Math 310 (33886), Fall 2016
Instructor: Chris Skalit
Exam 1

No calculators or any other electronic devices are permitted on this exam. **Show your work.**

1. (15 points) Find all solutions to the following system of equations:

$$\begin{array}{rcrcrcr} x_1 & + & 3x_2 & + & x_3 & = & -2 \\ -2x_1 & - & 2x_2 & & & = & -2 \\ & & x_2 & + & x_3 & = & -2 \end{array}$$

Solution: The augmented matrix for this system is $J = \begin{bmatrix} 1 & 3 & 1 & -2 \\ -2 & -2 & 0 & -2 \\ 0 & 1 & 1 & -2 \end{bmatrix}$.

Since rref $J = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$, we conclude that

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}$$

2. Let $A = \begin{bmatrix} 1 & 2 & 0 & -1 \\ 1 & 2 & 1 & -4 \\ 3 & 6 & 0 & -3 \end{bmatrix}$. Note that $\text{rref}(A) = \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Let $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix}$.

(a) (5 points) Write down all solutions $\mathbf{x} \in \mathbb{R}^4$ to the homogeneous equation $A\mathbf{x} = \mathbf{0}$.

Solution: Note that it suffices to solve the system $(\text{rref } A)\mathbf{x} = \mathbf{0}$, so if $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$, we obtain the relations

$$\begin{aligned} x_1 + 2x_2 - x_4 &= 0 \\ x_3 - 3x_4 &= 0 \end{aligned}$$

Our dependent variables correspond to pivot columns and thus are x_1 and x_3. Writing these in terms of the other (free) variables we see that our solutions are

$$\mathcal{S} = \left\{ \begin{bmatrix} -2x_2 + x_4 \\ x_2 \\ 3x_4 \\ x_4 \end{bmatrix} : x_2, x_4 \in \mathbb{R} \right\}$$

(b) (3 points) If $\mathbf{b} = A\mathbf{x}_0$, what is $\mathbf{b}$?

Solution:

$$A\mathbf{x}_0 = \begin{bmatrix} 1 & 2 & 0 & -1 \\ 1 & 2 & 1 & -4 \\ 3 & 6 & 0 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 6 \\ 9 \\ 18 \end{bmatrix}$$

(c) (3 points) Write down all solutions $\mathbf{x} \in \mathbb{R}^4$ to the matrix equation $A\mathbf{x} = \mathbf{b}$. (Hint: Use part (a).)

Solution: A generic solution $\mathbf{y}$ may be written as $\mathbf{y} = \mathbf{x}_0 + \mathbf{x}$ where $\mathbf{x}$ is a solution to the homogeneous equation $A\mathbf{x} = \mathbf{0}$. Hence,

$$\mathcal{S} = \left\{ \begin{bmatrix} -2x_2 + x_4 \\ x_2 \\ 3x_4 \\ x_4 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix} : x_2, x_4 \in \mathbb{R} \right\}$$

(d) (3 points) Are the columns of A linearly independent vectors in $\mathbb{R}^3$? Explain in one sentence.

Solution: NO. Since $A\mathbf{x} = \mathbf{0}$ has non-trivial solutions, we conclude that there are non-trivial relations among the columns of A.

(e) (3 points) If $T : \mathbb{R}^n \to \mathbb{R}^m$ is the linear transformation defined by $T(\mathbf{x}) = A\mathbf{x}$, what are m and n?

Solution: T is defined by the matrix A, which has 4 columns and 3 rows. Thus, A acts on vectors $\mathbf{x} \in \mathbf{R}^4$ and returns vectors in $\mathbf{R}^3$. That is $m = 3$, $n = 4$.

(f) (3 points) Is T surjective (onto)? Explain in one sentence.

Solution: No. This is clear becuase rref A does not have a pivot in each row.

3. (a) (8 points) Let $S : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear map such that $S\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x+y+z \\ 2x-y \\ x+3z \end{bmatrix}$.

What is the matrix C such that $S(\mathbf{v}) = C\mathbf{v}$ for all $\mathbf{v} \in \mathbb{R}^3$?

Solution:

$$S\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \quad S\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad S\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$$

Hence, the matrix that represents S is $C = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 0 \\ 1 & 0 & 3 \end{bmatrix}$.

