Math 310 (35180), Fall 2015
Instructor: Chris Skalit
Quiz 7

Name: ______________________ UIN: ______________________

1. Consider the vector space $V = \mathbb{R}^3$, equpped with the basis $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ where

$$\mathbf{v}_1 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

   (a) (1 point) Write the down the vector $\mathbf{x} \in \mathbb{R}^3$ whose coordinate vector with repsect to $\mathcal{B}$ is $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$.

   Solution: Reading off the coordinates, we have

$$\mathbf{x} = \mathbf{v}_1 + 2\mathbf{v}_2 - 3\mathbf{v}_3 = \begin{bmatrix} -5 \\ 1 \\ -2 \end{bmatrix}$$

   (b) (5 points) If $\mathbf{w} = \begin{bmatrix} -5 \\ 7 \\ 3 \end{bmatrix}$, what is the the coordinate vector $[\mathbf{w}]_{\mathcal{B}}$?

   Solution: We need to find scalars a, b, c such that $\mathbf{w} = a\mathbf{v}_1 + b\mathbf{v}_2 + c\mathbf{v}_3$. In other words, we solve the system

$$\begin{bmatrix} -1 & -2 & 0 \\ 2 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -5 \\ 7 \\ 3 \end{bmatrix} \Rightarrow [\mathbf{w}]_{\mathcal{B}} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$$

2. Suppose that $A = \begin{bmatrix} 2 & 2 & 0 & 2 \\ -4 & -4 & 7 & 17 \\ 2 & 2 & 7 & 23 \end{bmatrix}$. Note that $\operatorname{rref} A = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

   (a) (1 point) Write down a basis for $\operatorname{Row}(A)$. What is the dimension of this space?

   Solution: A and $\operatorname{rref} A$ have the same rowspace, so the basis is manifestly $\left\{ \begin{bmatrix} 1 & 1 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 3 \end{bmatrix} \right\}$. Hence $\dim \operatorname{Row}(A) = 2$.

(b) (1 point) Write down a basis for Col(A). What is the dimension of this space?

Solution: Since there are pivots in the first and third rows, the column space has basis $\left\{ \begin{bmatrix} 2 \\ -4 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 7 \\ 7 \end{bmatrix} \right\}$ and $\dim \operatorname{Col}(A) = 2$.

(c) (2 points) Write down a basis for Nul(A). What is the dimension of this space?

Solution: We solve $A\mathbf{x} = \mathbf{0}$ where $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$. From rref A, we see that $x_1 + x_2 + x_4 = 0$ and $x_3 + 3x_4 = 0$ where x_2 and x_4 are free. Hence,

$$\operatorname{Nul}(A) = \left\{ \begin{bmatrix} -x_2 - x_4 \\ x_2 \\ -3x_4 \\ x_4 \end{bmatrix} : x_2, x_4 \in \mathbb{R} \right\} = \left\{ x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ -3 \\ 1 \end{bmatrix} : x_2, x_4 \in \mathbb{R} \right\}.$$

Hence, our basis is $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -3 \\ 1 \end{bmatrix} \right\}$ and $\dim \operatorname{Nul}(A) = 2$.