Math 310, Fall 2015
Instructor: Chris Skalit
Quiz 2

Name: ______________________ UIN: ______________________

1. Compute each of the products $A\mathbf{v}$ or explain why the product is undefined.

   (a) (1 point) $\begin{bmatrix} 1 & 0 & -1 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$

   Solution: $\begin{bmatrix} 1 & 0 & -1 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 3 \end{bmatrix} + 2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 7 \end{bmatrix}$

   (b) (1 point) $\begin{bmatrix} 4 & 0 & 1 \\ 5 & 3 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

   Solution: The product isn't defined because the matrix has 3 columns while the vector only has 2.

2. Let $A = \begin{bmatrix} 2 & 2 & 4 & 0 \\ -1 & 5 & 10 & 6 \\ 0 & 1 & 2 & 1 \end{bmatrix}$

   (a) (4 points) Find all solutions to the homogeneous equation $A\mathbf{x} = \mathbf{0}$. Express your solutions in vector parametric form.

   Solution: We know that $\text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. If $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$, then by reading off the entries of $\text{rref}(A)$, we see that $x_1 - x_4 = 0$ and $x_2 + 2x_3 + x_4 = 0$. Our dependent variables, corresponding to our two pivot columns, are x_1 and x_2. By setting $x_3 = s$ and $x_4 = t$, we can write our solution set as

$$\mathcal{S} = \left\{ \begin{bmatrix} t \\ -2s - t \\ s \\ t \end{bmatrix} : s, t \in \mathbb{R} \right\} = \left\{ s \begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} : s, t \in \mathbb{R} \right\}$$

(b) (1 point) Find all solutions to $A\mathbf{x} = \mathbf{b}$ where $\mathbf{b} = \begin{bmatrix} 4 \\ 4 \\ 1 \end{bmatrix}$. **Hint:** Use part (a) and the observation that $A\mathbf{v} = \mathbf{b}$ where $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$. No computation is needed.

Solution: We know that the every solution $\mathbf{w}$ of $A\mathbf{x} = \mathbf{b}$ may be expressed in the form $\mathbf{w} = \mathbf{v} + \mathbf{h}$ where $\mathbf{h}$ is some solution to the homogeneous equation $A\mathbf{x} = \mathbf{0}$ and $\mathbf{v}$ is one particular solution to $A\mathbf{x} = \mathbf{b}$. Our solutions are therefore of the form

$$\mathcal{S} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} : s, t \in \mathbb{R} \right\}$$

3. (3 points) Balance the chemical equation for the combustion of pure acetic acid:

$$\mathrm{CH_3\,COOH + O_2 \longrightarrow H_2\,O + CO_2}$$

Solution: We want to find constants x_1, x_2, x_3, x_4 so that the number of each type of atom is conserved during the reaction:

$$x_1\,\mathrm{CH_3\,COOH} + x_2\,\mathrm{O_2} \longrightarrow x_3\,\mathrm{H_2\,O} + x_4\,\mathrm{CO_2}$$

Counting atoms yields the relations

$$\begin{array}{rcll} 2x_1 & = & x_4 & \text{(Carbon)} \\ 4x_1 & = & 2x_3 & \text{(Hydrogen)} \\ 2x_1 + 2x_2 & = & x_3 + 2x_4 & \text{(Oxygen)} \end{array}$$

which means that we must solve the homogeneous equation:

$$\begin{bmatrix} 2 & 0 & 0 & -1 \\ 4 & 0 & -2 & 0 \\ 2 & 2 & -1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Solving this system gives $x_1 = (1/2)x_4$, $x_2 = x_4$, and $x_3 = x_4$ where x_4 is a free variable. If we set $x_4 = 2$, then we have

$$\mathrm{CH_3\,COOH} + 2\,\mathrm{O_2} \longrightarrow 2\,\mathrm{H_2\,O} + 2\,\mathrm{CO_2}$$