(b) (7 points) If $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, write the vector $\mathbf{w} = \begin{bmatrix} 5 \\ 8 \end{bmatrix}$ as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$.

Solution: We want to solve the equation $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{w}$, or equivalently,

$$\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \end{bmatrix}.$$

We find that $c_1 = 1$, $c_2 = 2$.

(c) (5 points) Let $U : \mathbb{R}^2 \to \mathbb{R}^3$ be the linear transformation such that $U(\mathbf{v}_1) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $U(\mathbf{v}_2) = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$. Use your answer to part (b) to find $U(\mathbf{w})$.

Solution: From part (b), we see that

$$U(\mathbf{w}) = U(\mathbf{v}_1 + 2\mathbf{v}_2) = U(\mathbf{v}_1) + 2U(\mathbf{v}_2) = \begin{bmatrix} 5 \\ 1 \\ -1 \end{bmatrix}$$

4. (a) (15 points) Let $D = \begin{bmatrix} 1 & 0 & -2 \\ -1 & 1 & 3 \\ 1 & 0 & -1 \end{bmatrix}$. Compute D^{-1}.

Solution: We write the 3×6 matrix whose first three columns are those of D and whose latter three is the 3×3 identity matrix:

$$Q = \begin{bmatrix} 1 & 0 & -2 & 1 & 0 & 0 \\ -1 & 1 & 3 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 & 0 & 1 \end{bmatrix}$$

Since we have rref $Q = \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 2 \\ 0 & 1 & 0 & 2 & 1 & -1 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{bmatrix}$, we see that $D^{-1} = \begin{bmatrix} -1 & 0 & 2 \\ 2 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}$

(b) (5 points) Let $E = \begin{bmatrix} -1 & -1 & 0 \\ 2 & 1 & 0 \\ 5 & 1 & 1 \end{bmatrix}$. Using the fact that $E^{-1} = \begin{bmatrix} 1 & 1 & 0 \\ -2 & -1 & 0 \\ -3 & -4 & 1 \end{bmatrix}$, find the unique solution $\mathbf{x} \in \mathbb{R}^3$ to $E\mathbf{x} = \mathbf{y}$ where $\mathbf{y} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Solution: If $E\mathbf{x} = \mathbf{y}$, then $\mathbf{x} = E^{-1}\mathbf{y} = \begin{bmatrix} 1 & 1 & 0 \\ -2 & -1 & 0 \\ -3 & -4 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ -6 \end{bmatrix}$.

5. Let $F = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$, $G = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}$.

(a) (7 points) Compute FG and GF.

Solution:

$$FG = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 3 \\ -1 & 4 \end{bmatrix}$$

$$GF = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 7 \\ 0 & 1 \end{bmatrix}$$

(b) (4 points) What are $\det(F)$ and $\det(G)$?

Solution:

$$det(F) = 1 \quad \det(G) = 3$$

(c) (4 points) What is $\det(FGFGF)$? Is $FGFGF$ invertible? Explain in one sentence.

Solution: Remember that for square matrices A and B, $\det(AB) = \det(A)\det(B)$. Hence, $\det(FGFGF) = \det(F)^3 \det(G)^2 = 9$. The matrix is invertible as its determinant is nonzero. Note, however, that $FGFGF \neq F^3G^2$.

6. (10 points) Let $H = \begin{bmatrix} 2 & 2 & 2 & 2 \\ 1 & 2 & 1 & 2 \\ 0 & 1 & -1 & 2 \\ 1 & 3 & 1 & 0 \end{bmatrix}$. Find $\det(H)$. (Hint: You might want to try row-reducing H to a triangular matrix and note how the determinant changes with each operation you make.)

Solution: Dividing the first row by 2 gives $H' = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 2 \\ 0 & 1 & -1 & 2 \\ 1 & 3 & 1 & 0 \end{bmatrix}$ Note that $\det(H') = \frac{1}{2}\det(H)$ By performing a sequence of operations that add multiples of one row to another, we get $H'' = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -3 \end{bmatrix}$. Operations of this type do not change the determinant, so $\det(H') = \det(H'')$. Since H'' is triangular, its determinant is the product of the diagonal entries: $\det(H'') = 3$. Hence, $\det(H) = 6$.