Math 520: Fall 2016

Problem Set 2

1. Is a subring of a Noetherian ring Noetherian? Give a proof or counterexample.

2. Let R be Noetherian and let $I \subseteq R$ be an ideal. Show that $\operatorname{Spec} R/I$ is irreducible if and only if $\sqrt{I}$ is prime.

3. Let $f \in R$. Show that $R_f \cong R[X]/(Xf - 1)$. Thus, R_f is a finitely-generated R-algebra.

4. Let $S \subset T$ be multiplicative subsets of a ring R. Let $\widetilde{T}$ be the image of T in $S^{-1}R$. Show that $T^{-1}R \cong \widetilde{T}^{-1}(S^{-1}R)$.

5. A ring R is called *reduced* if its nilradical is zero. Show that R is reduced if and only if $R_\mathfrak{p}$ is reduced for every prime ideal $\mathfrak{p}$.

6. Suppose that every prime ideal of R is maximal. Show that $\operatorname{Spec} R$ is a Hausdorff space. *Hint: Reduce to the case where R has trivial nilradical. Given primes $\mathfrak{p}$ and $\mathfrak{q}$, choose $f \in \mathfrak{p} - \mathfrak{q}$, and note that f vanishes in $R_\mathfrak{p}$...*

7. Let A be a Noetherian ring and M a finitely-generated A-module. Show that if $M_\mathfrak{p}$ is a free $A_\mathfrak{p}$ module then M_f is free over A_f for some $f \in A$.

8. If R is a reduced ring, show that f is a zerodivisor if and only if f is contained in some minimal prime.

9. Let A be Noetherian aand M a finitely-generated module. Let $\mu_M : \operatorname{Spec} A \to \mathbb{R}$ be such that $\mu_M(\mathfrak{p}) = \dim(M_\mathfrak{p}/\mathfrak{p}M_\mathfrak{p})$ (where the dimension is being taken over the field $k(\mathfrak{p}) := A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}$).
   (a) Show that μ_M is upper-semicontinuous. That is, for all $\mathfrak{p} \in \operatorname{Spec} A$ and $\epsilon > 0$, there is an open neighborhood $U \subset \operatorname{Spec} A$ of $\mathfrak{p}$, such that $\mu_M(\mathfrak{q}) \leq \mu_M(\mathfrak{p}) + \epsilon$ for all $\mathfrak{q} \in U$.
   (b) If A is reduced and μ_M is constant, show that M is locally free (i.e. $M_\mathfrak{p}$ is free for all $\mathfrak{p}$).
   (c) If A is reduced, show that μ_M is continuous if and only if M is locally free.

10. Let A be a Noetherian ring and M a finitely generated module. We define the *support* of M via $\operatorname{Supp} M = \{\mathfrak{p} \in \operatorname{Spec} A : M_\mathfrak{p} \neq 0\}$. Show the following:
    (a) $\operatorname{Supp} M = V(\operatorname{ann}(M))$ where $\operatorname{ann}(M) = \{a \in A : ax = 0 \text{ for all } x \in M\}$.
    (b) If $0 \to M' \to M \to M'' \to 0$ is exact, then $\operatorname{Supp} M = \operatorname{Supp} M' \cup \operatorname{Supp} M''$.