Math 310 (33886), Spring 2016
Instructor: Chris Skalit
Quiz 8

Name: ______________________ UIN: ______________________

1. Suppose that V is a two-dimensional vector space with bases $\mathcal{A} = \{\mathbf{a}_1, \mathbf{a}_2\}$ and $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ such that

$$\mathbf{a}_1 = \mathbf{b}_1 + 5\mathbf{b}_2 \quad \mathbf{a}_2 = \mathbf{b}_1 + 6\mathbf{b}_2.$$

   (a) (1 point) Write down the change of basis matrix $P_{\mathcal{B}\leftarrow\mathcal{A}}$ which converts $\mathcal{A}$-coordinates to $\mathcal{B}$-coordinates.

   Solution:

$$P_{\mathcal{B}\leftarrow\mathcal{A}} = [[\mathbf{a}_1]_{\mathcal{B}}\,[\mathbf{a}_2]_{\mathcal{B}}] = \begin{bmatrix} 1 & 1 \\ 5 & 6 \end{bmatrix}$$

   (b) (2 points) Write down the change of basis matrix $P_{\mathcal{A}\leftarrow\mathcal{B}}$ which converts $\mathcal{B}$-coordinates to $\mathcal{A}$-coordinates.

   Solution:

$$P_{\mathcal{A}\leftarrow\mathcal{B}} = (P_{\mathcal{B}\leftarrow\mathcal{A}})^{-1} = \begin{bmatrix} 6 & -1 \\ -5 & 1 \end{bmatrix}$$

   (c) (2 points) If $\mathbf{v} = 4\mathbf{b}_1 - \mathbf{b}_2$, write down $[\mathbf{v}]_{\mathcal{A}}$ and $[\mathbf{v}]_{\mathcal{B}}$. (Hint: Use part (b).)

   Solution: We have, from inspection, that $[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$. Next we have

$$[\mathbf{v}]_{\mathcal{A}} = P_{\mathcal{A}\leftarrow\mathcal{B}}[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} 6 & -1 \\ -5 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ -1 \end{bmatrix} = \begin{bmatrix} 25 \\ -21 \end{bmatrix}$$

2. (3 points) Let $A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 9 \\ 0 & 1 & 1 \end{bmatrix}$. Find all eigenvalues of A.

   Solution: We compute the characteristic polynomial

$$P_A(t) = \det(A - tI) = \det\left(\begin{bmatrix} 5-t & 0 & 0 \\ 0 & 1-t & 9 \\ 0 & 1 & 1-t \end{bmatrix}\right) = (5-t)[(1-t)^2-9] = (5-t)[t^2-2t-8]$$

$$P_A(t) = (5-t)(t-4)(t+2)$$

   Hence, our eigenvalues are 5,4, and -2.

3. (2 points) Let $B = \begin{bmatrix} 3 & 0 & -2 \\ 1 & 2 & -2 \\ 2 & 0 & -2 \end{bmatrix}$. Compute the eigenspace E_2 corresponding to the eigenvalue 2. (You do not need to find the other eigenvalues.)

Solution:

$$E_2 = \operatorname{Nul}(B - 2I) = \operatorname{Nul}\left(\begin{bmatrix} 1 & 0 & -2 \\ 1 & 0 & -2 \\ 2 & 0 & -4 \end{bmatrix}\right)$$

As the nullspace of a matrix coincides with the nullspace of its reduced row-echelon form, we have

$$\operatorname{Nul}\left(\begin{bmatrix} 1 & 0 & -2 \\ 1 & 0 & -2 \\ 2 & 0 & -4 \end{bmatrix}\right) = \operatorname{Nul}\left(\begin{bmatrix} 1 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right).$$

Thus, if $\mathbf{x} \in \mathbb{R}^3$ satisfies $(B - 2I)\mathbf{x} = \mathbf{0}$, then $x_1 - 2x_3 = 0$ with x_2, x_3 free. Hence,

$$E_2 = \operatorname{Nul}(B - 2I) = \left\{ \begin{bmatrix} 2x_3 \\ x_2 \\ x_3 \end{bmatrix} : x_2, x_3 \in \mathbb{R} \right\} = \operatorname{span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\}.$$