Math 310 (33886), Fall 2016
Instructor: Chris Skalit
Quiz 2

Name: ______________________________ UIN: ______________________________

1. Compute the product or state that it is not defined.

   (a) (1 point) $\begin{bmatrix} 1 & 2 & -1 \\ 2 & 3 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$.

   Solution:

$$\begin{bmatrix} 1 & 2 & -1 \\ 2 & 3 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} - 2 \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} + 0 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ -4 \\ -2 \end{bmatrix}$$

   (b) (1 point) $\begin{bmatrix} 1 & 2 \\ 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

   Solution: This product is not defined; the matrix has two columns while the vector has three entries.

2. (4 points) Let $\mathbf{v} = \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$ show that $\mathbf{y} = \begin{bmatrix} -1 \\ 14 \\ 6 \end{bmatrix}$ belongs to span $\{\mathbf{v}, \mathbf{w}\}$ by finding scalars $\lambda_i \in \mathbb{R}$ such that

$$\mathbf{y} = \lambda_1 \mathbf{v} + \lambda_2 \mathbf{w}$$

Solution: We are being asked to solve the system

$$\lambda_1 \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix} + \lambda_2 \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ 14 \\ 6 \end{bmatrix}$$

or, equivalently, the matrix equation

$$\begin{bmatrix} -1 & 2 \\ 4 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 14 \\ 6 \end{bmatrix}$$

Since $\begin{bmatrix} -1 & 2 & -1 \\ 4 & 2 & 14 \\ 1 & 3 & 6 \end{bmatrix}$ row-reduces to $\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, we see that $\lambda_1 = 3$, $\lambda_2 = 1$. Hence $\mathbf{y}$ may be written as a linear combination of $\mathbf{v}$ and $\mathbf{w}$.

(PLEASE TURN OVER)

3. Note: This question requires virtually **no** computation if done correctly.

(a) (2 points) Let $A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Write down all solutions $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ to the homogeneous equation $A\mathbf{x} = \mathbf{0}$.

Solution: Our pivots occur in columns 1 and 3, so x_1 and x_3 are our dependent variables; the others are free. From the relations

$$\begin{aligned} x_1 + 2x_2 + x_4 &= 0 \Leftrightarrow x_1 = -2x_2 - x_4 \\ x_3 - x_4 &= 0 \Leftrightarrow x_3 = x_4 \end{aligned}$$

Our solutions are

$$\mathcal{S} = \left\{ \begin{bmatrix} -2x_2 - x_4 \\ x_2 \\ x_4 \\ x_4 \end{bmatrix} : x_2, x_4 \in \mathbb{R} \right\} = \left\{ x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix} : x_2, x_4 \in \mathbb{R} \right\}$$

(b) (1 point) Given $\mathbf{b} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, find all solutions to $A\mathbf{x} = \mathbf{b}$ by using your solution to part (a) and noting that $\mathbf{x}_p = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$ is a solution.

Solution: We know that all solutions $\mathbf{y}$ to this equation may be written as $\mathbf{y} = \mathbf{x}_p + \mathbf{z}$ where $\mathbf{z}$ is some solution to $A\mathbf{x} = \mathbf{0}$. Hence, from our answer in the previous part, our solution set looks like

$$\mathcal{S} = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix} : x_2, x_4 \in \mathbb{R} \right\}$$

(c) (1 point) Do the columns of A span $\mathbb{R}^3$? (In other words, is the equation $A\mathbf{x} = \mathbf{h}$ solvable for every choice of $\mathbf{h} \in \mathbb{R}^3$?) Explain in one sentence.

Solution: No. The reduced row-echelon form of A does not contain a pivot in every row.