Math 310 (35180), Fall 2015
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Quiz 8

Name: ____________________ UIN: ____________________

1. (4 points) Let V be a vector space with bases $\mathcal{A} = \{\mathbf{u}_1, \mathbf{u}_2\}$ and $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$. Suppose that

$$\mathbf{u}_1 = \mathbf{v}_1 - \mathbf{v}_2 \qquad \mathbf{u}_2 = \mathbf{v}_1 - 2\mathbf{v}_2$$

Find the change of coordinates matrix $P_{\mathcal{A}\to\mathcal{B}}$ from $\mathcal{A}$ to $\mathcal{B}$ and compute the change of coordinates matrix $P_{\mathcal{B}\to\mathcal{A}}$ from $\mathcal{B} \to \mathcal{A}$.

Solution: We have the basis vectors of $\mathcal{A}$ written in terms of $\mathcal{B}$, so we can write down

$$P_{\mathcal{A}\to\mathcal{B}} = \begin{bmatrix}[\mathbf{u}_1]_{\mathcal{B}} & [\mathbf{u}_2]_{\mathcal{B}}\end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}$$

The other change of coordinates matrix is just the inverse:

$$P_{\mathcal{B}\to\mathcal{A}} = (P_{\mathcal{A}\to\mathcal{B}})^{-1} = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$$

2. (3 points) Let $A = \begin{bmatrix} 2 & 1 & 2 \\ 0 & 1 & 0 \\ 1 & 1 & 3 \end{bmatrix}$. Compute the characteristic polynomial and find all eigenvalues of A.

Solution: $P_A(t) = \det(A - tI) = \det\left(\begin{bmatrix} 2-t & 1 & 2 \\ 0 & 1-t & 0 \\ 1 & 1 & 3-t \end{bmatrix}\right)$. By expanding the determinant along the second row, we have

$$P_A(t) = (1-t)[(2-t)(3-t) - 2] = (1-t)[t^2 - 5t + 4] = (1-t)(1-t)(4-t).$$

Hence, our eigenvalues are 1 (with an algebraic multiplicity of two) and 4 (with an algebraic multiplicity of one).

3. (3 points) Let $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$. Compute the eigenspace E_1 corresponding to the eigenvalue 1.

Solution: $E_1 = \text{Nul}(A - 1I) = \text{Nul}\left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}\right) = \text{Nul}\left(\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\right)$. In other words, E_1 is just the solution set of the homogeneous system

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

so $x_1 = -x_2$, $x_3 = 0$, and x_2 is free. Hence,

$$E_1 = \left\{ \begin{bmatrix} -x_2 \\ x_2 \\ 0 \end{bmatrix} : x_2 \in \mathbb{R} \right\} = \text{span}\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}.$$