Math 310 (33886), Spring 2016
Instructor: Chris Skalit
Exam II

Name: ______________________________ UIN: ______________________________

1. Put $A = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 2 & 0 & 2 & 1 & -1 \\ 1 & 1 & 3 & 0 & 0 \\ 3 & 1 & 5 & 1 & -1 \end{bmatrix}$ and note that $\text{rref}(A) = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

(a) (2 points) Write down a basis for $\text{Row}(A)$. What is the dimension of this space?

Solution: Since the row space of a matrix is unchanged under elementary row operations, we have $\text{Row}(A) = \text{Row}(\text{rref } A)$. Thus, it's clear that $\text{Row}(A)$ has basis $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} \right\}$. Hence, $\dim(\text{Row}(A)) = 3$.

(b) (3 points) Write down a basis for $\text{Col}(A)$. What is the dimension of this space?

Solution: The columns of A span the column space. We can extract a basis for this space by selecting those columns which correspond to pivot columns in rref A: $\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$. Hence, $\dim(\text{Col}(A)) = 3$.

(c) (5 points) Write down a basis for $\text{Nul}(A)$. What is the dimension of this space?

Solution: If a generic vector $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$ belongs to $\text{Nul}(A) = \text{Nul}(\text{rref } A)$, then we see from rref A that the following relations are imposed:

$$\begin{aligned} x_1 + x_3 &= 0 \\ x_2 + 2x_3 &= 0 \\ x_4 - x_5 &= 0 \end{aligned}$$

Solving for the dependent (pivot) variables, we get

$$\text{Nul}(A) = \left\{ \begin{bmatrix} -x_3 \\ -2x_3 \\ x_3 \\ x_5 \\ x_5 \end{bmatrix} : x_3, x_5 \in \mathbb{R} \right\} = \text{span} \left\{ \begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

This set is clearly linearly independent; $\dim(\text{Nul } A) = 2$.

(d) (2 points) If B is a 4×7 matrix with $\dim(\text{Col}(B)) = 2$, what are the dimenions of $\text{Row}(B)$ and $\text{Nul}(B)$?

Solution: B has 7 columns. By the rank-nullity theorem, $\dim(\text{Col}(B)) + \dim(\text{Nul}(B)) = 7$. Hence, $\dim(\text{Nul}(B)) = 5$. Since the row and column space of a matrix always have the same dimension, $\dim(\text{Row}(B)) = 2$.

(e) (3 points) Is there a 4×8 matrix C with linearly independent columns? Explain in one sentence.

Solution: If the columns of C were linearly independent, then the nullspace would be zero-dimensional. This would imply that the rank of C would be equal to 8. But this is impossible since C has 4 rows and therefore, at most, 4 pivots.

2. Let V be a 3-dimensional vector space with bases $\mathcal{A} = \{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ and $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$, subject to the relations:

$$\begin{aligned} \mathbf{a}_1 &= \mathbf{b}_2 \\ \mathbf{a}_2 &= \mathbf{b}_1 + 2\mathbf{b}_3 \\ \mathbf{a}_3 &= 2\mathbf{b}_1 + \mathbf{b}_2 + 3\mathbf{b}_3 \end{aligned}$$

(a) (4 points) Write down the change of basis matrix $P_{\mathcal{B}\leftarrow\mathcal{A}}$ which transforms $\mathcal{A}$ to $\mathcal{B}$ coordinates.

Solution: The i-th column of $P_{\mathcal{B}\leftarrow\mathcal{A}}$ is $[\mathbf{a}_i]_{\mathcal{B}}$, whence

$$P_{\mathcal{B}\leftarrow\mathcal{A}} = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & 3 \end{bmatrix}.$$

(b) (8 points) Compute the change of basis matrix $P_{\mathcal{A}\leftarrow\mathcal{B}}$.

Solution:

$$P_{\mathcal{A}\leftarrow\mathcal{B}} = (P_{\mathcal{B}\leftarrow\mathcal{A}})^{-1} = \begin{bmatrix} -2 & 1 & 1 \\ -3 & 0 & 2 \\ 2 & 0 & -1 \end{bmatrix}$$

(c) (4 points) If $\mathbf{x} = \mathbf{a}_1 - 2\mathbf{a}_3$, what are the coordinate vectors $[\mathbf{x}]_{\mathcal{A}}$ and $[\mathbf{x}]_{\mathcal{B}}$?

Solution: We can read off the $\mathcal{A}$-coordinate vector immediately: $[\mathbf{x}]_{\mathcal{A}} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$.

For the latter, we have

$$[\mathbf{x}]_{\mathcal{B}} = P_{\mathcal{B}\leftarrow\mathcal{A}}[\mathbf{x}]_{\mathcal{A}} = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} = \begin{bmatrix} -4 \\ -1 \\ -6 \end{bmatrix}.$$

(d) (3 points) Let $T : V \to V$ be linear such that $T(\mathbf{a}_1) = 2\mathbf{a}_1 - \mathbf{a}_2$, $T(\mathbf{a}_2) = \mathbf{a}_1 - \mathbf{a}_3$, and $T(\mathbf{a}_3) = \mathbf{0}$. Find the matrix $[T]_{\mathcal{A}}$ which expresses T in $\mathcal{A}$-coordinates.

