Math 310 (33886), Spring 2016
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Quiz 11

Name: ______________________ UIN: ______________________

1. (2 points) Suppose that $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$ is an orthogonal basis for $\mathbb{R}^3$ such that $||\mathbf{a}_1|| = 1$, $||\mathbf{a}_2|| = 2$, and $||\mathbf{a}_3|| = 3$. Let $\mathbf{x} = c_1\mathbf{a}_1 + c_2\mathbf{a}_2 + c_3\mathbf{a}_3$. Compute each scalar c_i provided that $\mathbf{x} \cdot \mathbf{a}_i = 1$ for all $1 \leq i \leq 3$.

   Solution: Since $\mathbf{a}_i \cdot \mathbf{a}_j = 0$ for $i \neq j$, we see that

$$\mathbf{a}_i \cdot \mathbf{x} = \mathbf{a}_i \cdot (c_1\mathbf{a}_1 + c_2\mathbf{a}_2 + c_3\mathbf{a}_3) = c_i(\mathbf{a}_i \cdot \mathbf{a}_i) \Rightarrow c_i = \frac{\mathbf{a}_i \cdot \mathbf{x}}{||\mathbf{a}_i||^2}$$

   Hence, $c_1 = 1, c_2 = 1/4, c_3 = 1/9$.

2. Consider the two-dimensional subspace V of $\mathbb{R}^3$ that is cut out by $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}$.

   Note that these vectors are orthogonal.

   (a) (3 points) Let $\mathbf{y} = \begin{bmatrix} 2 \\ -6 \\ 4 \end{bmatrix}$. Compute the orthogonal projection $\text{proj}_V(\mathbf{y})$.

   Solution:

$$\text{proj}_V(\mathbf{y}) = \frac{\mathbf{y} \cdot \mathbf{v}_1}{||\mathbf{v}_1||^2}\mathbf{v}_1 + \frac{\mathbf{y} \cdot \mathbf{v}_2}{||\mathbf{v}_2||^2}\mathbf{v}_2 = \left(\frac{10}{5}\right) \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + \left(\frac{6}{6}\right) \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix}$$

   (b) (2 points) What is the distance from $\mathbf{y}$ to V (i.e. the distance from $\mathbf{y}$ to the point in V nearest to $\mathbf{y}$)?

   Solution: The shortest distance is the so-called "perpendicular distance" which is given by

$$||\mathbf{y} - \text{proj}_V(\mathbf{y})|| = || \begin{bmatrix} 2 \\ -6 \\ 4 \end{bmatrix} - \begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix} || = \sqrt{30}.$$

3. (3 points) Let W be spanned by the linearly independent vectors $\mathbf{w}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$ and $\mathbf{w}_2 = \begin{bmatrix} 2 \\ 5 \\ 0 \\ -1 \end{bmatrix}$. Use Gram-Schmidt to obtain an orthogonal basis for W.

Solution: We take as our first member of our orthogonal basis $\mathbf{u}_1 = \mathbf{w}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$. We let $U_1 = \text{span}\{\mathbf{u}_1\}$. Next,

$$\mathbf{u}_2 = \mathbf{w}_2 - \text{proj}_{U_1}(\mathbf{w}_2) = \mathbf{w}_2 - \frac{\mathbf{w}_2 \cdot \mathbf{u}_1}{||\mathbf{u}_1||^2}\mathbf{u}_1 = \mathbf{w}_2 = \begin{bmatrix} 2 \\ 5 \\ 0 \\ -1 \end{bmatrix} - \left(\frac{8}{4}\right) \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ -2 \\ 1 \end{bmatrix}.$$