Math 310 (35180), Fall 2015
Instructor: Chris Skalit
Quiz 4

Name: ______________________________ UIN: ______________________________

1. (2 points) Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 - x_3 \\ 2x_1 + x_3 \end{bmatrix}$$

You do **not** need to check that T is linear. Find the matrix A such that $T(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^3$.

Solution: We begin by computing $T(\mathbf{e}_i)$ for $i = 1, 2, 3$:

$$T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

whence we conclude that A, the matrix representing T, is given by

$$A = \begin{bmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) & T(\mathbf{e}_3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 0 & 1 \end{bmatrix}$$

2. Let $T : \mathbb{R}^4 \to \mathbb{R}^2$ be defined by $T(\mathbf{x}) = A\mathbf{x}$ where $A = \begin{bmatrix} -1 & -2 & 1 & 1 \\ -2 & -4 & 1 & -2 \end{bmatrix}$.

   (a) (2 points) Compute $\text{rref}(A)$.

   Solution: $\text{rref}(A) = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 4 \end{bmatrix}$

   (b) (1 point) Based on your answer to (a), is T onto? Why or why not?

   Solution: T is onto because every row of $\text{rref}(A)$ contains a pivot. In other words, $A\mathbf{x} = \mathbf{b}$ is consistent for all $\mathbf{b} \in \mathbb{R}^2$, meaning that the range of T is all of $\mathbb{R}^2$.

   (c) (1 point) Based on your answer to (a), is T one-to-one? Why or why not?

   Solution: T is not one-to-one because not every column of $\text{rref}(A)$ is a pivot column. Therefore, the homogeneous system $A\mathbf{x} = \mathbf{0}$ has free variables and hence infinitely many solutions.

3. (4 points) Let $A = \begin{bmatrix} 2 & 0 & 0 \\ -3 & 1 & -1 \\ 3 & 0 & 1 \end{bmatrix}$. Compute A^{-1} (if it exists).

Solution: We let $B = \left[\begin{array}{c|c} A & I_3 \end{array}\right] = \left[\begin{array}{rrr|rrr} 2 & 0 & 0 & 1 & 0 & 0 \\ -3 & 1 & -1 & 0 & 1 & 0 \\ 3 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$. Since we have that

$$\text{rref}(B) = \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -3/2 & 0 & 1 \end{array}\right]$$

we can read off $A^{-1} = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1 & 1 \\ -3/2 & 0 & 1 \end{bmatrix}$.