Math 310 (33886), Spring 2016
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Quiz 9

Name: ______________________ UIN: ______________________

1. Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

   (a) (1 point) Find all eigenvalues of A

   Solution: Our characteristic polynomial is

   $$P_A(t) = \det(A - tI) = \det\left(\begin{bmatrix} 1-t & 2 \\ 2 & 1-t \end{bmatrix}\right) = (1-t)^2 - 4$$

   Hence, our eigenvalues are 3 and -1.

   (b) (3 points) Compute the eigenspace for each eigenvalue.

   Solution: We first compute the 3-eigenspace:

   $$E_3 = \text{Nul}(A - 3I) = \text{Nul}\left(\begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix}\right) = \text{Nul}\left(\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}\right)$$

   Now, $\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{0}$ if and only if $x_1 - x_2 = 0$, so

   $$\text{Nul}\left(\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}\right) = \left\{\begin{bmatrix} x_2 \\ x_2 \end{bmatrix} : x_2 \in \mathbb{R}\right\} = \text{span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$$

   Through a similar calculation, we find that the (-1)-eigenspace is

   $$E_{(-1)} = \text{Nul}(A + I) = \text{span}\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$$

   (c) (2 points) Using parts (a) and (b), diagonalize A. That is, find matrices S and D where D is diagonal and $A = SDS^{-1}$.

   Solution: D is simply the matrix with the eigenvalues of A running down the diagonal: $D = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$. The columns of S are merely the eigenvectors of A with the same ordering as their corresponding eigenvalues in D: $S = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$. $S^{-1} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}$

2. Let $\mathcal{P}_2$ be the space of polynomials having degree at most 2. Consider the standard basis $\mathcal{B} = \{1, t, t^2\}$. Suppose that $T : \mathcal{P}_2 \to \mathcal{P}_2$ is the linear transformation defined via

$$T(f) = t^2 \frac{d^2 f}{dt^2} + 2\frac{df}{dt}$$

(a) (2 points) Write down the matrix $[T]_{\mathcal{B}}$ for T with respect to the basis $\mathcal{B}$.

Solution: We need only check what T does to each basis vector:

$$T(1) = 0 \Rightarrow [T(1)]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$T(t) = 2 \Rightarrow [T(t)]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$

$$T(t^2) = 4t + 2t^2 \Rightarrow [T(t^2)]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix}$$

And thus, we have $[T]_{\mathcal{B}} = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 4 \\ 0 & 0 & 2 \end{bmatrix}$

(b) (2 points) What are the eigenvalues of T?

Solution: Note that with respect to $\mathcal{B}$, the matrix for T is triangular. Its characteristic polynomial is $P_T(t) = t^2(2 - t)$, meaning that the eigenvalues are 0 and 2.