Math 520: Fall 2016
Problem Set 12

1. A ring map $A \to B$ is called 0-smooth if the following condition holds: Given any ring C with ideal I such that $I^2 = 0$ and a diagram

$$\begin{array}{ccc} B & \longrightarrow & C/I \\ \uparrow & & \uparrow \\ A & \longrightarrow & C \end{array}$$

then there exists a map $B \to C$ making the diagram commute.

   (a) Show that if $B = A[X]$, then $A \to B$ is 0-smooth.

   (b) Show that if $B = S^{-1}A$ for some multiplicative $S \subset A$, then $A \to B$ is 0-smooth.

   (c) Show that if $B = A[X]/(f)$ where f is a monic polynomial with $(f, df/dx) = A[X]$, then $A \to B$ is 0-smooth. (In particular, any finite, separable field extension is 0-smooth.)

   (d) Show that a composite of 0-smooth maps is 0-smooth.

2. Let E/k be a finitely-generated extension of fields. If k is perfect, show that there exist $x_1, \cdots x_n \in E$ which are algebraically-independent over k and such that $k(x_1, \cdots, x_n) \subset E$ is a finite, separable extension.

3. Let k be a perfect field and let R be a finitely-generated k-algebra. Let $\mathfrak{p} \in \operatorname{Spec} R$ and let $(A, \mathfrak{m})$ be the completion $\widehat{R_{\mathfrak{p}}}$. If K is the residue field of A, use Questions 1 and 2 to show that there is a map $K \to A$ that maps isomorphically onto the residue field. *Hint: Show that $k \to K$ is 0-smooth and inductively construct maps $K \to A/\mathfrak{m}^n$.*

4. If M and N are R-modules, show that $\operatorname{Supp}(\operatorname{Tor}_i^R(M, N)) \subset \operatorname{Supp}(M) \cap \operatorname{Supp}(N)$ for all $i \geq 0$.

5. A directed set $(\mathcal{D}, \leq)$ is a partially-ordered set such that for all $i, j \in \mathcal{D}$, there exists a $k \in \mathcal{D}$ such that $i, j \leq k$. A directed system of R-modules $\{M_i\}_{(i \in \mathcal{D})}$ is a family modules with maps $\phi_{ij} : M_i \to M_j$ if $i \leq j$ and subject to the relations: $\phi_{jk} \circ \phi_{ij} = \phi_{ik}$ if $i \leq j \leq k$ and $\phi_{ii} = Id_{M_i}$. The direct limit $\varinjlim M_i$ is a module equipped with maps $\psi_j : M_j \to \varinjlim M_i$ which satisfies the following universal property: For any module N with homomorphisms $\pi_j : M_j \to N$ such that $\pi_k \circ \phi_{jk} = \pi_j$, there is a unique $\gamma : \varinjlim M_i \to N$ such that $\gamma \circ \psi_j = \pi_j$. Let $\mathbf{M} = \left(\bigoplus_{i \in \mathcal{D}} M_i \right) / \mathcal{R}$ where $\mathcal{R}$ is the submodule generated by the relations $x_j - \phi_{ij}(x_i)$ whenever x_i maps to x_j via ϕ_{ij}. Show that $\mathbf{M}$ satisfies the universal property.

6. If $\{M_i\} \to \{N_i\} \to \{Q_i\}$ is a sequence of directed systems which is exact in the middle, show that $\varinjlim M_i \to \varinjlim N_i \to \varinjlim Q_i$ is also exact in the middle. That is, "taking direct limits preserves exactness."

7. If $\{M_i\}$ is a directed system and N is a fixed R-module, we can form a new directed system by applying the functor $- \otimes_R N$. Show that there is a canonical isomorphism

$$\varinjlim (M_i \otimes_R N) \cong (\varinjlim M_i) \otimes_R N$$

Hint: You can construct canonical maps in both directions via the universal properties for direct limit and tensor product.

8. If $\{M_i\}$ is a directed system and N is a fixed R-module, show that

$$\varinjlim \operatorname{Tor}_i^R(M, N) = \operatorname{Tor}_i^R(\varinjlim M_i, N) \text{ for all } i \geq 0.$$

Hint: Take a projective resolution $P. \to N \to 0$ *and consider the chain-complex of directed systems* $M_i \otimes_R P.$*. Since taking direct limits is exact, it preserves homology.*

9. If N is an R-module, show that N is flat if and only if $\operatorname{Tor}_1^R(M', N) = 0$ for all **finitely-generated** modules M'. *Hint: If* M *is an arbitrary module, realize it as a direct limit of its finitely-generated submodules.*

10. If R is a Dedekind domain, show that an R-module N is flat if and only if it is torsion-free. *Hint: Reduce to the local case and use the previous exercise...*