Math 520: Fall 2016

Problem Set 1

1. Let $\phi : A \to B$ is a surjective morphism of rings.
   (a) Produce a bijection between the following sets:
   $$\{I \subseteq A : I \text{ an ideal}, \ \ker\phi \subseteq I\} \leftrightarrow \{J \subseteq B : J \text{ an ideal}\}$$
   (b) Show that the induced map $\phi^* : \operatorname{Spec} B \to \operatorname{Spec} A$ establishes a homeomorphism between $\operatorname{Spec} B$ and $V(\ker\phi)$.

2. Let $k = \overline{k}$ be a field. Describe all maximal ideals of $A = k[X]$.

3. Fix a prime $p \in \mathbb{Z}$. Let $H = \left\{ \dfrac{a}{p^n} \in \mathbb{Q} : a \in \mathbb{Z}, n \geq 0 \right\}$, regarded as an Abelian group. Show that $H/\mathbb{Z}$ has no maximal subgroups. Conclude that H is not Noetherian as a $\mathbb{Z}$-module. (What happens if you try to blindly invoke Zorn's Lemma as in the proof of the existence of maximal ideals?)

4. If M is a finitely-generated Abelian group, show that there exists a subgroup N such that $M/N \cong \mathbb{Z}/p\mathbb{Z}$ for some prime p.

5. Let R be a ring and let $I \subseteq R$ be an ideal.
   (a) Show that $\sqrt{I}$ is the intersection of all primes Q containing I. *Hint: Think about the nilradical of R/I and use Problem 1.*
   (b) Show that $V(I) = V(\sqrt{I})$.
   (c) Conclude that if J is another ideal, $V(I) \subseteq V(J)$ if and only if $\sqrt{J} \subseteq \sqrt{I}$.

6. When $f \in R$, we typically write $D(f)$ for the complement of $V(f)$ in $\operatorname{Spec} R$. Show that open sets of the form $D(f)$ form a basis for the Zariski topology on $\operatorname{Spec} R$. (Reminder: A basis $\mathcal{B}$ for a topological space X is a family of open sets such that for each open $U \subseteq X$ and each $x \in U$, there is a $B \in \mathcal{B}$ with $x \in B \subseteq U$.)

7. Let Q be a prime ideal of R. Use Zorn's Lemma to show that Q contains a minimal prime P.

8. Show that if $\operatorname{Spec} R$ is not connected, then there exists a decomposition $R = S \times T$.

9. (Optional) Use a simple cardinality argument to show that $\prod_{i \in \mathbb{N}} \mathbb{Q}$ does not have a countable set of generators (as a $\mathbb{Q}$ - module).