Solution: The i-th column of $[T]_{\mathcal{A}}$ is $[T(\mathbf{a}_i)]_{\mathcal{A}}$, so

$$[T]_{\mathcal{A}} = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}.$$

(e) (3 points) If $\mathbf{y} = T(\mathbf{x})$, what is $[\mathbf{y}]_{\mathcal{A}}$?

Solution:

$$[\mathbf{y}]_{\mathcal{A}} = [T(\mathbf{x})]_{\mathcal{A}} = [T]_{\mathcal{A}}[\mathbf{x}]_{\mathcal{A}} = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$$

(f) (3 points) Express $[T]_{\mathcal{B}}$ as a product of matrices. You do not need to multiply the matrices.

Solution:

$$[T]_{\mathcal{B}} = P_{\mathcal{B}\leftarrow\mathcal{A}}[T]_{\mathcal{A}}P_{\mathcal{A}\leftarrow\mathcal{B}} = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & 3 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} -2 & 1 & 1 \\ -3 & 0 & 2 \\ 2 & 0 & -1 \end{bmatrix}$$

3. Let $F = \begin{bmatrix} 2 & 1 & 2 \\ -1 & 4 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

(a) (6 points) Compute the characteristic polynomial of F and find all eigenvalues.

Solution: Our characteristic polynomial is

$$P_F(t) = \det(F - tI) = \det\left(\begin{bmatrix} 2-t & 1 & 2 \\ -1 & 4-t & 1 \\ 0 & 0 & 1-t \end{bmatrix}\right).$$

By using cofactor expansion along the bottom row, we deduce that

$$P_F(t) = (1-t)\det\left(\begin{bmatrix} 2-t & 1 \\ -1 & 4-t \end{bmatrix}\right) = (1-t)(t^2 - 6t + 9) = (1-t)(t-3)^2.$$

Our eigenvalues are therefore 1 and 3.

(b) (8 points) For each eigenvalue computed in part (a), find the corresponding eigenspace.

Solution: For the eigenvalue 1, we have

$$E_1 = \text{Nul}(F - I) = \text{Nul}\left(\begin{bmatrix} 1 & 1 & 2 \\ -1 & 3 & 1 \\ 0 & 0 & 0 \end{bmatrix}\right) = \text{Nul}\left(\begin{bmatrix} 1 & 0 & 5/4 \\ 0 & 1 & 3/4 \\ 0 & 0 & 0 \end{bmatrix}\right) = \text{span}\left\{\begin{bmatrix} -5/4 \\ -3/4 \\ 1 \end{bmatrix}\right\}.$$

For the eigenvalue 3, we have

$$E_3 = \text{Nul}(F - 3I) = \text{Nul}\left(\begin{bmatrix} -1 & 1 & 2 \\ -1 & 1 & 1 \\ 0 & 0 & -2 \end{bmatrix}\right) = \text{Nul}\left(\begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\right) = \text{span}\left\{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right\}.$$

(c) (1 point) Is F diagonalizable? Explain in one sentence.

Solution: No, there are only two linearly independent eigenvectors. In particular, the eigenvalue 3 has algebraic multiplicity 2 while its corresponding eigenspace is only 1-dimensional.

4. Let $\mathcal{P}_2$ be the space of polynomials having degree at most two. Let $\mathcal{C} = \{1, t, t^2\}$ be the standard basis.

(a) (6 points) Let $\phi : \mathcal{P}_2 \to \mathcal{P}_2$ be the linear map defined by $\phi(f) = 2\frac{d^2 f}{dt^2} + t\frac{df}{dt} + f$. Write down $[\phi]_{\mathcal{C}}$.

Solution: We have the functions $h_0(t) = 1$, $h_1(t) = t$ and $h_2(t) = t^2$. These form a basis for $\mathcal{P}_2$.

$$\phi(h_0) = 2\frac{d^2 h_0}{dt^2} + t\frac{dh_0}{dt} + h_0(t) = 2 \cdot 0 + t \cdot 0 + 1 = 1 \Rightarrow [\phi(h_0)]_{\mathcal{C}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\phi(h_1) = 2\frac{d^2 h_1}{dt^2} + t\frac{dh_1}{dt} + h_1(t) = 2 \cdot 0 + t \cdot 1 + t = 2t \Rightarrow [\phi(h_1)]_{\mathcal{C}} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$$

$$\phi(h_2) = 2\frac{d^2 h_2}{dt^2} + t\frac{dh_2}{dt} + h_2(t) = 2 \cdot 2 + t \cdot 2t + t^2 = 4 + 3t^2 \Rightarrow [\phi(h_2)]_{\mathcal{C}} = \begin{bmatrix} 4 \\ 0 \\ 3 \end{bmatrix}$$

Hence, $[\phi]_{\mathcal{C}} = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

(b) (7 points) Use your answer to part (a) to find a degree-two polynomial $g \in \mathcal{P}_2$ such that $\phi(g) = 3g$. (Hint: Compute the 3-eigenspace of the matrix $[\phi]_{\mathcal{C}}$ and then recast your answer in terms of a polynomial.)

