Math 310 (33886), Fall 2016
Instructor: Chris Skalit
Quiz 5

Name: ____________________________ UIN: ____________________________

1. (6 points) Let $A = \begin{bmatrix} 1 & -2 & 0 \\ 3 & -3 & 0 \\ 2 & -7 & 1 \end{bmatrix}$. Find an upper-triangular matrix U and a lower-triangular L such that $A = LU$.

   Solution: To find U, we put A into row-echelon form, using only operations which add a multiple of one row to another below it:

$$\begin{bmatrix} 1 & -2 & 0 \\ 3 & -3 & 0 \\ 2 & -7 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 0 \\ 0 & 3 & 0 \\ 2 & -7 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 0 \\ 0 & 3 & 0 \\ 0 & -3 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} = U$$

   To obtain U, we performed the following operations:

   1. Add (-3)R1 to R2
   2. Add (-2)R1 to R3
   3. Add $(+1)$R2 to R3

   To obtain L, we start with the identity matrix and perform the inverse of these operations in the reverse order:

   1. Add (-1)R2 to R3
   2. Add $(+2)$R1 to R3
   3. Add $(+3)$R1 to R2

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix} = L$$

2. Let $B = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 2 & 3 \end{bmatrix}$.

(a) (2 points) Compute $\det B$.

Solution: By cofactor expansion along the top row, we have

$$\det B = 2(1 \cdot 3 - 1 \cdot 2) + 1(1 \cdot 2 - 1 \cdot 0) = 4$$

(b) (2 points) Suppose that $\mathbf{y} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ where $a, b, c \in \mathbb{R}$. Find the unique $\mathbf{x} \in \mathbb{R}^3$ such that $B\mathbf{x} = \mathbf{y}$ by using Cramer's Rule and the fact that

$$\det\left(\begin{bmatrix} a & 0 & 1 \\ b & 1 & 1 \\ c & 2 & 3 \end{bmatrix}\right) = 3 \quad \det\left(\begin{bmatrix} 2 & a & 1 \\ 1 & b & 1 \\ 0 & c & 3 \end{bmatrix}\right) = 5 \quad \det\left(\begin{bmatrix} 2 & 0 & a \\ 1 & 1 & b \\ 0 & 2 & c \end{bmatrix}\right) = 8.$$

Solution: If we denote by $B_i(\mathbf{y})$ the matrix obtained by replacing the i-th column of B with $\mathbf{y}$, then Cramer's Rule says that $x_i = \frac{\det B_i(\mathbf{y})}{\det B}$. Hence,

$$x_1 = \frac{3}{4} \quad x_2 = \frac{5}{4} \quad x_3 = \frac{8}{4}$$