Math 310 (33886), Spring 2016
Instructor: Chris Skalit
Quiz 6

Name: ______________________ UIN: ______________________

1. Let $A = \begin{bmatrix} -3 & -9 & 3 & 0 & 0 \\ 2 & 6 & -2 & 3 & -6 \\ 1 & 3 & -1 & 1 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 3 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. Note that $B = \text{rref}(A)$.

   (a) (2 points) Write down a basis for the column space, $\text{Col}\, A$.

   Solution: We can extract a basis for $\text{Col}\, A$ by simply including those vectors whose corresponding columns in $\text{rref}\, A$ have pivots:

$$\mathcal{B} = \left\{ \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} \right\}$$

   (b) (3 points) Write down a basis for the nullspace, $\text{Nul}(A)$.

   Solution: $\text{Nul}\, A$ is just the solution set to $B\mathbf{x} = \mathbf{0}$, whence we obtain the relations

$$\begin{aligned} x_1 + 3x_2 - x_3 &= 0 \\ x_4 - 2x_5 &= 0 \end{aligned}$$

   with x_2, x_3, x_5 free. If we write the solutions in vector-parametric form, we get

$$\text{Nul} = \left\{ \begin{bmatrix} -3x_2 + x_3 \\ x_2 \\ x_3 \\ 2x_5 \\ x_5 \end{bmatrix} : x_2, x_3, x_5 \in \mathbb{R} \right\} = \left\{ x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \\ 1 \end{bmatrix} : x_2, x_3, x_5 \in \mathbb{R} \right\}.$$

   These vectors are linearly independent, so our basis is

$$\mathcal{B} = \left\{ \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \\ 1 \end{bmatrix} \right\}.$$

2. (3 points) Consider the space of linear polynomials $\mathcal{P}_1$. Let $T : \mathcal{P}_1 \to \mathbb{R}$ be the linear transformation defined by

$$T(f) = \int_2^4 f(t)\, dt$$

If $f(x) = ax + b \in \mathcal{P}_1$ and $f \in \ker(T)$, then necessarily, $a = kb$ for some $k \in \mathbb{R}$. What is this value of k?

Solution: We have $f(x) = ax + b$. The condition that $f \in \ker(T)$ means precisely that

$$\begin{aligned} 0 &= T(f) \\ &= \int_2^4 (at + b)\, dt \\ &= \frac{a}{2}t^2 + bt\Big|_2^4 \\ &= 6a + 2b \end{aligned}$$

Thus, when we solve for a, we get $a = -\frac{1}{3}b$.

3. (2 points) Let $W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x^2 - y = 0 \right\}$. Show that W is **not** a subspace of $\mathbb{R}^2$ by finding $\mathbf{v}, \mathbf{w} \in W$ such that $\mathbf{v} + \mathbf{w} \notin W$.

Solution: We can take $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$. Note that $\mathbf{v} + \mathbf{w} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$, but $5 \neq (3)^2$; hence $\mathbf{v} + \mathbf{w} \notin W$.