Solution: We compute the 3-eigenspace of the matrix $J = [\phi]_{\mathcal{C}} = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$:

$$E_3 = \mathrm{Nul}(J - 3I) = \mathrm{Nul}\left(\begin{bmatrix} -2 & 0 & 4 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \mathrm{span}\left\{ \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

We note that the function $g(t) = 2 + t^2$ has $\mathcal{C}$-coordinates $[g]_{\mathcal{C}} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$.

(c) (5 points) Consider the linear transformation $\theta : \mathcal{P}_2 \to \mathbb{R}$ defined via $\theta(f) = \int_0^1 6tf(t)\, dt$. Suppose that $h(t) = at + 1$ and that $h \in \ker(\theta)$. Find a.

Solution: To say that $h \in \ker\theta$ means precisely that $\theta(h) = 0$. We therefore solve the equation

$$\begin{aligned} 0 &= \theta(h) \\ 0 &= \int_0^1 6t(h(t))\,dt \\ 0 &= \int_0^1 6at^2 + 6t\,dt \\ 0 &= 2at^3 + 3t^2\big|_0^1 \\ 0 &= 2a + 3 \end{aligned}$$

Hence, $a = -3/2$.

5. (7 points) Let $\mathbf{z}_0 = \begin{bmatrix} 0.58 \\ 0.42 \end{bmatrix}$. Consider the Markov Chain $(\mathbf{z}_n)$ defined via $\mathbf{z}_{n+1} = G\mathbf{z}_n$ where G is the stochastic matrix $\begin{bmatrix} 0.20 & 0.10 \\ 0.80 & 0.90 \end{bmatrix}$. Find $\lim_{n\to\infty} \mathbf{z}_n$.

Solution: Since G is a stochastic matrix with all non-zero entries, the general theory of Markov Chains asserts that G admits a unique steady state vector $\mathbf{q}$ to which $(\mathbf{z}_n)$ converges. To find the steady state we just compute the 1-eigenspace of G:

$$E_1 = \text{Nul}(G - I) = \text{span}\left\{ \begin{bmatrix} 1 \\ 8 \end{bmatrix} \right\}$$

The steady state $\mathbf{q} \in E_1$ has the definiting property that its entries sum to 1. We therefore rescale $\begin{bmatrix} 1 \\ 8 \end{bmatrix}$ so that

$$\mathbf{q} = \frac{1}{1+8} \begin{bmatrix} 1 \\ 8 \end{bmatrix} = \begin{bmatrix} 1/9 \\ 8/9 \end{bmatrix}.$$

6. Let $H = \begin{bmatrix} 7 & -3 \\ 10 & -4 \end{bmatrix}$, $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$. Note that $H\mathbf{v}_1 = \mathbf{v}_1$ and $H\mathbf{v}_2 = 2\mathbf{v}_2$. Thus, $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis for $\mathbb{R}^2$ consisting of eigenvectors for H.

(a) (6 points) Using the eigenvectors and eigenvalues that have already been computed for you above, diagonalize H. That is, obtain a factorization $H = SDS^{-1}$ where D is diagonal.

Solution: S is the matrix whose columns are the eigenvectors: $S = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}$. We place the corresponding eigenvales along the diagonal of D: $D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$. Hence,

$$H = SDS^{-1} = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix}.$$

(b) (7 points) Compute H^{100}. Your answer should be a single 2×2 matrix whose entries involve 2^{100}.

Solution: Recall that raising a diagonal matrix to the n-th power is the same thing as raising each entry to the n-th power – something that is wildly false in general. Thus, to compute H^{100} we apply the factorization obtained in part (a):

$$H^{100} = SD^{100}S^{-1} = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2^{100} \end{bmatrix} \begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} -5 + 6(2^{100}) & 3 - 3(2^{100}) \\ -10 + 10(2^{100}) & 6 - 5(2^{100}) \end{bmatrix}.$$

(c) (3 points) Consider the differential equation $\dfrac{d\mathbf{x}}{dt} = H\mathbf{x}(t)$. Write the generation solution to this equation.

Solution:

$$\mathbf{x}(t) = c_1 e^t \mathbf{v}_1 + c_2 e^{2t} \mathbf{v}_2 = c_1 e^t \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

(d) (4 points) Find the particular solution subject to the condition that $\mathbf{x}(0) = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$.

Solution: We need to find the particular values of c_1, c_2 for which we have

$$\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \mathbf{x}(0) = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$

The solution to this linear system of equations is $c_1 = -4$ and $c_2 = 2$.