Math 520: Fall 2016

Problem Set 10

1. Let R be a Noetherian ring, M a finitely-generated R-module, and I an ideal. Let $F^n M$ be an I-filtration on M.

   (a) Let $\mathcal{R}(I) = \bigoplus_{n=0}^{\infty} I^n$ be the Rees algebra on I and let $\widetilde{M} = \bigoplus_{n=0}^{\infty} F^n M$ be the associated $\mathcal{R}(I)$-module. Show that $\widetilde{M}$ is finitely-generated if and only if the filtration on M is I-stable.

   (b) Conclude that if $N \subset M$ is a submodule and $F^n N := F^n M \cap N$, then the induced filtration on N is I-stable if the one on M is.

2. Let R be a Noetherian domain with fraction field K and let M be a finitely-generated R module.

   (a) Show that $M \otimes_R K = 0$ if and only if M is torsion.

   (b) In general, $M \otimes_R K$ is a vector space with basis $\{m_1/f_1, \cdots m_n/f_n\}$ with $m_i \in M$, $f_i \in R$. If $R^n \to M$ is the map sending $e_i \mapsto m_i$, show that this gives an exact sequence
   $$0 \to R^n \to M \to C \to 0$$
   where C is torsion.

3. Let R be a Noetherian ring and that M is a finitely-generated R-module. Show that M is Artinian if and only if $\operatorname{Supp} M$ is a finite set of closed points.

4. For any finitely-generated module M over a Noetherian ring A, we define $\dim M = \dim(A/\operatorname{ann} M)$. When $(A, \mathfrak{m})$ is local, we can also define
   $$\delta(M) = \inf\{r : \exists x_1, \cdots, x_r \in \mathfrak{m} \text{ such that } \ell(M/(x_1, \cdots, x_r)M) < \infty\}$$
   $d(M) = \deg p_{\mathfrak{m},M}$ where $p_{\mathfrak{m},M}$ is the polynomial such that $p_{\mathfrak{m},M}(n) = \ell(M/\mathfrak{m}^n M)$ for all $n >> 0$

   Show that $d(M) = \dim M = \delta(M)$ when A is local by completing the following outline:

   (a) Let $B = A/\operatorname{ann} M$. Note that M is naturally a B-module. Since we already know the dimension theorem for B, it will suffice to prove $\delta(B) = \delta(M)$ and $(B) = \delta(M)$.

   (b) Show that M is supported at every point of $\operatorname{Spec} B$.

   (c) Using question 3, show that $M \otimes_B B/I$ is artinian if and only if B/I is. Conclude that $\delta(M) = \delta(B)$.

   (d) Deduce $d(B) \geq d(M)$ from the fact that B^n surjects on to M.

   (e) Let $\mathfrak{p} \subseteq B$ be a minimal prime such that $\dim B = \dim(B/\mathfrak{p})$. Let $B' = B/\mathfrak{p}$ and $M' = M/\mathfrak{p} = M \otimes_B (B/\mathfrak{p})$. Show that M', as a B'-module, is supported at every point of $\operatorname{Spec} B'$. Conclude that $\dim M = \dim M' = \dim B'$

(f) Since B' is a domain, question 2 says there is an exact sequence $0 \to (B')^n \to M' \to N \to 0$ with N a torsion module. Show that $\dim N < \dim B'$ and deduce from the theorem of additivity of Hilbert functions that $d(M') = d(B')$.

(g) Since $\dim B' = \dim B$, the dimension theorem for local rings says that $d(B') = d(B)$. Since $d(M) \geq d(M')$, we have $d(M) \geq d(M') = d(B') = d(B)$. The proof is now complete.

5. Let $(A, \mathfrak{m})$ be local Noetherian and let M be finitely-generated over A. Suppose that $f \in \mathfrak{m}$ lies outside of every minimial prime of $\operatorname{Supp} M$ (this is the case, for example, when f is a non-zerodivisor on M). Show that $\dim M - 1 = \dim M/fM$.

6. For any numerical function $f : \mathbb{N} \to \mathbb{N}$, we can take iterated discrete derivatives:

$$\begin{aligned} \Delta f(n) &= f(n+1) - f(n) \\ \Delta^2 f(n) &= \Delta f(n+1) - \Delta f(n) = f(n+2) - 2f(n+1) + f(n) \\ &\vdots \end{aligned}$$

Let $(A, \mathfrak{m})$ be a local Noetherian ring and let M be a finitely-generated A-module of dimension d. Let I be $\mathfrak{m}$-primary and put $h_{I,M}(n) = \ell(M/I^n M)$. Show that for all $t >> 0$ and all $r \geq d$,

$$\Delta^r h_{I,M}(t) = e_I(M, r) := \lim_{n \to \infty} \frac{r!}{n^r} \ell(M/I^n M).$$

Conclude that $e_I(M) = e_I(M, d)$ is a positive integer.