}_2 \\ \mathbf{v}_3 & = & \mathbf{w}_1 + 2\mathbf{w}_2 \end{array}$$

(a) (7 points) Write down the change of basis matrix $P_{\mathcal{A} \to \mathcal{B}}$, which transforms $\mathcal{A}$-coordinates to $\mathcal{B}$-coordinates.

Solution:

$$P_{\mathcal{A}\to\mathcal{B}} = \begin{bmatrix}[\mathbf{v}_1]_{\mathcal{B}} & [\mathbf{v}_2]_{\mathcal{B}} & [\mathbf{v}_3]_{\mathcal{B}}\end{bmatrix} = \begin{bmatrix}0 & 1 & 1\\ 0 & 1 & 2\\ 1 & 0 & 0\end{bmatrix}$$

(b) (8 points) Compute $P_{\mathcal{B}\to\mathcal{A}}$.

Solution:

$$P_{\mathcal{B}\to\mathcal{A}} = (P_{\mathcal{A}\to\mathcal{B}})^{-1} = \begin{bmatrix}0 & 0 & 1\\ 2 & -1 & 0\\ -1 & 1 & 0\end{bmatrix}$$

(c) (5 points) If $\mathbf{x} \in V$ is a vector whose $\mathcal{A}$-coordinates are $[\mathbf{x}]_{\mathcal{A}} = \begin{bmatrix}1\\ 2\\ -1\end{bmatrix}$, what is $[\mathbf{x}]_{\mathcal{B}}$?

Solution:

$$[\mathbf{x}]_{\mathcal{B}} = (P_{\mathcal{A}\to\mathcal{B}})[\mathbf{x}]_{\mathcal{A}} = \begin{bmatrix}0 & 1 & 1\\ 0 & 1 & 2\\ 1 & 0 & 0\end{bmatrix}\begin{bmatrix}1\\ 2\\ -1\end{bmatrix} = \begin{bmatrix}1\\ 0\\ 1\end{bmatrix}$$

(d) (5 points) Let $T : V \to V$ be linear such that $T(\mathbf{v}_1) = \mathbf{v}_1 - \mathbf{v}_3$, $T(\mathbf{v}_2) = \mathbf{v}_1 + \mathbf{v}_2$, and $T(\mathbf{v}_3) = 0$. Find the matrix $[T]_{\mathcal{A}}$ for T with respect to the basis $\mathcal{A}$.

Solution:

$$[T]_{\mathcal{A}} = \begin{bmatrix}[T(\mathbf{v}_1)]_{\mathcal{A}} & [T(\mathbf{v}_2)]_{\mathcal{A}} & [T(\mathbf{v}_3)]_{\mathcal{A}}\end{bmatrix} = \begin{bmatrix}1 & 1 & 0\\ 0 & 1 & 0\\ -1 & 0 & 0\end{bmatrix}$$

(e) (5 points) Write $[T]_{\mathcal{B}}$ as a product of three matrices. You do **NOT** need to compute the product.

Solution:

$$[T]_{\mathcal{B}} = (P_{\mathcal{A}\to\mathcal{B}})[T]_{\mathcal{A}}(P_{\mathcal{B}\to\mathcal{A}}) = \begin{bmatrix}0 & 1 & 1\\ 0 & 1 & 2\\ 1 & 0 & 0\end{bmatrix}\begin{bmatrix}1 & 1 & 0\\ 0 & 1 & 0\\ -1 & 0 & 0\end{bmatrix}\begin{bmatrix}0 & 0 & 1\\ 2 & -1 & 0\\ -1 & 1 & 0\end{bmatrix}$$

4. Let $E = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & -1 \\ 0 & 0 & 2 \end{bmatrix}$.

(a) (6 points) Compute the characteristic polynomial of E and find all eigenvalues.

Solution: The characteristic polynomial is

$$P_E(t) = \det(E - tI) = \det\left(\begin{bmatrix} 1-t & 0 & 1 \\ 1 & 2-t & -1 \\ 0 & 0 & 2-t \end{bmatrix}\right) = (2-t)(1-t)(2-t).$$

The eigenvalues are therefore 1 and 2.

(b) (11 points) For each eigenvalue λ found in part (a), compute the corresponding eigenspace E_λ.

Solution: To compute the 1-eigenspace we have

$$\mathbf{E}_1 = \text{Nul}(E - I) = \text{Nul}\left(\begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}\right) = \text{Nul}\left(\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\right)$$

Hence $\mathbf{E}_1 = \text{Span}\left\{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}\right\}$.

Moving on to the 2-eigenspace gives

$$\mathbf{E}_2 = \text{Nul}(E - 2I) = \text{Nul}\left(\begin{bmatrix} -1 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}\right) = \text{Nul}\left(\begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right)$$

In other words, we have a system of linear equations in three variables with $x_1 - x_3 = 0$ where x_2 and x_3 are free. Thus, $\mathbf{E}_2 = \text{Span}\left\{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right\}$.

(c) (3 points) Is E diagonalizable? Why or why not?

Solution: We have found three linearly independent eigenvectors for A. Thus $\mathbb{R}^3$ has a basis consisting of eigenvectors for A, so A is diagonalizable.

5. Let $H = \begin{bmatrix} 1 & 0 \\ 6 & -1 \end{bmatrix}$. Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Observe that

$$H\mathbf{v}_1 = \mathbf{v}_1 \qquad H\mathbf{v}_2 = (-1)\mathbf{v}_2$$

so $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis of eigenvectors for H.

(a) (6 points) Diagonalize H. That is, find a factorization $H = SDS^{-1}$ where D is diagonal.

Solution: We let the columns of S be the eigenvectors of A (which have already been computed for us): $S = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$. $S^{-1} = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$. D is a diagonal matrix whose diagonal entries are the eigenvalues which correspond to the eigenvectors in the columns of S: $D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

(b) (6 points) Compute H^{1001}. **Hint:** Use part (a).

Solution: We know that $H^{1001} = SD^{1001}S^{-1}$. Since D is diagonal, we have $D^{1001} = \begin{bmatrix} 1^{1001} & 0 \\ 0 & (-1)^{1001} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. It then follows that

$$H^{1001} = SD^{1001}S^{-1} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 6 & -1 \end{bmatrix}$$

Remark: For this example, it's actually possible to compute H^{1001} without explicitly knowing S^{-1}. We note that D^{1000} is just the identity matrix I, so $H^{1000} = SIS^{-1} = I$, and $H^{1001} = H(H^{1000}) = H$.

(c) (8 points) Consider the system of differential equations $\frac{d\mathbf{x}}{dt} = H\mathbf{x}(t)$. Find the particular solution $\mathbf{x}(t)$ subject to the initial condition $\mathbf{x}(0) = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$.

Solution: Since our matrix is diagonalizable, our system of differential equations has the following general solution:

$$\mathbf{x}(t) = c_1 e^t \begin{bmatrix} 1 \\ 3 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

To find the particular solutions (i.e. determine the unknown constants c_1 and c_2, we solve

$$\begin{bmatrix} 5 \\ 3 \end{bmatrix} = \mathbf{x}(0) = c_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

From this it's easily seen that $c_1 = 5$, $c_2 = -12$.