Math 520: Fall 2016
Problem Set 13

1. Let $A \to B$ be a flat ring map. For M and N A-modules, show that $\mathrm{Tor}_i^A(M,N) \otimes_A B \cong \mathrm{Tor}_i^B(M \otimes_A B, N \otimes_A B)$ for all $i \geq 0$.

2. Let R be a Dedekind ring. Let M be a finitely-generated module. Show that $\mathrm{pd}_R(M) \leq 1$. *Hint: Choose a surjection $R^n \to M$. Show that K is locally-free.*

3. Let $(A, \mathfrak{m})$ be a Noetherian local ring with $\mathfrak{m} = (x_1, \cdots, x_d)$. Suppose that the x_i form a regular sequence for A. Prove that A is regular (in the sense that the embedding dimension and Krull dimension of A coincide).

4. A Noetherian ring R is called regular if for every $\mathfrak{p} \in \operatorname{Spec} R$ $R_\mathfrak{p}$ is a regular local ring. Prove that R is a regular ring if and only if every finitely-generated R module M has finite-projective dimension by completing the following outline:
   (a) If R is Noetherian and M is finitely-generated over $S^{-1}R$, show that there is a finitely-generated R-module M_0 such that $M \cong S^{-1}M_0$. *Hint: M is defined by a matrix over $S^{-1}R$. Lift this matrix to R.*
   (b) Assume that every finitely-generated R module has finite projective dimension. Let k be the residue field of some local ring $R_\mathfrak{p}$. Deduce that $\mathrm{pd}_{R_\mathfrak{p}}(k) < \infty$ and conclude that $R_\mathfrak{p}$ is regular.
   (c) Now assume that every local ring $R_\mathfrak{p}$ is regular. If N is a finitely-generated R-module, take any free resolution $F. \to N \to 0$ with each F_i having finite rank. If $E_i = \ker(P_i \to P_{i-1})$, show that for each $\mathfrak{p}$ $(E_i)_\mathfrak{p}$ is free for $i >> 0$.
   (d) With the assumptions of part (c), prove the following more precise statement: For each $\mathfrak{p} \in \operatorname{Spec} R$, there exists an integer $j_\mathfrak{p} > 0$ such that for all $i > j_\mathfrak{p}$, there exists an $0 \neq f_{\mathfrak{p},i} \in R$, such that $(E_i)_{f_{\mathfrak{p},i}}$ is free over $R_{f_{\mathfrak{p},i}}$.
   (e) Use the quasi-compactness of $\operatorname{Spec} R$ to conclude that E_i is locally-free (and hence projective) for $i >> 0$.

5. Let $R \to A$ be a faithfully-flat extension of Noetherian rings. Show that R is regular if A is.

6. Give a direct proof that $\mathbb{Q}$ is not projective over $\mathbb{Z}$.

7. Let R be a ring and let M be an R-module. Let $x \in R$ be a non-zerodivisor on M and R. Put $\overline{R} = R/xR$ and $\overline{M} = M/xM$. Show that for all $\overline{R}$-modules N, $\mathrm{Tor}_n^{\overline{R}}(\overline{M}, N) = \mathrm{Tor}_n^R(M, N)$ for all $n \geq 0$.