Math 310 (35180), Fall 2015
Instructor: Chris Skalit
Quiz 9

Name: ______________________ UIN: ______________________

1. (6 points) Let $A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$. Diagonalize A. That is, find a factorization $A = SDS^{-1}$ where D is diagonal and the columns of S are eigenvectors for A.

Solution: The characteristic polynomial for A is $P_A(t) = \det(A - tI) = (1-t)(2-t) - 2 = t(t-3)$. Hence the eigenvalues are 0 and 3. Our eigenspaces

$$E_0 = \mathrm{Nul}(A) = \mathrm{Nul}\left(\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}\right) = \mathrm{Nul}\left(\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}\right) = \mathrm{span}\left\{\begin{bmatrix} -2 \\ 1 \end{bmatrix}\right\}$$

$$E_3 = \mathrm{Nul}(A - 3I) = \mathrm{Nul}\left(\begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix}\right) = \mathrm{Nul}\left(\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}\right) = \mathrm{span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$$

Hence, we can put $S = \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix}$, $S^{-1} = \begin{bmatrix} -1/3 & 1/3 \\ 1/3 & 2/3 \end{bmatrix}$, and $D = \begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix}$.

2. (4 points) Denote by $\mathbb{P}_2$ the space of polynomials having degree at most 2. Let $\mathcal{B} = \{g_1, g_2, g_3\}$ be the standard basis:

$$g_1(x) = 1 \qquad g_2(x) = x \qquad g_3(x) = x^2$$

Let $T : \mathbb{P}_2 \to \mathbb{P}_2$ be the linear transformation defined by $T(f) = f + \frac{df}{dx}$. Write down the matrix $[T]_{\mathcal{B}}$ for T with respect to $\mathcal{B}$, and use it to find all eigenvalues of T (you do **not** need to compute the corresponding eigenvectors).

Solution:

$$\begin{aligned} T(g_1) &= 1 &&= g_1 \\ T(g_2) &= x + 1 &&= g_1 + g_2 \\ T(g_3) &= x^2 + 2x &&= 2g_2 + g_3 \end{aligned}$$

Thus, $[T]_{\mathcal{B}} = \begin{bmatrix} [T(g_1)]_{\mathcal{B}} & [T(g_2)]_{\mathcal{B}} & [T(g_3)]_{\mathcal{B}} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$. With $B = [T]_{\mathcal{B}}$, we compute the characteristic polynomial $P_B(t) = \det(B - tI) = (1-t)^3$. Hence the only eigenvalue of T is 1 (with corresponding eigenspace spanned by g_1).