Math 520: Fall 2016
Problem Set 11

1. Suppose we have a commutative diagram in R-mod with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \\ 0 & \longrightarrow & A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \longrightarrow & 0 \end{array}$$

   (a) If $y \in \ker\gamma$, take $x \in B$ such that $g(x) = y$. Show that $\beta(x) = f'(w)$ for some $w \in A'$ and that $y \mapsto \overline{w}$ gives a well-defined morphism $\phi : \ker\gamma \to \operatorname{coker}\alpha$.

   (b) Show that there is an exact sequence

$$0 \to \ker\alpha \to \ker\beta \to \ker\gamma \xrightarrow{\phi} \operatorname{coker}\alpha \to \operatorname{coker}\beta \to \operatorname{coker}\gamma \to 0$$

2. Suppose that we have an exact sequence of inverse systems of R-modules

$$0 \to \{A_n\} \to \{B_n\} \to \{C_n\} \to 0$$

   Assume all inverse systems are indexed over $\mathbb{N}$.

   (a) Show that the induced sequence on inverse limits is left-exact.

   (b) Show that if the maps in $\{A_n\}$ are all surjective, then the induced sequence on inverse limits is exact. (Use the Snake Lemma from Question 1).

3. Let M be an R-module with decreasing filtration F^nM. Let $(\overline{x}_n) \in \varprojlim_n M/F^nM$ be a coherent sequence.

   (a) Show that if $x_n \in M$ has residue $\overline{x}_n$ in M/F^nM, then the sequence (x_n) is Cauchy.

   (b) Show that if $y_n \in M$ is a second lift of $\overline{x}_n$, then $\lim\limits_{n\to\infty} x_n - y_n = 0$.

4. A local ring $(A, \mathfrak{m})$ is called *Hensellian* if it satisfies the following condition: Given a monic polynomial $f \in A[X]$, if there exists a factorization $\overline{f} = \overline{g}\overline{h}$ in $(A/\mathfrak{m})[X]$ with $\overline{g}$ and $\overline{f}$ coprime, then they lift to a factorization $f = gh$ in $A[X]$. Show that an $\mathfrak{m}$-adically complete, Noetherian local ring $(A, \mathfrak{m})$ is Hensellian.

5. Let $\phi : (A, \mathfrak{m}) \to (B, \mathfrak{n})$ be a local homomorphism of local rings. (A local homomorphism means that $\mathfrak{m}B \subset \mathfrak{n}$.) Show that if ϕ is flat then it is faithfully flat. Conclude that if $(A, \mathfrak{m})$ is a Noetherian local ring then $A \to \hat{A}_{\mathfrak{m}}$ is faithfully flat.

6. Suppose that $(A, \mathfrak{m})$ is a Noetherian local ring and that $\hat{A}$ is the $\mathfrak{m}$-adic completion. Let R be an arbitrary ring and suppose that $I \subset R$ is an ideal such that $I^2 = 0$. Suppose we have a commutative diagram

$$\begin{array}{ccc} \hat{A} & \xrightarrow{\phi} & R/I \\ \uparrow & & \uparrow \\ A & \longrightarrow & R \end{array}$$

where $\phi(\mathfrak{m}^n \hat{A}) = 0$ for some $n > 0$. Show that there is a map $\hat{A} \to R$ making the diagram commute.

7. Let $(A, \mathfrak{m})$ be a DVR. Show that $e_{\mathfrak{m}}(A) = 1$.