Math 310 (33886), Spring 2016
Instructor: Chris Skalit
Quiz 12

Name: ______________________ UIN: ______________________

1. (5 points) Use the method of least squares to find the best-fit line $y = a_0 + a_1 x$ for the points $\begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}2\\4\end{bmatrix} \in \mathbb{R}^2$.

   Solution: If it were true that every point were contained on $y = a_0 + a_1 x$, then we would have equations

$$\begin{aligned} 0 &= a_0 + a_1(0) \\ 1 &= a_0 + a_1(1) \\ 4 &= a_0 + a_1(2) \end{aligned}$$

   We can encode this as $M\mathbf{x} = \mathbf{y}$ where $M = \begin{bmatrix}1 & 0\\1 & 1\\1 & 2\end{bmatrix}$, $\mathbf{x} = \begin{bmatrix}a_0\\a_1\end{bmatrix}$, $\mathbf{y} = \begin{bmatrix}0\\1\\4\end{bmatrix}$. This equation does not have a solution. We instead compute the least-squares solution – that is, a solution to $M^T M\mathbf{x} = M^T\mathbf{y}$:

$$\begin{bmatrix}3 & 3\\3 & 5\end{bmatrix}\begin{bmatrix}a_0\\a_1\end{bmatrix} = \begin{bmatrix}5\\9\end{bmatrix} \Rightarrow a_0 = -1/3,\ a_1 = 2$$

   Hence, the best-fit line is $y = (-1/3) + 2x$.

2. (5 points) Let $A = \begin{bmatrix}1 & 2\\2 & 1\end{bmatrix}$. Orthogonally diagonalize A: that is, find an orthogonal matrix P such that $A = PDP^T$ where D is diagonal.

   Solution: The characteristic polynomial is $P(t) = \det(A - tA) = (1-t)^2 - 4$, so our eigenvalues are -1 and 3. Our eigenspaces are

$$E_{(-1)} = \mathrm{Nul}(A + I) = \mathrm{Nul}\left(\begin{bmatrix}2 & 2\\2 & 2\end{bmatrix}\right) = \mathrm{Nul}\left(\begin{bmatrix}1 & 1\\0 & 0\end{bmatrix}\right) = \mathrm{span}\left\{\begin{bmatrix}-1\\1\end{bmatrix}\right\}$$

$$E_3 = \mathrm{Nul}(A - 3I) = \mathrm{Nul}\left(\begin{bmatrix}-2 & 2\\2 & -2\end{bmatrix}\right) = \mathrm{Nul}\left(\begin{bmatrix}1 & -1\\0 & 0\end{bmatrix}\right) = \mathrm{span}\left\{\begin{bmatrix}1\\1\end{bmatrix}\right\}$$

   Our eigenvectors are already orthogonal to each other; we need to renormalize them to get an orthonormal basis for $\mathbb{R}^2$:

$$\mathbf{v}_{(-1)} = \frac{1}{\sqrt{2}}\begin{bmatrix}-1\\1\end{bmatrix} = \begin{bmatrix}-1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix}$$

$$\mathbf{v}_3 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix}$$

We put $P = \begin{bmatrix}-1/\sqrt{2} & 1/\sqrt{2}\\ 1/\sqrt{2} & 1/\sqrt{2}\end{bmatrix}$ and $D = \begin{bmatrix}-1 & 0\\ 0 & 3\end{bmatrix}$, so

$$A = PDP^T = \begin{bmatrix}-1/\sqrt{2} & 1/\sqrt{2}\\ 1/\sqrt{2} & 1/\sqrt{2}\end{bmatrix}\begin{bmatrix}-1 & 0\\ 0 & 3\end{bmatrix}\begin{bmatrix}-1/\sqrt{2} & 1/\sqrt{2}\\ 1/\sqrt{2} & 1/\sqrt{2}\end{bmatrix}.$$